**Control Engineering**
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**Lecture – 20**

Let me mention one important thing, electrical engineering course or program in general and in particular a course or a lecture series on control system requires a fairly good bit of mathematical knowledge and so, one should not shy away from the mathematics or get unnecessarily scared by it. Like any subject, if you devote enough attention, you can learn the necessary mathematic and it should not be thought of as a you know an unfortunate burden that you know.

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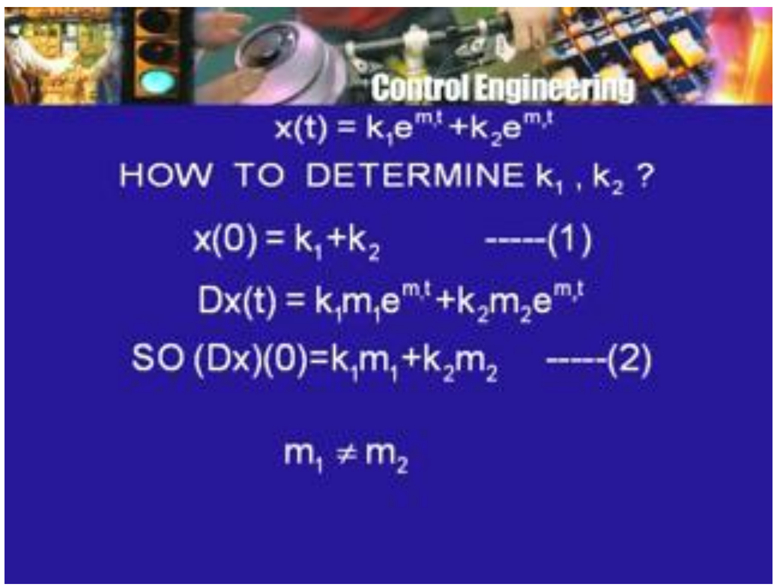


I would like to study the course but what to do there is a lot of mathematics in it and you know, is it really necessary and things like that. Well, it is absolutely necessary one is not using mathematics just because one has studied it or one wants to show off or anything like that. The mathematics is there simply because the systems cannot be described or studied without the appropriate mathematical concepts and ideas, perhaps you can go to the laboratory and start a DC shunt motor without using any mathematical knowledge in that process and of course, mechanics and workshop people all over the world, they have been doing this kind of a things.

They have a been operating equipment without perhaps having the necessary mathematical knowledge to understand, how the equipment works, why somethings where not working satisfactorily therefore, changes were made, new designs where introduced and so on and so forth. For the control engineer, a differential equation is something with which, a he or she has to be very familiar and not afraid of that is, if I see a differential equation, I should not turn away from it or feel that it is an unnecessary bother because as I told you earlier that unless, you are looking at things only in the steady state, where everything is constant and therefore only algebra and a little bit of arithmetic all that are involved, when you are looking at changes and the whole business of control and in particular feedback control is to cope with these changes, some unpredictable changes, variations and so on.

So, when you are not looking only at the steady state but you are going to look at the transient performance, things are going to be changing with time and therefore, effects like inertia, either mechanical, the moment of inertia or electrical inductance cannot be ignored have to be considered and the only way of considering them is through the use of the derivative. Of course, associated with an inductor, let us say is flux flux linkages or what have you and production of EMF, when the current in the inductor changes a certain voltage appears across the inductor and so on.

But this is only qualitative, we need to relate the voltage associated with the inductor or the voltage across the inductor with the current, associated with the inductor or the current in the inductor and the only way in which, we can relate it is through the derivative. By saying okay, the voltage of the inductor is l times, the rate of change of current and there comes the derivative and there is no way of avoiding it. Similarly, when a rotating body is being accelerated, there is an angular acceleration, there is a torque required for that and if you think in terms of speed then, the relationship between torque and speed involves derivative because acceleration is derivative of the torque.

So, let us not be afraid of the mathematics and let us learn it, as an when it is required some of which you already have. In fact, you may have studied it in mathematics courses without knowing, why you are studying it. For example, you may have look at differential equation just like that first order, second order or whatever and been asked to solve 50 differential equations of the same kind, without being told well, where do these differential equations occur. Of course, some examples from mechanics may have been given but that they occur in electrical engineering or chemical engineering or aerospace engineering etcetera. Those examples may not have been presented to you, so it is time now to look at that.

So, we are looking at the solution of a ordinary linear differential equation with constant coefficient which was homogeneous, meaning that there is no input or strictly speaking the input was 0 or the forcing function or stimulus was the 0 function. Then, how does the system behave. So, we looked at the first order differential equation and we found out that it could be solved easily because fortunately, we were familiar with the exponential function. By the way, this also means that you have to go beyond the simpler functions which one may have studied in school or even in the 11th and 12th standard. You do have to look in to these more complicated functions, not too complicated but the exponential, the sinusoidal function, the polynomial functions and combinations of them, as we will see. These functions are unavoidable; in fact you have studied a more complicated function namely the logarithm in school.

So, the first order differential equation solution just consists of an exponential with an appropriate coefficient of the exponent or an appropriate exponent and in order to determine the solution uniquely, you need to have an initial or a boundary condition. Then, we went on to study the second order differential equation and one method that I talked about was, to assume that the solution might be an exponential function and then, you discover that the exponent must satisfy a particular equation called the indicial equation. Fortunately, that equation is of the second degree or a quadratic and one knows therefore, how to find such an exponent or index m it turns out that in general, you will get 2 numbers m 1 and m 2. As we show earlier and we looked at 3 or 4 different cases and therefore, the solution consists of a sum of a 2 terms sum of 2 exponentials with different exponents and with coefficients that multiply them which will be different in general and to determine the solution, you need 2 initial condition.

So, for example if x t is the solution then, it has k 1 and k 2 as the 2 parameters or 2 unknown or undetermined coefficients, as they are sometimes called. Then, we need 2 equations hopefully, they will be enough to find out k 1 and k 2 and what are they x 0 is known or given. So that gives you one equation and the derivative of x at 0 is also given. So that gives you the other equation hopefully these 2 equations can be solved to get a unique pair of numbers k 1 and k 2 and if that is done then, we have got the solution and we can confidently say, that the system which was described by the second order differential equation with these coefficients a 0 and a 1 and with these initial values x 0 and D x 0 will behave like this, where m 1, m 2 are these numbers, k 1, k 2 are these numbers. We can then plot it either by hand or you can get a computer plot and things like that. So the differential can be solved.

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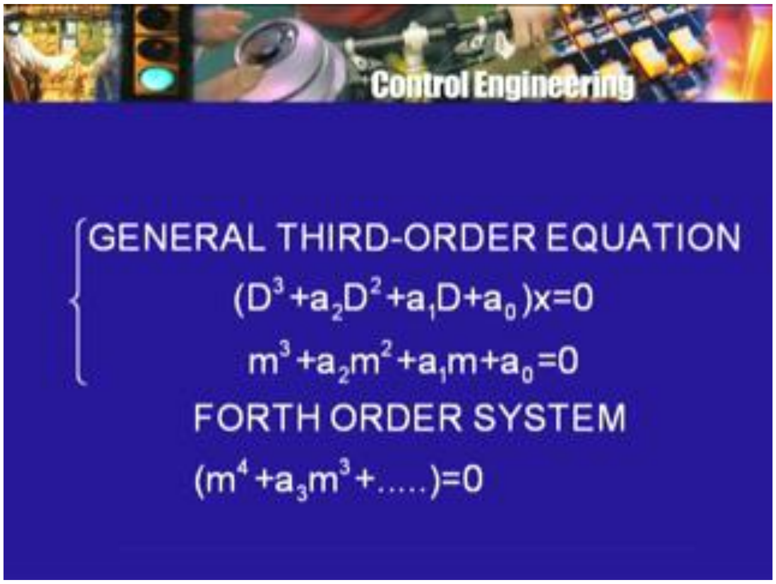


Now, this is for second order or second differential equation, what do you do with a third order or third degree differential equation. It should be clear that this approach can be tried once again, the differential equation may be of third order but I can try a solution which is again e raise to m t or k multiplied by e raise to m t and then, I will find that this m has to satisfy a third degree polynomial equation. It will be perhaps something like m cube plus a 2 m square plus a 1 m plus a 0 equals 0. Find an m such that this is true a 2, a 1 and a 0 are 3 coefficients that appear in the differential equation. So what must the differential equation be, the differential equation would be in our notation D cube plus a 2 D square plus a 1 D plus a 0. This operator polynomial differential operator acting on x produces the 0 function there is no input or 0 input.

So, x satisfies this condition if x is assumed to be an exponential function with exponent m with a multiplying coefficient k then, m must satisfy this polynomial equation of third degree. Now, unfortunately you may not remember the formula for the solution or for the roots of cubic equation although there is a formula but that formula is fairly involved not only that, today one will not bother to use a formula, but one would rather use a computer program based on some numerical analysis technique, to find out the solutions of such an equation or to find out the roots of the 0s of a polynomial and there is a good reason for that and that is as follows.

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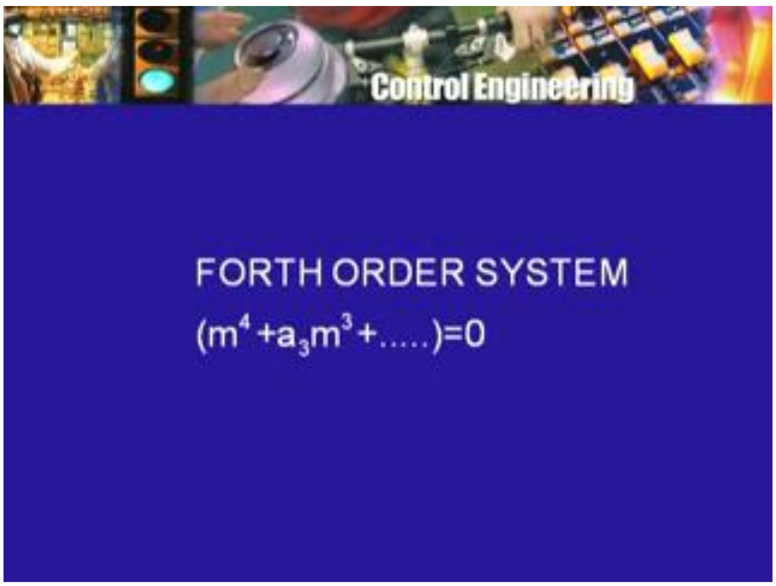


If I have a 4th order differential equation, I will get a polynomial of degree 4 m to the power 4 plus a 3, a m cube plus etcetera acting on x equal to 0. Note the way I am labeling the coefficients of the powers, a 2 goes with m raise 2, a 1 goes with m which of course is m raise to 1 and a 0, I can think of going with a 0 in to m raise to 0. Therefore, in the 4th degree polynomial the coefficient of m cube, I am labeling it as a 3 that is when I want to talk about the general equation, I will use this notation. For 4th degree polynomial also there are formulas for the roots, the formulas are even more complicated then, the formulas for the or the formula for the cubic but there are formulas. Unfortunately, for degree greater than 4 that is 5, 6, 7 or whatever there is no general formula unlike the quadratic case where you had, if you remember for the equation in the usual form a x squared plus b x plus c, in school algebra. The roots where given by x equal to minus b plus minus square root of b square minus 4 a c divided by 2 a.

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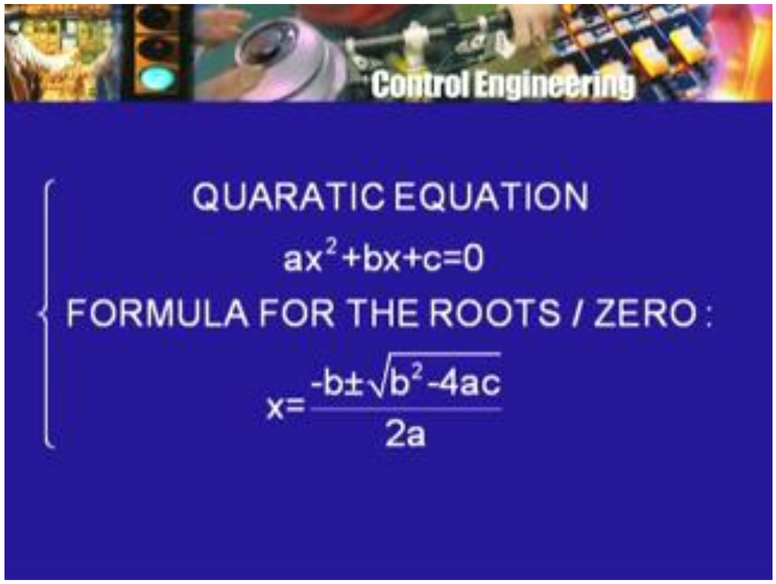


So you had a formula which involved only the coefficients a, b, c all you have to do is put those numbers in place of a, b, c and you work out the required things square, subtract, square root whatever and you get not 1 but 2 solution. So there is similar formula for a cubic and for the 4th degree polynomials by similar, I do not mean necessarily involving only square roots but there is a formula which involves these operation, addition, multiplication subtraction, division, square root, cube root, 4th root and so on.

But for degree higher than 4, there is no formula and not only there is not formula, there is no hope of anybody being smart enough to find a general formula. This is an important result in mathematics which we perhaps do not know in at school level, it is not easy to prove it. Of course, because one is saying that no matter, how clever anybody may be there is no such, no formula for the roots of a polynomial of degree greater than 4 involving the coefficients of the polynomial and only these operations of addition, subtraction, multiplication, division and extraction of a root, square root, cube root, 4th root, 5th root whatever, if these are the only things that you are allowed to do and these are the only computation that one can perform at school level also, certainly with a calculator one can do them. Then, there is not such formula and therefore, you have to use some approximation technique to find the roots of a polynomial of degree greater than 4 and these techniques can be used for polynomials of degree 4 and 3 also.

Although for a quadratic the formula is fairly simple and its, we have learnt it at an early age and therefore, we are not any longer afraid of using this formula rather than write a computer program or use a computer program that finds the roots of a quadratic. So in general then, one will have to use a computer program which is based on some numerical analysis technique of finding the roots or because these are numbers which make the polynomials 0s, they are called the 0s of a polynomial.

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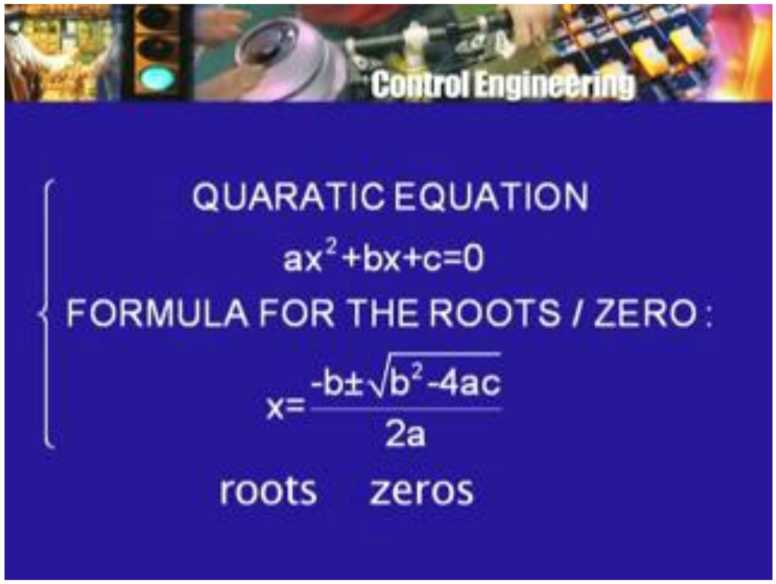


Now, this turns out to be a very basic problem in a lot of control system theory, in particular the control system the theory of what are called linear time invariant control system such as the motor problem that we are looking at .You come across this problem of finding the roots or 0s of a polynomial. If we use this e raise m t approach then, of course we come across it right away but we will see that when we use a Laplace transformation method, we will again come across the same problem of finding the roots of a polynomial.

So, this is a problem which you have to live with and you have to find out or you have know rather, numerical methods of finding the roots of a polynomial. Fortunately, there are ready made programs available, all you have to input to these programs is the coefficients of the polynomial, of course you have to do it in the proper order and then, use a command key or use a command and you will get the numbers printed out which are the roots of the polynomial. So it is as easy as that but this is something which is involved in the solution of these problem and of course, as I said in the quadratic case, there were 4 different cases and that is because the roots may not be purely real, they could be purely imaginary or they could be complex also, the quadratic may have only one root therefore, we say it is one root repeated twice or of multiplicity 2 or 2 coincidence roots but all I get is only one number from the formula.

Similarly, the cubic will have, could have only root, it is quite easy to cook up examples. For example, I simply write m plus 1 cube expand it out as a polynomial and again from school days everybody remembers, what this is so m cube plus 3 m square plus 3 m plus 1 and now, if I am looking at the polynomial equation this thing is equal to 0 and there is only one number that will do the job, it is clear because its m plus 1 cube. So the only root of this is m equal to minus 1.

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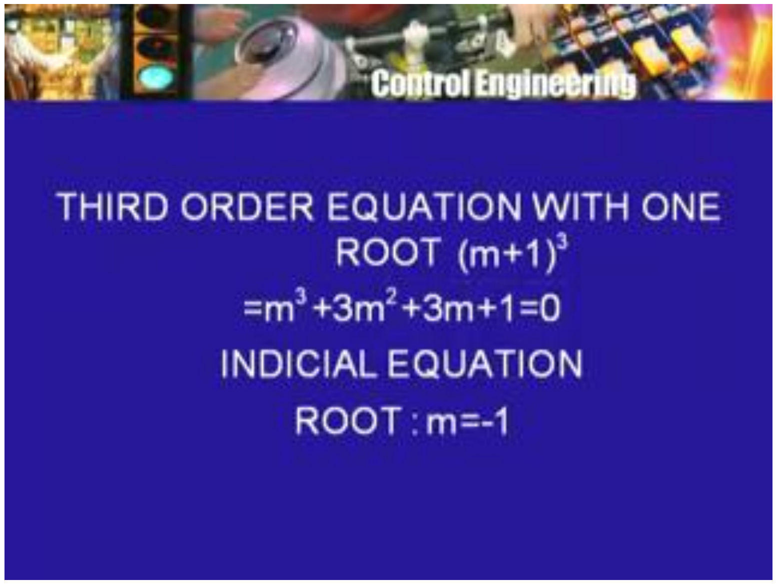


Now, I may pretend that it is minus 1 repeated 3 times and there is a reason for thinking of it that way but what I get is only one number. Now, if this is associated with a third order differential equation then from this I will get only e raise to minus the or k, e raise to minus t as a possible solution. But that may not be enough because you will actually find that this may not fit the behavior, mathematically that is because we can find out another function which also satisfies the same differential equation, but it is not k e to the minus t, but it is k t, e raise to minus t.

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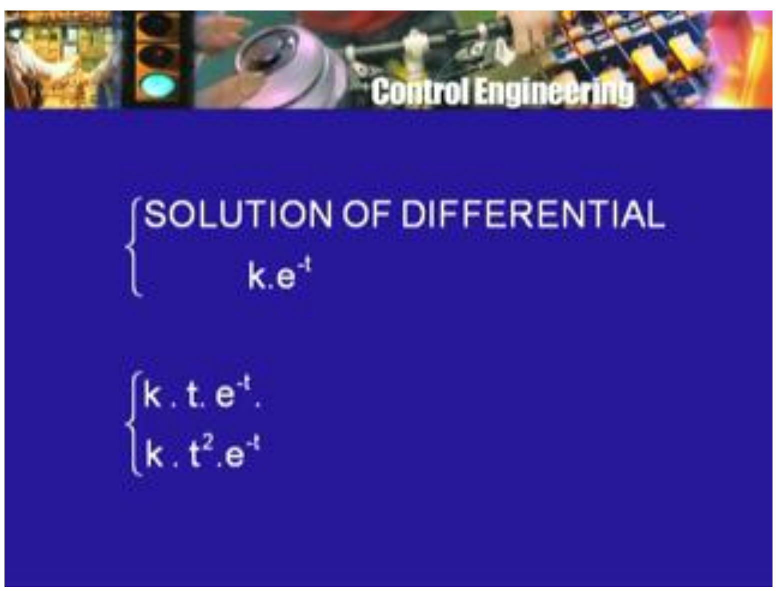

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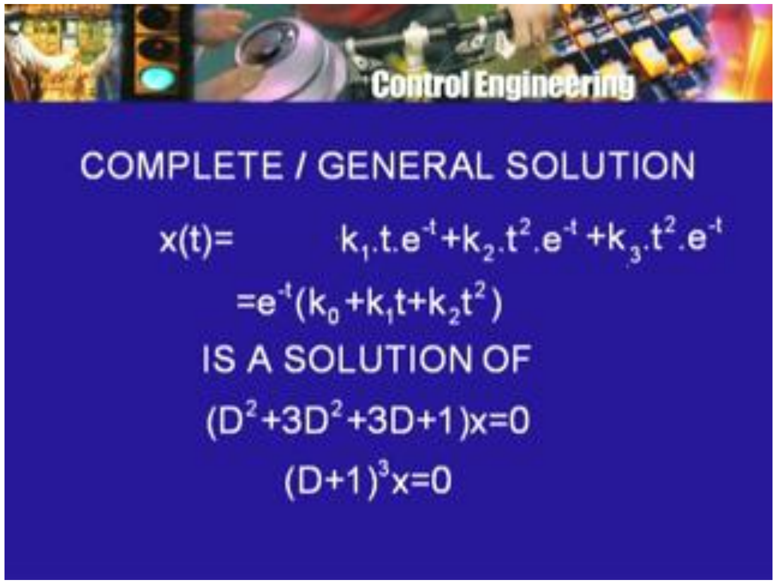


In fact, we saw this earlier for the second degree problem, for the third degree problem then, I will have to go one step further and I will also have to think of k t square in to e raise to minus t and so, what is called sometimes the complete solution that is a formula which will include all solutions will be given by k 1, e raise to minus t plus k 2 in to t in to e raise to minus t plus k 3 in to t square in to e raise to minus t or pulling out that e raise to minus t outside, e raise to minus t multiplied by and I will change the notation for the reason that you know now, instead of k 1, I will call it k 0 because it goes with the 0 th power of t plus k 1 t plus k 2 t square.

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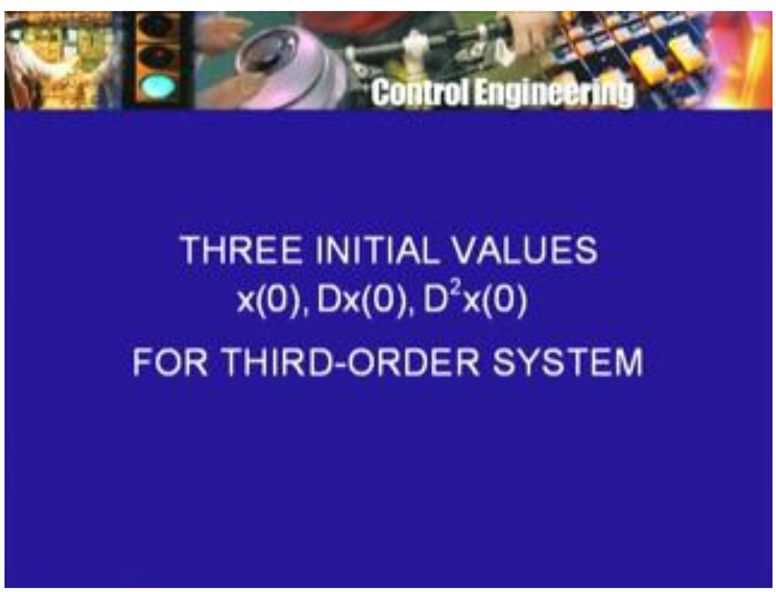

So, this will be the general solution that means the set of these functions for various value of k 0, k 1, k 2, all of them will satisfy one and the same third order differential equation which is D cube plus 3 D square plus 3 D plus 1, acting on x equal to 0 or even more compactly, I can write D plus cube acting on x equal to 0, one can show of course, the solutions of this equation are all included in the set, all of these are solutions. So this is the complete set of solution. So this is called usually in math's course, the general solution but that gives me infinitely many solutions whereas, I want to know what is going to happen to this particular system.

Now, for that once again we need initial conditions and this time, there are 3 unknown or undetermined coefficients. So, I may have to look for 3 initial conditions and indeed one can show that given the 3 initial conditions x 0 that is the value of x, the response at 0, the value of D x at 0 and the value of D square x at 0, given these initial condition, one can uniquely determine, these 3 numbers k 0, k 1, k 2, what were unknown or undetermined coefficients, can be determined or will become known uniquely and therefore, it will give you a unique solution of the differential equation but subject to the specified initial value or boundary value. Now, this of course will apply to higher order differential equations of the kind that we are thinking of.

So, in general for the n th order linear differential equation with constant coefficients and which is homogeneous, I will require n, if n is the order I will require n initial values, x 0, D x 0, D square x 0 and derivatives up to order n minus 1 and there will be these exponential function e raise to m t, with different values of m and if, there are not enough number of them because of this phenomenon of repeated root. Then, they will be multiplied by expressions of this kind t, t square and so on. So this a result that we have but 0 input is a very very special case, in fact it is hardly interesting to operate a control system, you are going to apply some input its only as a special case that you may want to look at the 0 input situation.

So what when the input is not 0 how do you get a solution or the solution if the system equation and therefore you will say that will this will what will happen in the laboratory. If I go and setup and apply this input, this is the response that I expect to see, how does one solve therefore, LDEWCC which is not homogeneous or which is said to be non-homogeneous meaning that there is an input which is not 0, there is a forcing function, there is a stimulus, knowing the stimulus function, knowing the forcing function of the input that is going to be applied. Can I find out the solution of the system or the differential equation, how, that is the question.

Now, once again before we go to the Laplace transformation method which is the preferred method in control theory and which is the one which you will be using extensively, it is good to know that there are more elementary method, elementary from this point of view that you do not have to know what Laplace transformation is which also perhaps you have studied in your mathematics, introductory mathematics courses. Let us take the first order differential equations I will write it as D plus a 0 x, x is the response, x as a function of time and buying to control theory practice on the right hand side I will not write y or f which is very commonly done in mathematic courses but I will write u, this is the preferred symbol for the input function u. So my problem will be find out the solution x, if I am given u for example I may be told that ut because u, this u is an input which may not be constant ut equals sin t and of course, we expect that in order to get a unique solution I will required to know x 0 also.

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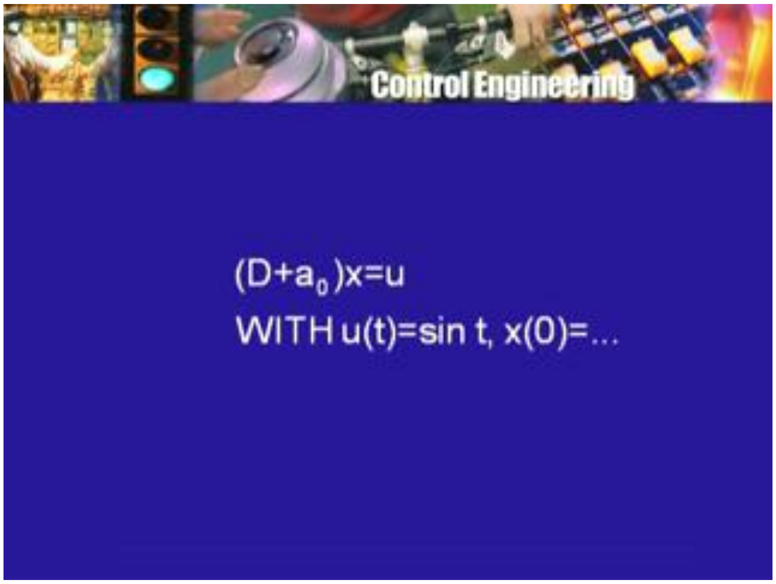


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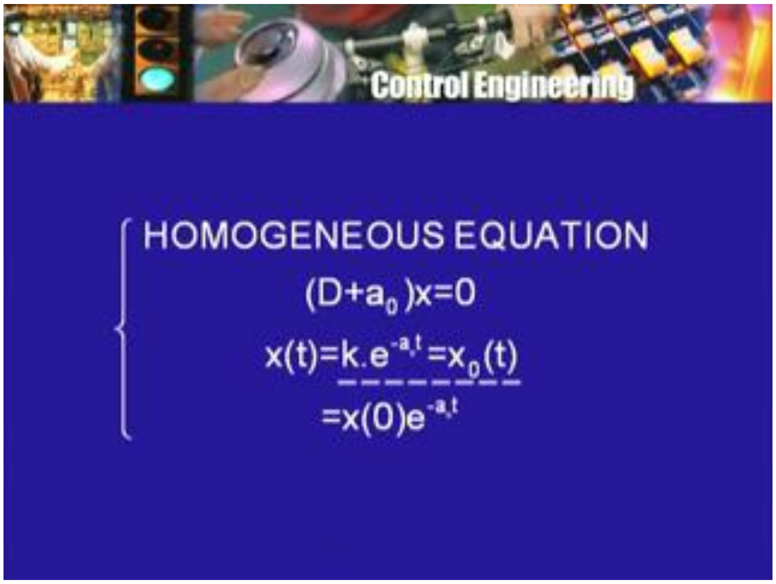


So x 0 will be some specified number the input is specified as some particular function like sin t, solve the differential equation D plus a 0 operating on x equal to u or D x plus a 0 x equal to u. Now, the technique that is used here which can be then, generalized for higher order equations of the same sought is called the method of variation of constant and this looks like something self-contradictory or saying something is constant and then, you are varying it or a better expression is variation of parameter and I will explain, what is involved here.

Let us look at the solution of the corresponding homogeneous equation as it is called that is let us look at the case, when u is 0 the equation is D plus a 0 x equal to 0 and we already know the solution of it. So the solution of its looks like k, e raise to minus a 0 t, so the equation D plus a 0 x equal to 0 is said to be the homogeneous equation corresponding to the non-homogeneous equation D plus a 0 x equal to u that is, you simply put u equal to 0, what you get is the corresponding homogeneous equation that equation we have solved knowing the exponential function and the solution appears like k, e raise to minus a 0 t. Let us call this solution, let us give it a name, let us call it x 0. So this is some known function of course k with depend of the value of x at 0.

In fact, I can write go further and write this as x 0 in to e raise to minus a 0 t, x at 0, e raise to minus a 0 t, let us call this function x 0. So, I know the solution of the corresponding homogeneous equation for the given initial value, this solution is x 0. For the first order case extremely easy for the second third $4^{th}$, we have to find roots of the polynomial and then, knowing the earlier theory, we know the general solution of the homogeneous equation, it will involve may be not just x 0 but also D x 0, D square x 0 or therefore, it will have constants like not only k but k 0, k 1, k 2, k and so on.

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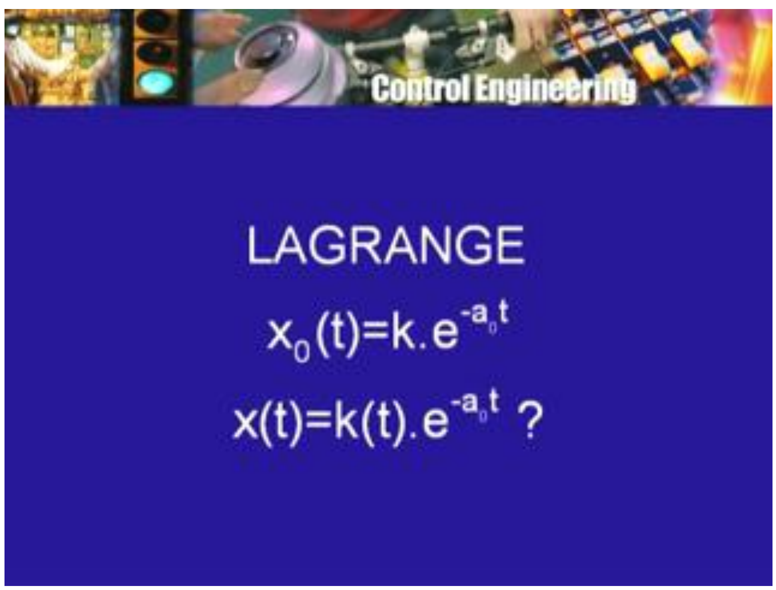


So, let us look at this solution k, e to the minus a 0 t, this is the solution of the homogeneous equation. Now, somebody I, if I remember correctly probably it was Lagrange thought that the following might be helpful in finding the solution of the non-homogeneous equation and he noted the following fact that in the solution x 0 of t, we have this coefficient k which is unknown, undetermined and it is a constant in the sense, this is not where, t occurs, t occurs in the exponent but now, I have ut on the right hand side. So things are going to change with time. So this will not of course be the solution. In fact it will be the solution only of the homogeneous equation.

So he made the guess that perhaps x t will look like k t in to e raise to minus a 0 t. So instead of just a number k, may be there is a function k t which multiplies the exponential function. Can this work as a solution of the non-homogeneous differential equation and then, you see the reason now, why it is called the method of variation of constants or variation of parameter, what was constant or the homogeneous equation is now going to be replaced by something which varies or which is another function of time for the non-homogeneous equation because this k, as I told is also called a parameter, parameter being a general term resistance, inductance values and so on, initial conditions, all of them are referred to as system parameter, in the general sense. So that is why, it is called a method of variation of parameter.

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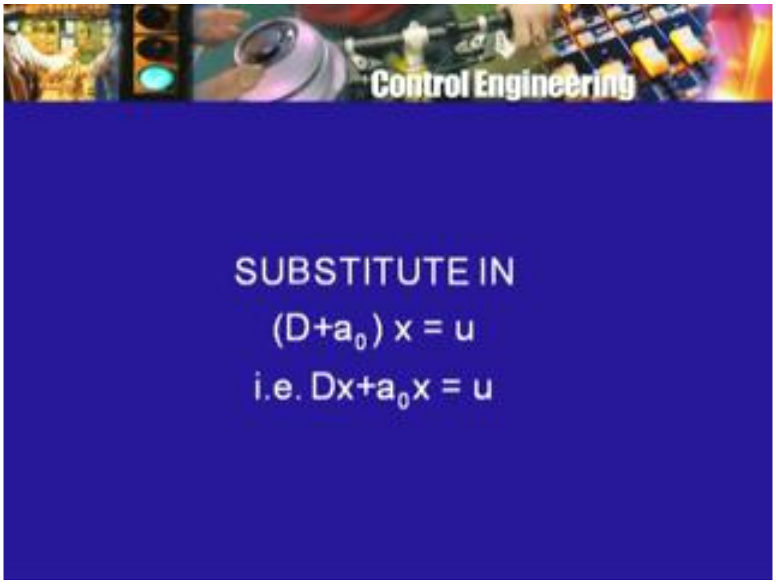


Now, when will this x t given by k t, e raise to minus a 0 t when will it be a solution of the differential equation D plus a 0 acting on x equal to u. Obviously, this k t cannot be taken to be anything you like for example, I cannot take it as constant then will be a solution only of the homogeneous equation. But, what could this k t be I have no idea. Well, let us try it out if we want this to be the solution of the differential equation then, I will substitute on the left hand side and then, equate it to the right hand side. So I will try D plus a 0 of x, with x given by k t in to e raise to minus a 0 t, try to work out, what it is spend the next minute or 2 just working this out. Essentially, all I have to do is D plus a 0 x is simply D x plus a 0 x, x is of course we have assumed it as k t, e raise to a 0 t so a 0 z is very easy to determine but the first terms I have to differentiate x.

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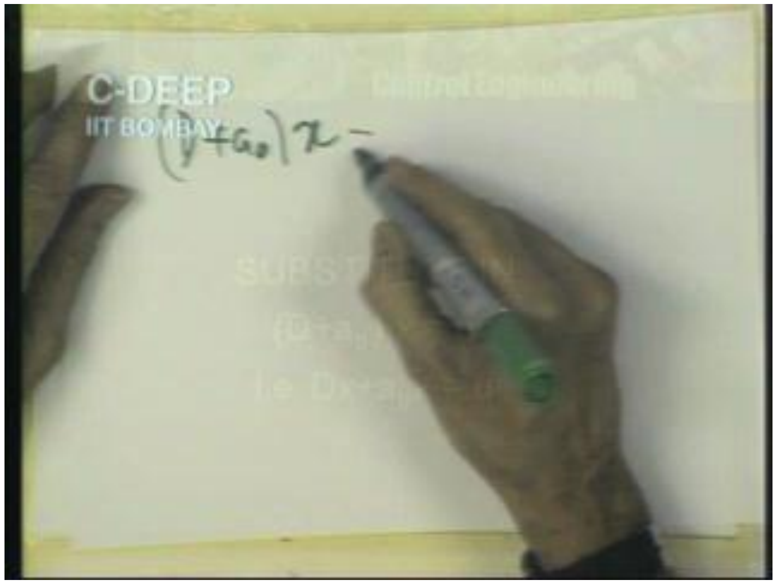


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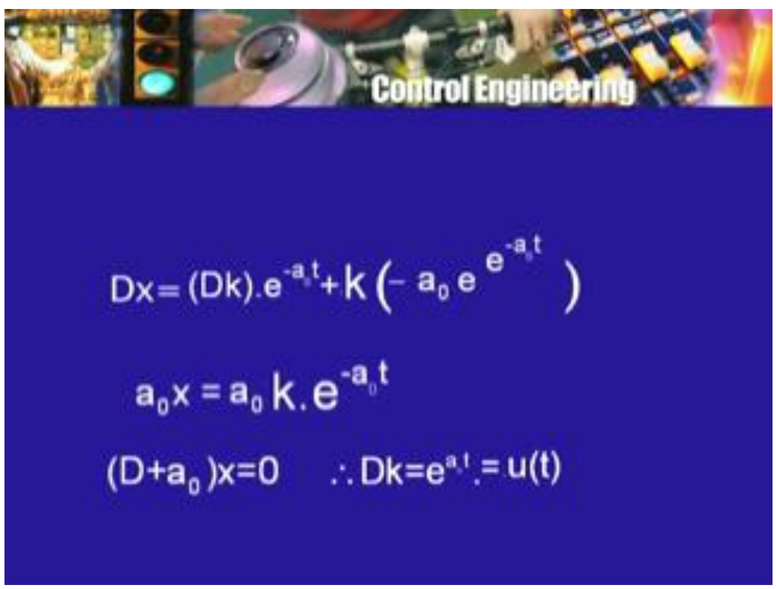


Now this is not a big order you have just have a product of 2 function and as I told you cannot do control theory without knowing some calculus better be able to work out the derivative of that product without making any mistake. Find out that derivative, it is easy, what is the derivative of x and I will not use that t all the time. I have a product of 2 functions k and the exponential function. So the derivative of x consists of 2 terms, one is the derivative of k, so I will write it as D k in to the other function which is unchanged e raise to minus a 0 t plus k in to, the derivative of the exponential function but what is the derivative of the exponential function, it is minus a 0, e raise to minus a 0 t. So this is the expression for D x to that I add a 0 x but a 0 x is simply, k e to raise minus a 0 t. So I add the

2. So I get D plus a 0 acting on x equals what interesting thing, is this terms cancels out I have minus a 0 k, e raise to minus a 0 t and I have plus a 0 k, e raise to minus a 0 t.

So, this e raise to a 0 minus a 0 t term, the exponential function which occurs in the homogeneous solution cancels out and I get therefore D plus a 0 x equal to simply D k, the derivative of k, k is not known in to e raise to minus a 0 t. So the a 0 t , e raise to minus a 0 t has not disappeared altogether it remains there but I get write this and this equals u t, if this x k t in to exponential is going to satisfy the differential equation then, this must be equal to u t. Now, this gives you a condition on D k and it is very easy to find out the k which satisfies that differential equation. Once again, you should try to think about it for a minute the problem is I am given the function u t. For example, u t may be sin t, I want to find the function k of t such that the derivative of k in to e raise to minus a 0 t will turn to be equal to my given function u t, find out such a function k.

Of course, it will turn out that such a function k will not be unique, there will be many solution. In fact, infinitely many as we will see but find out at least one solution. So spend a minute or 2 on this before looking at my answer. Fortunately, this equation is easy to solve for the reason that the factor e raise to minus a 0 t on the left hand side which multiples D k is not 0 and I told you that this was an important property of the exponential function graph that the exponential function graph never becomes equal to 0, no matter what anybody may tell you.

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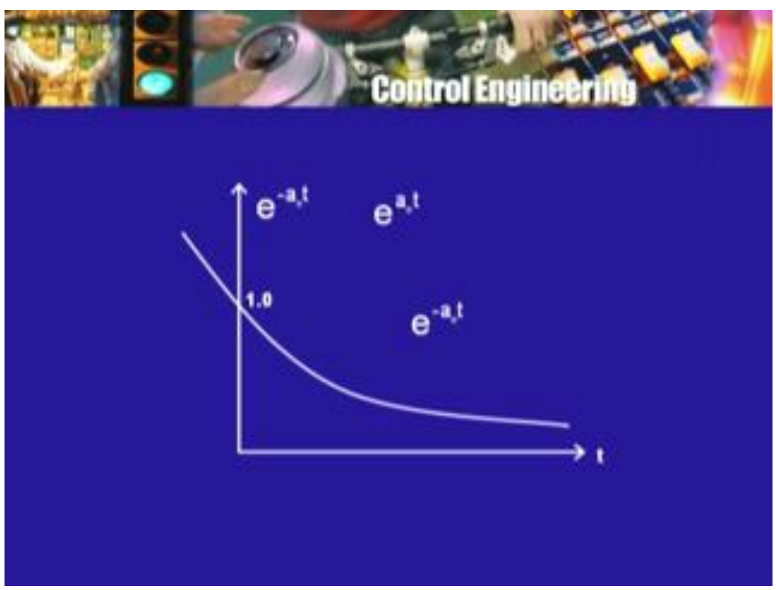


The exponential function which is a conceptual mathematical function behaves like this. You have no choice practically, the exponential function may be dead may be 0, after 5 time constants or after 10 time constants and I had talked about time constants earlier because practically 10 raise to minus 5 as good as 0 or may be 10 raise minus 10 certainly, is as good as 0. Hence, practically speaking but theoretically conceptually, no the exponential function is never 0 and therefore, I can talk about its a reciprocal and in fact, we all know that, we have done these manipulations the reciprocal of e raise to minus a 0 t, simply e raise to a 0 t that is 1 divided by e raise to minus a 0 t is e raise to a 0 t. Now this much knowledge of the exponential function is, obviously a must to proceed further and as I told you exponential sinusoidal functions, if you want to do control theory, better learn them very well.

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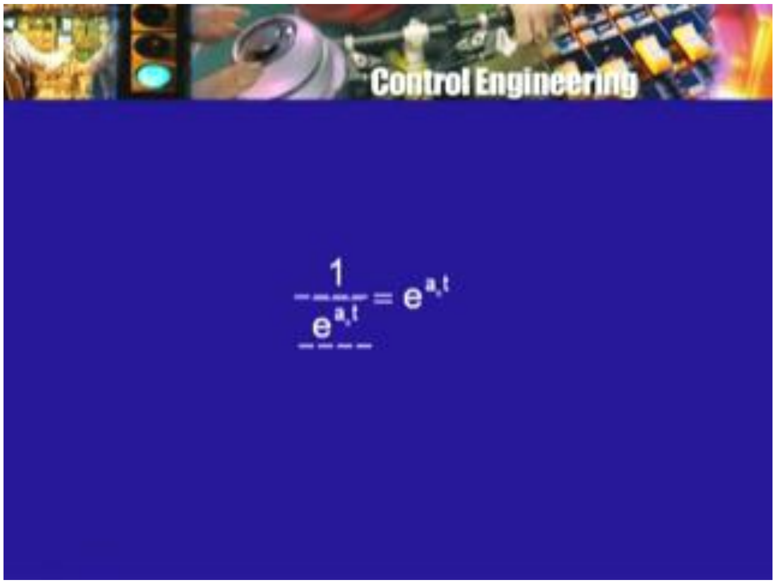


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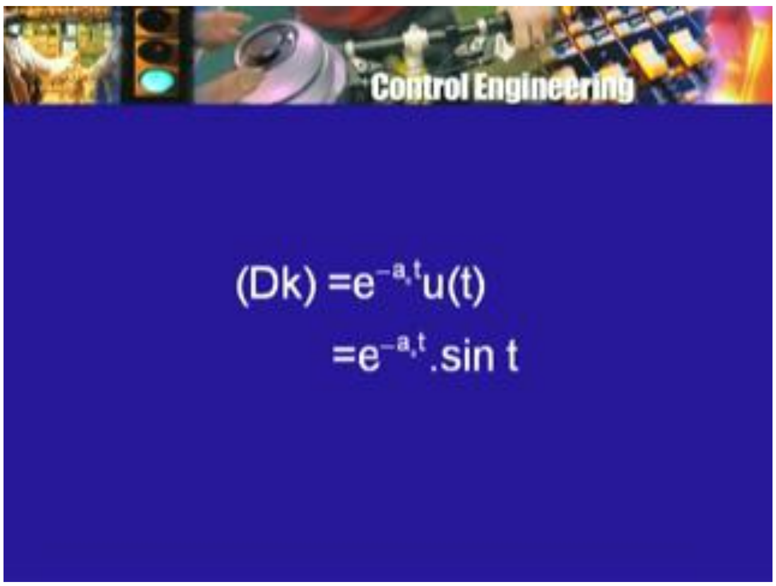


So I can solve that equation for D k, as D k, the derivative of k equals a product of 2 functions not e raise to minus a 0 t now, but e raise to minus a 0 t multiplied by u t. So in particular u t is given to be sin t then, I have e raise to minus a 0 t in to sin t and so, I have to find out a function k such that its derivative equals, this function and I made a mistake, if you have been noticing, what was my k t, when did by k t turn out to be or e raise to k t was multiplied by what, k t was multiplied by e raise to a 0 t. So in fact, I made a mistake which should have been corrected one by e raise to minus a 0 t equals e, e raise to a 0 t so when i get rid of the e raise to minus a 0 t on the left hand side, I do not get e raise to minus a 0 t on the left side, I get plus e raise to plus a 0 t on the right hand side. So make this correction.

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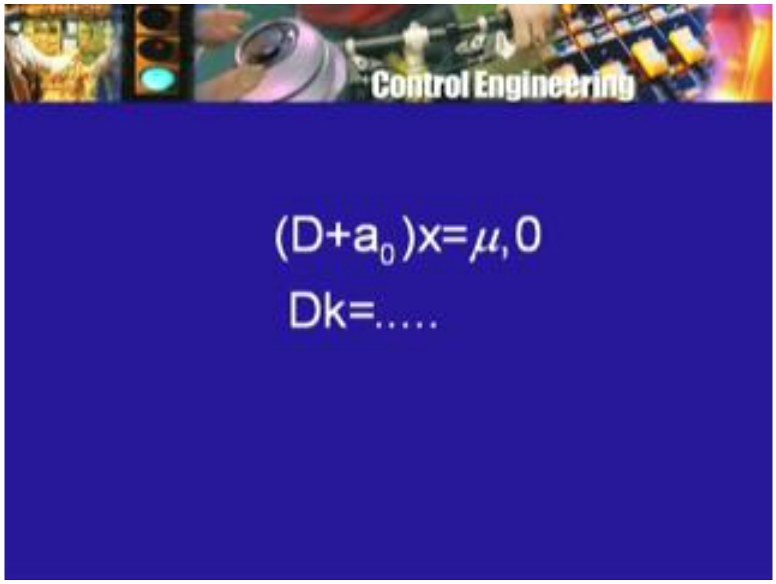


Now, this as you may realize is really a differential equation of the first order but not homogeneous because the right hand side is not 0, it is non-homogeneous but the left hand side is very special, unlike D plus a 0 x equal to u or 0, where we had D and a 0 this one has only D k equal to some known function. So this coefficient a 0 is 0 and because of this to solve it, I do not really need to know or use the exponential function but I what I need to know is the operation of integration.

Now, just as differentiation derivative use of it knowledge about it is a must similarly, integral, integration knowledge of how to integrate, what is in an integral, what are the properties of integration, all that is very essentially in control theorem and we will see, you will see right now, right way, why because I have D k equal to something I know the derivative of a function, I want to find the function. So what is the solution of this and again you should know that the solution is not unique because there is always that, there is something indefinite about the integral, there is always that constant which you have to put.

In addition to something else, in addition to the integral and so this k is not determined uniquely, what is the solution of this D k equal to some function which is known, k is the integral of that function plus a constant term and therefore I will write the solution, as k t the value of this function which we are looking for a t time t equals some constant, let me call it say b 0 because in some other cases, there may be more than one constant or more than one coefficient b 0 plus integral and I am writing, what is called the indefinite integral, what is appearing on the right hand side.

Now, this is where, we have to be very careful with the notation and with, what we are doing on the left hand side I have the value of the function at any particular time t that I may want to know, on the right hand side therefore, what should appear, first of all this indefinite integral is not adequate. So, we have to make it a definite integral usually, we choose the upper limit of the definite integral as t, the time at which you want the response to calculate the response, the lower limit can be any convenient time instance and since, I told you that most control system activity starts at some moment of time and then, you operate the system that is the so called initial time and we have chosen to call it or represent it by the number 0, So we can chose the lower limit as 0, so we have definite integral from 0 to t of a function.

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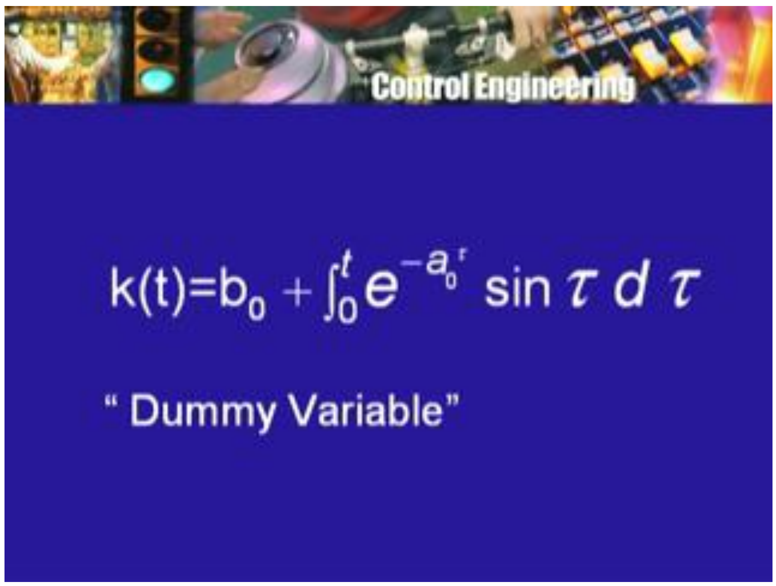


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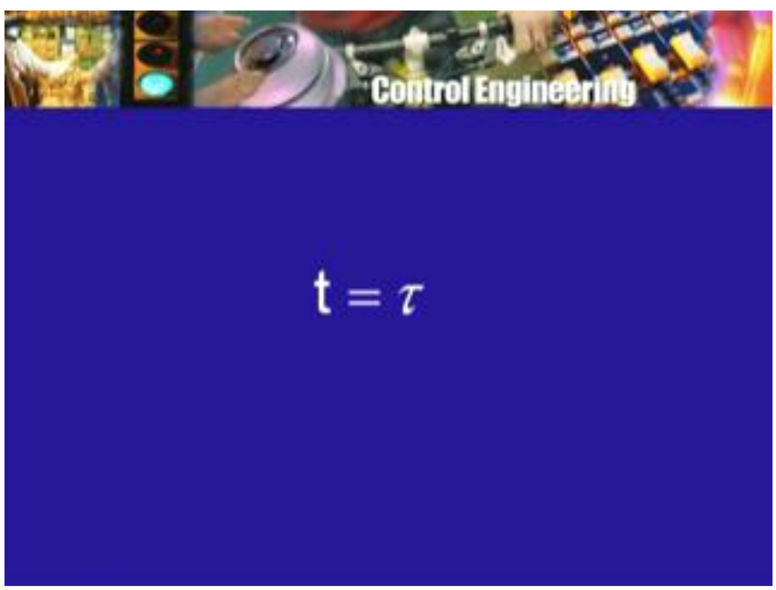

Now, unfortunately the integral notation is such that we write there things like D of something and therefore, I should not write for our particular problem e raise to minus a 0 t, sin t d t because this t that occurs in dt and therefore also occurs in sin t and e raise to minus a 0 t is what is called the dummy variable of integration. Unfortunately, the integral notation is such that we are use to it or we have to get used to it I would have preferred a different and much better notation for the integral but this is more or less the standard notation.

However, we have to make one change and this change is essentially for control theory because otherwise, without that you can get very wrong answer and that change will be, this dummy variable t which occurs under the integral sign, I will replace it by some other variable symbol or some other symbol and it is common practice to replace the t, instead t have the, t have the Greek letter tau or tau. So I will write the expression of the solution as k t equals b 0 plus integral 0 to t the upper limit is t which appears on the left hand side, the lower limit I am choosing as 0, I could choose it as 3 or 50 or minus 100 or whatever and what is the integrand function, what is being integrated e raise to minus a 0 not t but tau, sin tau, d tau.

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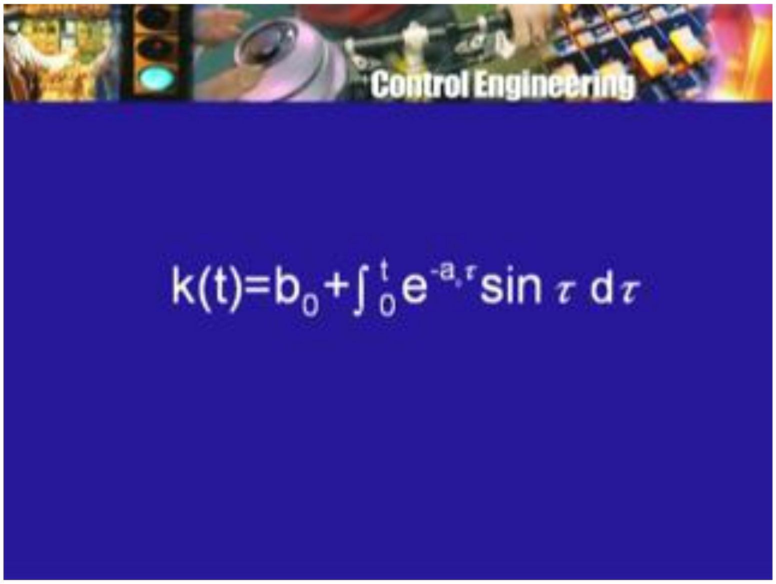


Now, here is very small exercise for you to do after the end of the lecture, calculate this integral, this definite integral, integrand function is e raise to minus a 0 tau, sin tau, d tau. The limits of the integration are 0 and t, find out this integral, you have done your calculus course successfully, you should be able to work out this problem. The result will be an expression that involves only t and tau that is why tau is called the dummy variable of integration. The expression will be something that involves only t and tau. This is just an exercise in finding the definite integral of a function, the only thing is the lower limit of the definite integral is a specific number 0, the upper limit is t where, t is any moment of time for which we want to know, what k t is. So this t appears on the left hand side it appears in the upper limit of the integral on the right hand side.

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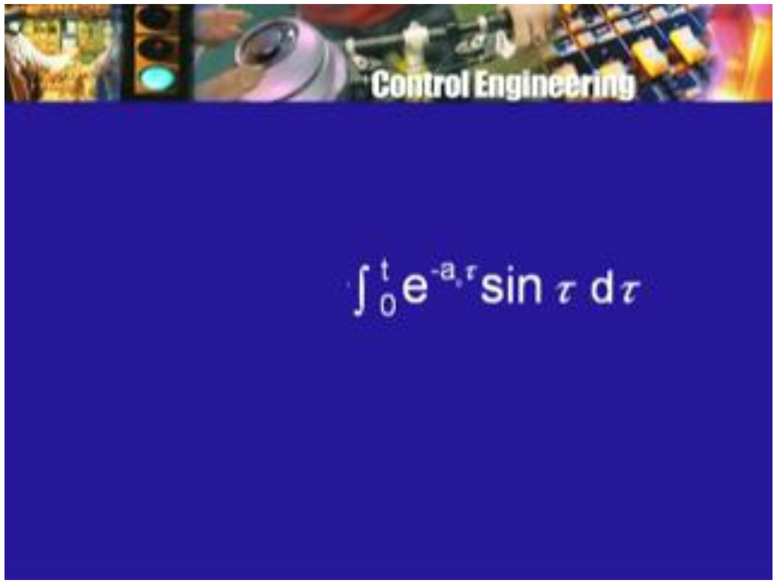


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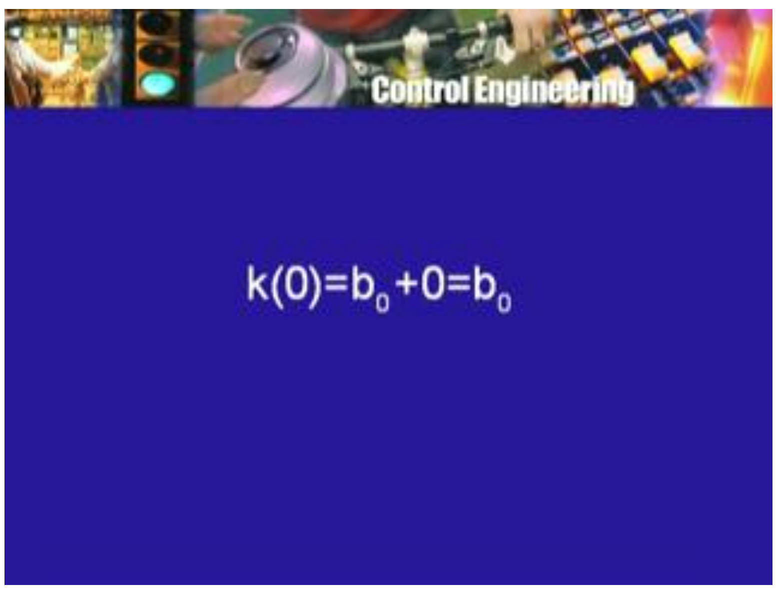


Now that is k t but my solution was not k t, it was k t, e raise to minus a 0 t. So I must put that secondly, what about this b 0, how do I determine this b 0. Well, put t equal to 0 then, I have k 0 equals b 0, what about the integral? The integral is simply 0 that is a special case of any integral whose upper and lower limits coincide and so the integral is 0 and therefore, I simply get b 0. So this b 0 is nothing but k 0 now if I can find out what k 0 is then, I know this b 0. So I know this arbitrary constant which is no longer arbitrarily but is determined as k 0. So if I can find out k 0 somehow I will get this b 0and so I will have find out the solution of the or I will be lead to the solution of the differential equation as k t, e raise to minus a 0 t.

So let us do this, so now my x t equal to k t, e raise to minus a 0 t will be a solution of the non-homogeneous equation D plus a 0 x equals u, I have chosen u t equal to sin t therefore, in this particular case I get k t which is b 0 plus integral 0 to t, e raise to minus a 0 tau, sin tau, D tau, this whole thing is multiplied by e raise to minus a 0 t. This is the expression for x of t, now this b 0 which also happens to be k 0 is the unknown constant, how to determine it, let us put t equal to 0.

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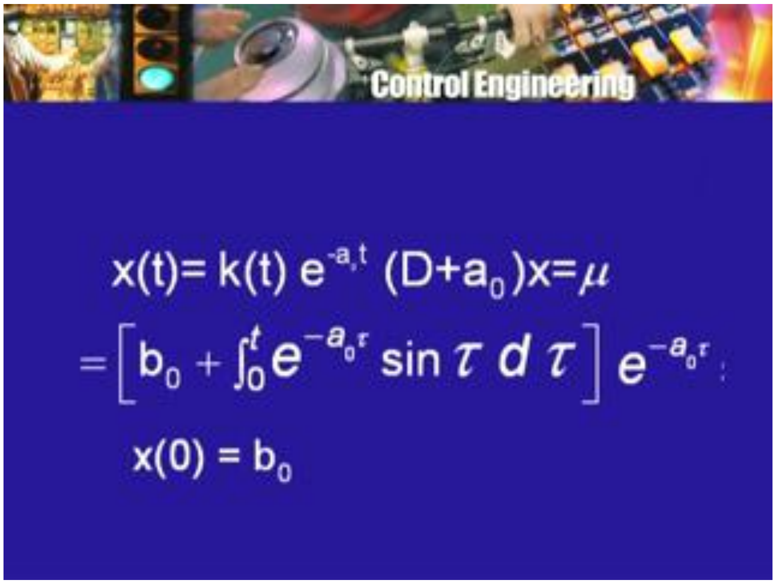


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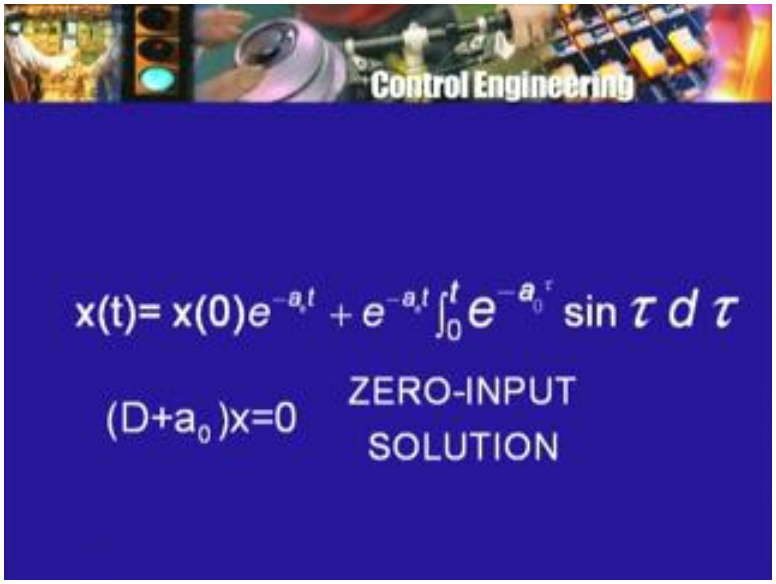


So on the left hand side, I have x 0 and on the right hand side what do I get. The expression inside the rectangular brackets is b 0 plus 0, so only b 0 e raise to minus a 0 t, at t equal to 0 is 1. So I will get simply b 0. So this unknown constant or the arbitrary constant b 0 is not arbitrary now, it has to be equal to x 0, the initial value of the response variable x. So, if substitute that here then I can write the solution as follows x of t equals b 0 in to e raise to minus a 0 t but b 0, we have determined to be x 0. So I will have x t equal to x 0, e raise to minus a 0 t plus I will write that multiplied e raise to minus a 0 t up front multiplying integral 0 to t, e raise to minus a 0 tau sin tau, D tau. This is the solution because now, there is nothing unknown x 0 is known a 0 is of course given the integral can be evaluated I will ask you to evaluate it. So this is the solution.

Now, we can see that the solution has 2 parts that is it consists of the sum of 2 terms that is this term x 0 e raise to minus a 0 t which we had already encountered. In fact, it was the solution of the corresponding homogeneous equation D plus a 0 x equal to 0, the solution of this was precisely x 0 e raise to minus a 0 t. So one part of the solution consists of just the 0 input or homogeneous equation, solution that corresponds to the case, when the input is 0 and therefore, it is quite natural to call it, the 0 input solution. This is the response that you will get when the input is 0. Even though, the input is 0 there is some response because x 0 may not be 0, if x 0 is 0 of course there is nothing or the response is 0.

So, we will get the 0 input solution which involves or which depends on the initial condition. In this case, x 0 and it involves again the exponential function which goes with that exponent m which appears in the indicial equation and so on. So this is the 0 input solution and naturally, we expected this in a way because when the input is 0, we should get something and this is what we get, what about the other term? The other term depends on the input that you have chosen. In this case, we have chosen ut equal to sin t or u tau equal to sin tau. So it is going to depend on the input but it is what remains, when the initial condition is put equal to 0, when x 0 is put equal to 0 this part disappears and I only get this part. So, what remains can legitimately, we call the 0, initial condition solution or I told you long time ago that one talks about the concept of state of a system and so it is called the 0 state solution and this 0 state solution depends on the input and so, the solution of the system or the response can be thought of as consisting of 2 parts or sum of 2 parts, one the 0 input part and the other the 0 state part.

Of course, when I go to the laboratory and make measurements, I am not going to get these 2 separate term, these I have a reason while we were solving the differential equation, what I will get only, will be there sum x of t. So one has to look at what these 2 terms look like and then, what there sum will look like and that is what you expect to see. If I connect an oscilloscope for seeing some particular voltage or current variable then, the kind of wave form that I will get will be determined by the sum of these 2 terms and not by them separately. I do not experimentally get this separately unless I put input equal to 0, I will do not get the 0 state response separately, unless I put the initial condition equal to 0. But in the actual practical case, I will have a non-zero initial condition and perhaps and a non-zero input. When I have both of them present, I do not get these 2 things separately, I only get there sum, what is our good fortune that x t consists of 2 parts which has its feature, one part the 0 input solution depends on the initial conditions only or it is a solution, when the input is 0, so for some other input of course this part will be there but there will be something more, the 0 initial condition, the 0 state solution depends only on the input and it will be there as long as the input is not 0. So we have a sum of 2 term.

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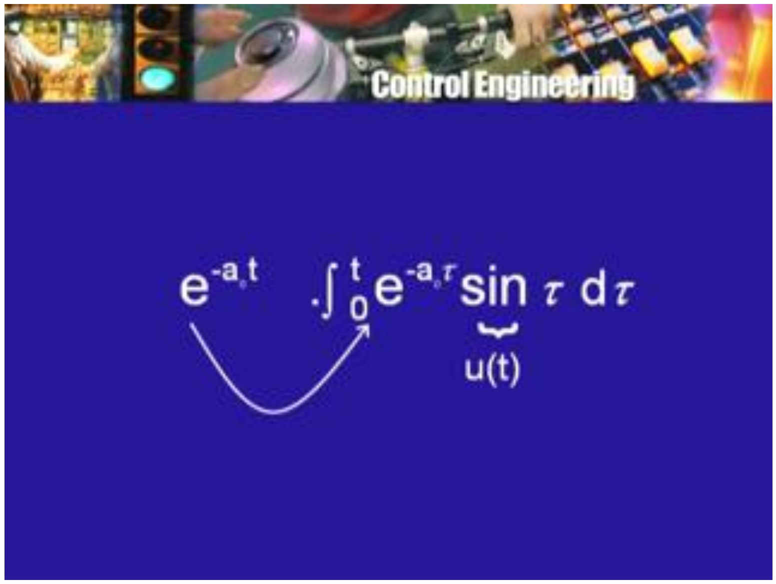


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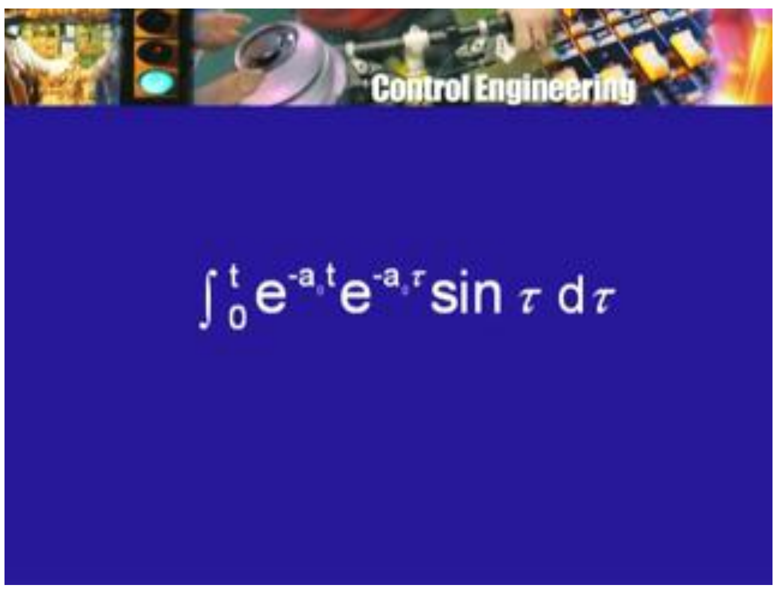


Now the second term, the 0 state solution can be written in the following way and I am not going to, go in to any greater details right now, later on when we may require it, I will use it and even spend a little more time on it. But, what can be done is, we have this e raise to minus a 0 t outside, the integral sign 0 to t, the upper limit is t inside we have a minus a 0 tau multiplying the input function sin tau, d tau, tau is only the dummy variable this d only tells you that you are integrating in place of sin tau In general, I can think of my input u tau in fact unless I am actually calculating makes no difference whether I put u tau there or sin tau. So might as well put u tau, so for an arbitrary input u tau, this will be the part.

Now, what one can do is, this e raise to minus a 0 t which appears outside the integral, can be pushed inside the integral and this is a some trickery is required and care, what is inside or under the integral sign has only tau and therefore, we are justified in pushing this e raise to minus a 0 t inside the integral. Now, this requires some careful argument and I am not going to give it or go in to it but if I do that then, what will I get integral 0 to t then, I will have e raise to minus a 0 t in the equations that you may have copied. So I have this e raise to minus a 0 t okay, but under the integral sign I have plus e raise to a 0 tau multiplied by sin tau d tau that is what I have.

Now, I will rewrite it further and right now, you may not see much reason for doing it. I will rewrite it as integral 0 to t, e raise to minus a 0 in to brackets t minus tau that is this a 0 multiplies an expression in parenthesis which looks like t minus tau, just check e raise to minus a 0 t, e raise to minus a,0 minus tau that is e raise to a0 tau. So that is okay, nobody can prevent me from writing it like this this multiplied by u tau, d tau.

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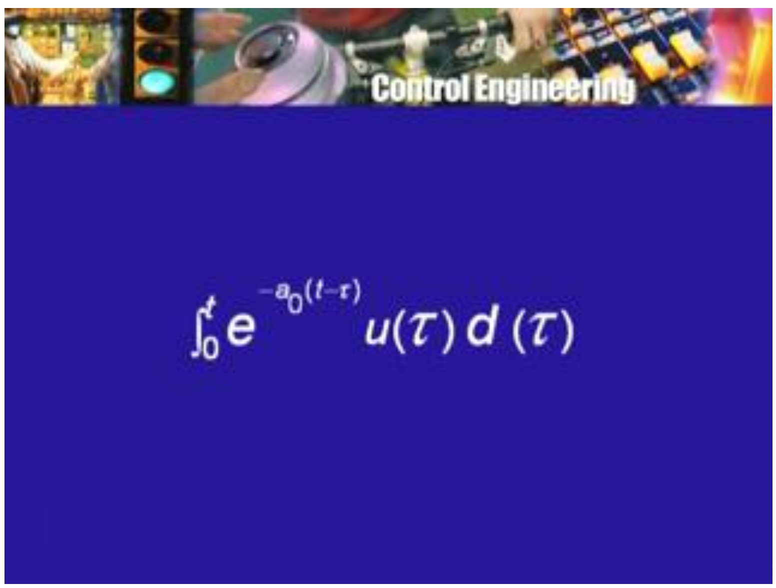


Now such an integral is called a convolution integral and the reason for writing this e raise to minus a 0 t minus tau is the following, my 0 input solution that I had talked about I had called it x 0 was simply, x t equal to x 0 in to e raise to minus a 0 t. So, if I look at the special case when x 0 equals 1 then, the solution is simply e raise to minus a 0 t. So this minus a 0 t is the solution of the homogeneous equation. When the initial condition is 1, x 0 is 1. Now, so this is one special solution of the homogeneous equation, homogeneous equation therefore, right hand side is 0, but initial condition equal to1. So, if I denote this by x 0 and this is what I did earlier because that k was being replaced by k t then, the above integral this one can be rewritten as integral 0 to t, this exponential term can be written as x of x 0 of t minus tau, u tau, d tau.

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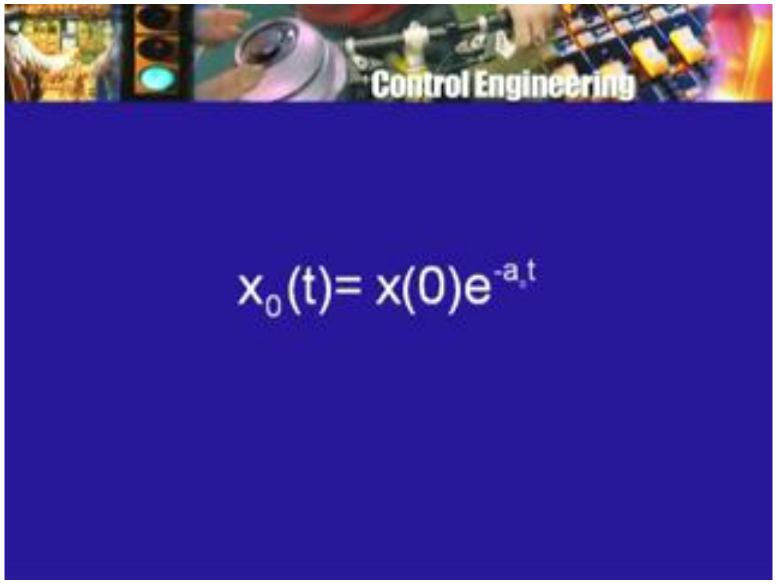


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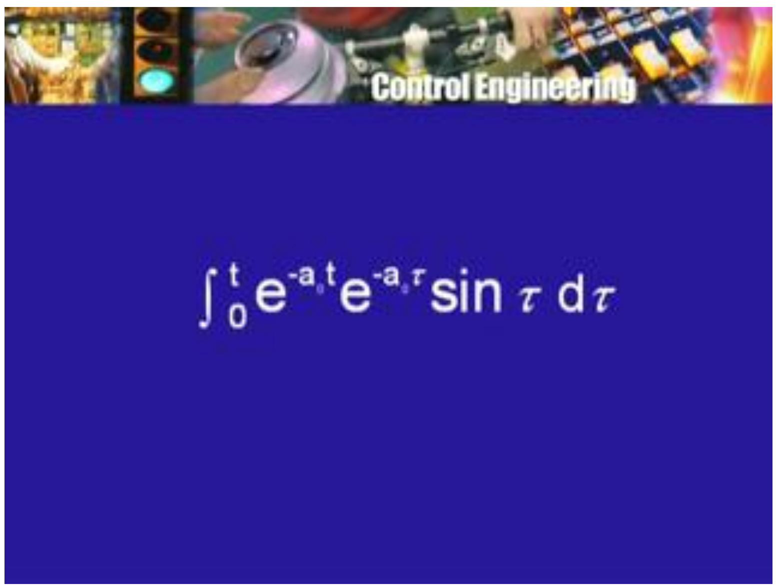


So, I have the function u which is the input function I have the function x 0, which is a particular solution of the homogeneous equation for a particular initial condition then, the integral looks like this. It is not product of the 2 function, it is not x 0 tau u tau, but its x 0 t minus tau in to u tau, integrated from 0 to t and this is said to be the convolution of x 0 with u, this operation is referred to as doing or taking the convolution of x 0 and u, the convolution of 2 function, the input function and a particular response of the system.

Now, this particular response for some historical reason gets associated with something which is called an impulse function and we will look at this, when we use the Laplace transform approach and therefore, this x 0 has been called the impulse response of the system. One has to be very careful in understanding these expressions because as we will see the impulse function, strictly speaking is no function at all, that is it something with which you cannot really do mathematics and therefore, you talk about response to the impulse function will be going 1 step beyond what you are allowed to do and in practice also, we do not have anything like the impulse function, input practically speaking.

However, let us keep aside that thing for the moment, simply remember the term impulse response and so the 0 initial condition response or the part of the response that depends on the input appears as a convolution of the impulse response with the input. This operation of convolution has some interesting properties, it is something which can be actually of course worked out and I ask you to solve that integral or find that integral. So without telling you that you were doing the operation of convolution, you would have done it. It can also be given a graphical interpretation of what is happening integral of a function as the interpretation of area under curve. Now, if you have the integral of a product and the product is this in this peculiar way, so what does it mean graphically, it can be given a meaning and that is the reason, why the term convolution is used.

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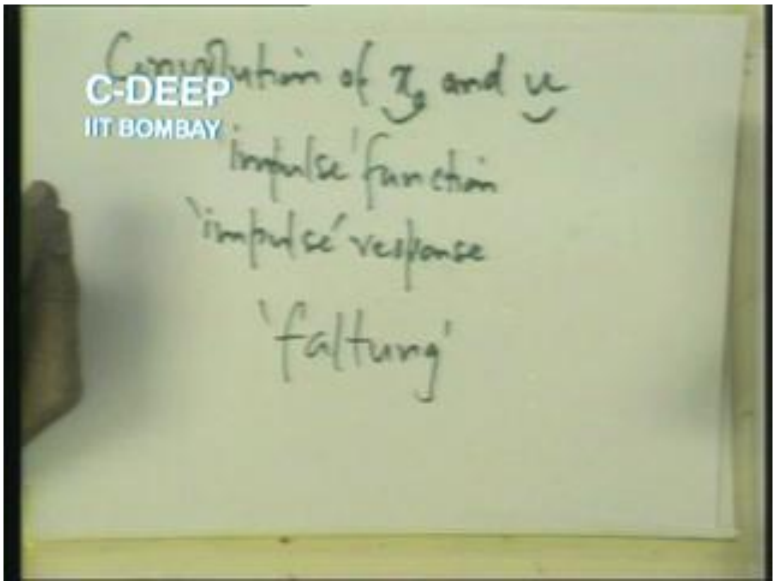


In fact, this was done, this idea this interpretation was done first by German mathematicians and they used the term which is called faltung, which means a folding that is in their interpretation or in the interpretation that of course, you and I can use and you will use something like folding, an operation of folding is involved, not only folding but then, folding, shifting, multiplying and then, finding the area the under the curve. So, they use the word faltung, the integral was known as the faltung integral and when this term was translated in to English, somebody invented the term convolution. Although convolution is not really exactly folding, it is really something which is twisted around in some various ways, it is so said to be something which is convoluted.

A convoluted surface is a surface which has all kinds of twists and turns and folds and depressions and projections and what is not, that is said to be a convoluted surface. So the name convolution integral and convolution of 2 functions arose out of that and so, this is what we have then, the solution of the system with a non-zero initial condition and with a non-zero input will be a sum of 2 term, one of which will be the solution of the homogeneous equation with the initial condition and the other will be the solution of the equation with 0 initial condition and input and that can also be thought of as the convolution of, what is called the impulse response of the system and the input and the impulse response can be thought of as response of the system to a certain function called the impulse function under certain very appropriate initial condition. Now, this approach, we did only for the first order differential equation.

The same approach extends to the second order differential equation and higher order differential equation. Of course, in the case of the second order differential equation, there are 2 values of m in general and there are2 constants or coefficients of parameters k 1 and kb2 and so you will have to think of varying both of them and then, get some expressions for those multiplier functions k 1 of t and k 2 of t and there by get the solution, one can put it in exactly the same form that we have got for the first order case a, 0 input part which depends on the initial condition and a, 0 state part which is a convolution.

So whether the differential equation is of first degree or second degree of fifth degree and therefore, n th degree, so to speak the solution is fortunately given by the same form, a 0 input form which of course will involve exponential and coefficients which are to be determined based on the initial condition and a 0 state part which requires knowledge of the input and which can be carried out through the operation of convolution or evaluating an integral, which involves a product of the input function with something, which goes or which appears in the 0 state response, in the 0 input response or specifically, it is the impulse response of the system.

So this is 1 way of solving the system differential equation. For an ordinary linear differential equation with constant coefficient which may be non-homogeneous, we can find the solution, if we know appropriate number of initial value and if we can evaluate certain integral involving the input function and the impulse response of the system finding out the 0 input response or the impulse response of the system requires finding the roots of a polynomial equation or the 0s of a polynomial or to go back to school algebra, it requires factorization of a polynomial. So that is one basic numerical problem that one has to solve, once you solve that you get these exponential function then, the other problem is of course carrying out the integration.

Now, today of course all these can be program. So that there are program packages, where you just specify the initial conditions, you specify the derivative coefficient, you specify the input function, it may be given explicitly as sin t or you may give its values at instance of time and the program will calculate and plot for you x of t, what has to be remembered is that no matter, how impressive that plot is, what the program does is only finds an approximate solution and not the exact solution. It looks very nice but that is only an approximation to the solution, how good the approximation is depends on what program you have used, how may discrete steps you have divided the time interval into, what is the accuracy of you computer or precision rather single precision arithmetic, double precision arithmetic and so on, what you get is only an approximate solution but the approximation is reasonably good most of the time.