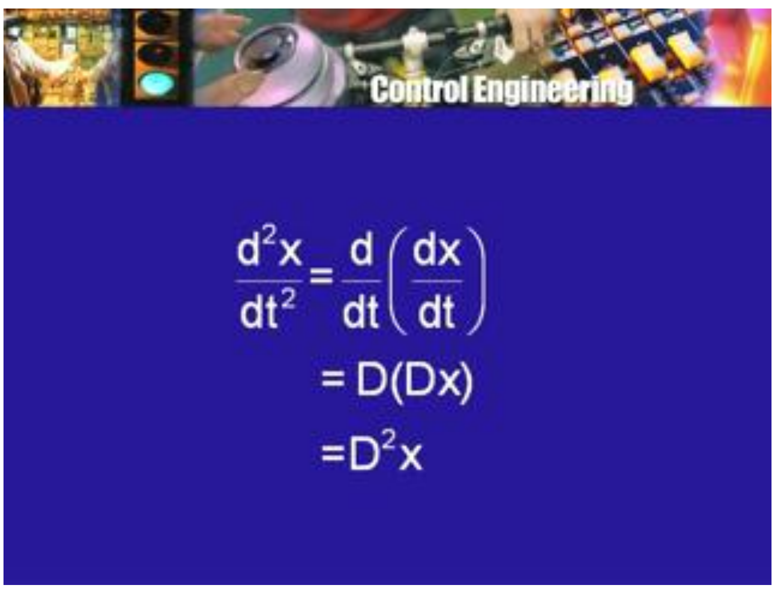
Control Engineering
Prof. S. D. Agashe
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture - 19

Let us look at the solution of the second order linear differential equation with constant coefficient which is also homogeneous. So the equation has the form a 1 D plus a 0 acting on X equal to 0. I have already introduced the notation D, for the derivative as an operator and I have gone 1 step further here, instead of writing the differential equation as let us say d square X by d t square, as we call it plus a 1 d X by d t plus a 0 X equal to 0 where, X is the unknown function of time which is to be determined the right hand side is 0. So the equation is called homogeneous so instead of using this more familiar d by d t or d square by d t square notation. I am using capital d to denote the derivative operator. So that d X d t will be replaced by D acting on X it is just the derivative operation taking place on X and similarly, d square X by d t square which is d by dt of d X by d t, therefore becomes D acting on D X and for a short or for brevity's sake we write this as D square X and I am reading it as D square X but what is really means is D operated 2 times, D acting on X and D acting on again on result which is D X.

(Refer Slide Time: 02:21)



So with this then we have d square X d t square. So I can write D square X plus a 1 d X d t, so I can write it as a 1 d X plus a 0 X equals the 0 function the 0 on the right hand side is not the number 0 but it is the function 0, the function is 0 for all time T. On the left hand side also we have functions X is the unknown function. For us it is a signal for it is a function of time something that changes with time perhaps. So all the left hand side the quantities are expressions or functions a 0 X is a 0 times the function X a 1 D X is a 1 times the function D X and D square X is the second derivative of X that is D acting 2 times consecutively on the function X.

So that is what, we have on the left hand side. Now instead of writing the 3 separate terms I have written them in the form with the which I started namely, I have written them as D square plus a 1 D plus a 0 and put that in bracket and X equal to 0 and now, we look upon this D square plus a 1 D plus a 0 as a more complex operator that is just as the symbol D denotes an operation of derivative taking place on the function X, to produce the function derivative of X. Similarly, this D square plus a 1 D plus a 0 is an operation taking place on X, the function X and the definition is simply this operation acting on X. By definition is simply the sum D square X plus a 1 D X plus a 0 X such an operator is called a polynomial differential operator for an obvious reason, it is an differential operator because the derivative occurs as a part of it derivative 2 times that is twice in succession D square and perhaps D cube and so on and it is called a polynomial differential operator because it looks like a polynomial in D.

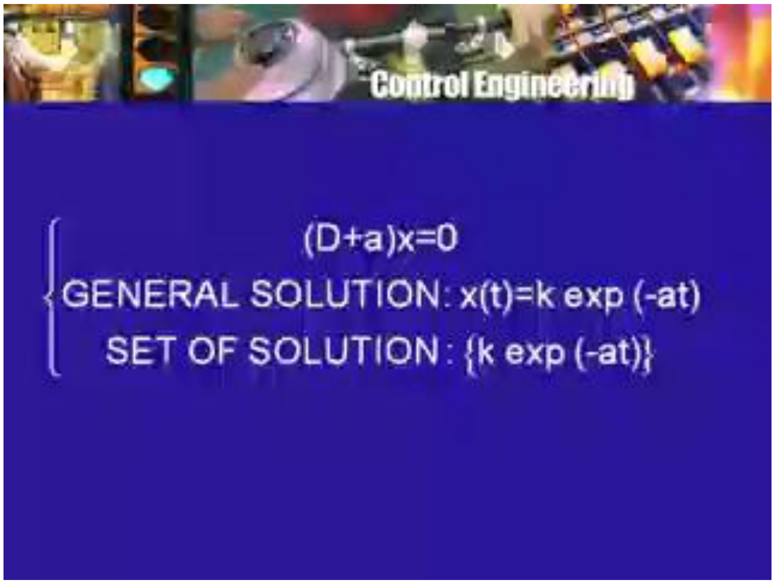
We are more use to polynomials in perhaps X and Y or T but, here is a polynomial in this operator D, it looks like a polynomial D square plus a 1 D plus a 0. For example, in particular numerical case it May be D square plus say, 5 D plus 6 this is a polynomial differential operator and one can talk about the degree of the polynomial differential operator just as one talks about the degree of the polynomial and the degree of the polynomial operator is 2 in this case and we also talk about power of D, that is how many times at the most D acts in this case D acts 2 times.

So, we say that the power of D is into e square is power of 2 and that is a highest power of 2 in this expression therefore, the degree of the polynomial differential operator is 2. If you did not know that D was an operator would you just thought that it was the symbol for a variable as an algebra then, you would have said that this is a polynomial of degree 2. But here for us D is the polynomial or the differential operator. So what you have is a polynomial operator of degree 2 mean talking about the differential equation, one talks sometimes about the order of the differential equation rather than degree and one says therefore, that the differential equation d square X plus a 1 d X d t plus a 0 X equal to 0 is a differential equation of the second order.

So, instead of talking about first order differential equation then, one talks about second order third order and so on, one could equally will call it the first degree or second degree differential equation and so on. In fact since, we are calling, they have with respect to the polynomial, we are using the word degree with respect to the differential equation also, we could the same word degree but some books do use the word order, all right. So we have this second order homogeneous linear differential equation with constant coefficient and we are trying to find out what are all the solutions of it because it may have more than one solution. Earlier, we saw the first order differential equation and then, we saw that that also the homogeneous equation had not 1 but infinitely many solutions just the differential equation by itself and therefore, you needed something more namely what is called an initial condition to determine the solution uniquely.

To repeat very quickly, the differential equation D plus a X equal to 0 has the solution X of t that is a function X whose, value at any time t is given by any number if you like k multiplied by exponential of minus a t. This was the solution of the set of all solutions of the differential equation first order of first degree differential equation D plus a X equal to 0. Here, k can be any number what so ever. So for each value of k, we get one solution and therefore we get infinitely when a solution on the set of solutions which can be written as follows. The 2 braces the curly brackets at the end are supposed to stand for the set of all k exponential e raise to minus a t for all possible values of k that is k any real number 1, 2 minus 50, 10,000 what have you, multiplied by exponential minus a t each one of them is a solution of the differential equation.

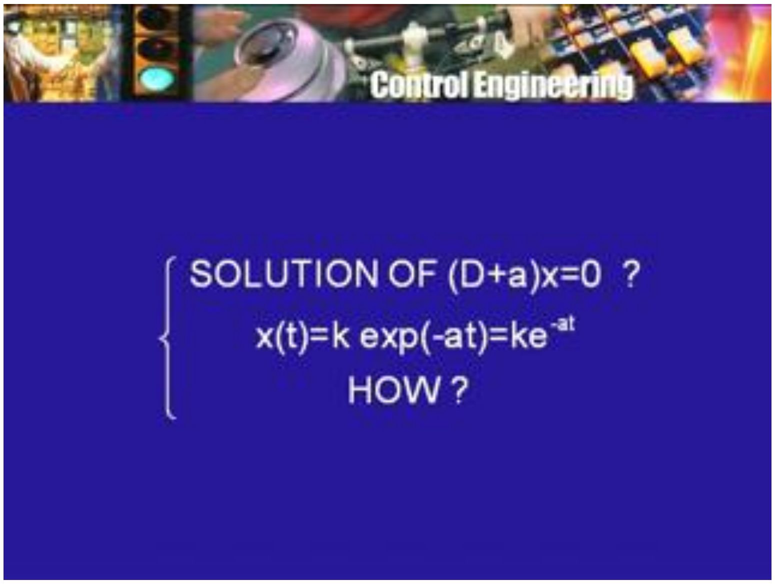
(Refer Slide Time: 07:19)



Now we have a differential equation of degree 2, so what do we do about it, how do we find it, set of all solution. I told you that the way in which one solved this differential equation D plus a X equal to 0 by actually remembering, the exponential function and remembering some simple properties of the exponential function that was one way of doing it that is suppose, you were presented the definition of the exponential function and that can be done in several ways. For example, one can define it in the form of an infinite sequence and so on, which you may recall from your calculus courses.

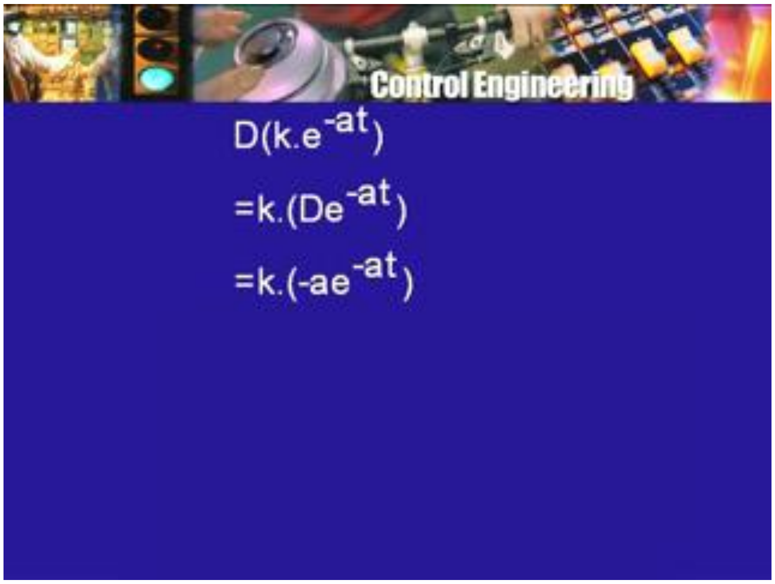
So when you have the definition of the exponential function that it is this function then, one can examine its properties for example, one can study I showed you the graphs of the exponential function, of the graph of the exponential function which is a very important graph that it is never equal to 0 at t equal to 0, it is 1 etcetera. One can study some more properties of the function for example, what about its derivative and this is one of the basic exercises or results in calculus that if I look at the function e raise to t then, its derivative is simply e raise to t itself, that is differentiating does not change the function e raise to t after differentiation gives you e raise to t only, the function e raise to is unchanged by differentiation operation. However, if I have e raise to a t is the exponent is now, a t exponent of e is a t then, the derivative of that is no longer e raise to a t but what is it a times e raise to a t. So the d operation does not change the function e raise to t but the d operation changes the function e raise to a t and what it does to the function is simply multiplies it by the number a. Similarly, if I have e raise to minus 3 t the derivative operator on that would multiplied by minus 3.

(Refer Slide Time: 10:34)



So their derivative of that is minus 3 into e raise to minus 3 t. This is the property of the exponential function which we use to see that D plus a X equal to 0 has X t equal to k exponential of minus a t or rewriting at also as k, e raise to minus a t for a solution because simply apply the derivative operator to k, e raise to minus a t. So D acting on k e raise to minus a t is equal to what function first of all the derivative operator has the property that if you have a multiple of function and take the derivative of that product that multiple then, it is the multiple of the derivative of the original function therefore, D acting on k times e to the minus a t is simply k multiplying D acting on e raise to minus a t but D acting on minus a t is in turn minus a times e raise to minus a t.

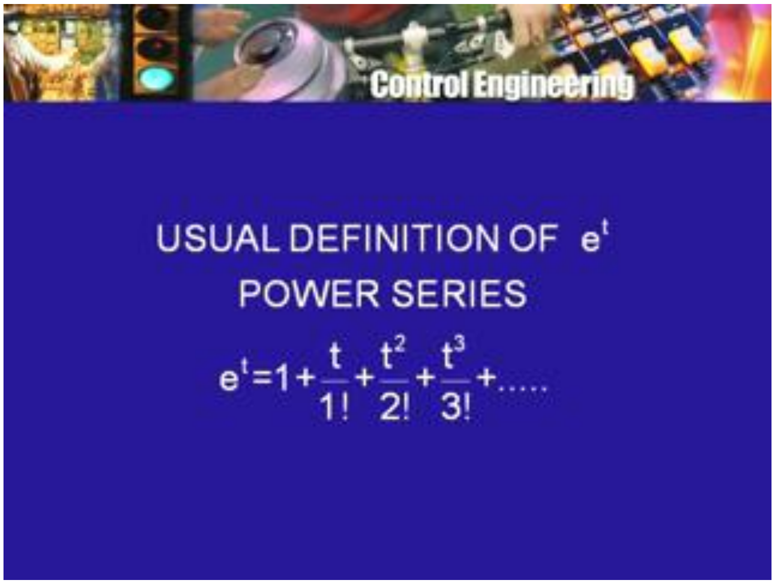
(Refer Slide Time: 11:19)



So the derivative of k times e raise to minus a t is minus k a, e raise to minus a t and therefore, we can simply now verify that e, k e raise to minus a t is the solution of d X plus a X equal to 0 because if X is k e raise to minus a t then D X is this. So I have minus k a, e raise to minus a t that is my D X if X is e raise to my k e raise to minus a t plus a times X but the X is k e raise to minus a t, this is a X and a simple addition shows that the result is 0. So the exponential function or any multiple operate any arbitrary multiple operate is a solution of the first order, first degree linear differential, homogeneous differential equation with constant coefficient.

So this is one way of obtaining the solution by simply knowing that the exponential function has a such and such property. In advance courses in mathematics of course, one can approach the matter little differently because there, you not only talk about linear differential equation with constant coefficients homogeneous or non-homogeneous but even non-linear differential equations and then, ask questions, does it have any solution, how shall we find the solution and so on and so, quite different technique has to be used for investigating that problem. For our purposes, we know the exponential function already, we have to remember, its property with respect to differentiation and then, we have the readymade solution, how to evaluate the exponential function is different matter.

(Refer Slide Time: 13:42)



Of course, your calculators do provide the exponential as one of the functions that are available, but you will recall that the exponential function is given either by an infinite series or by the limit of a sequence. Just to remind you, e raise to a t what is the infinity series for it now, this is again something every electrical engineer must know by heart, must not forget it, as long as he can afford to remain an electrical engineer, what is the exponential by the, so call power series expansion for e raise to t, which is also the Maclaurian series for e raise to t. It starts of at 1 plus t divided by one for the sake of generality, I am writing t divided by 1 factorial t square divided by 2 factorial plus t cube divided by 3 factorial plus and so on because the thing continues without any end.

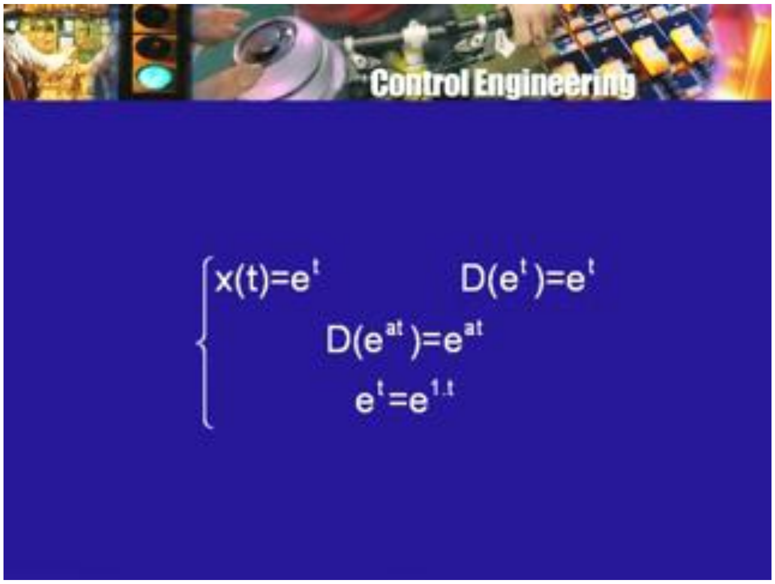
This is the infinite series expansion, for e raise to t and what is the other formula or other expression for e raise to t, it is as the limit as n tends to infinity, the natural number n tends to infinity of a power and the power is 1 plus t divided by n raise to n. The limit of this as n tends to infinity taking values 1, 2, 3 and endlessly is the same function e raise to t. So with this definition and some investigations of differentiability etcetera, one can show that the exponential function as the property that I have talked about and therefore, the first order equation is easily solved one does not use these expressions for calculating the exponential functions because first of all, in both cases this dot, dot, dot here means just go on adding and adding endlessly.

Similarly, here limit means that you take n equal to 50 with that is not good enough, you have to take n equal to 51 that is not enough take n equal to 1000 that cannot be enough. So this is also an endless process here, but there are ways of getting approximate value of e raise to t for any given number t and you are calculated as that. It does not give you the exact value but it gives you a very good approximation, may be up to 8 places of decimal or something of that sort. Now, back to the second order differential equation, what shall we do so one of the methods again uses that property of the exponential function, which we already made use of and ask suppose I have now, going back D square plus a 1 D plus a 0 acting on X equal to 0 and am looking at X, what shall X be. Can it be an exponential function? So what you can try x of t equal to e raise to t but, you can see that this will not be a solution or it may not be a solution why not for one should try.

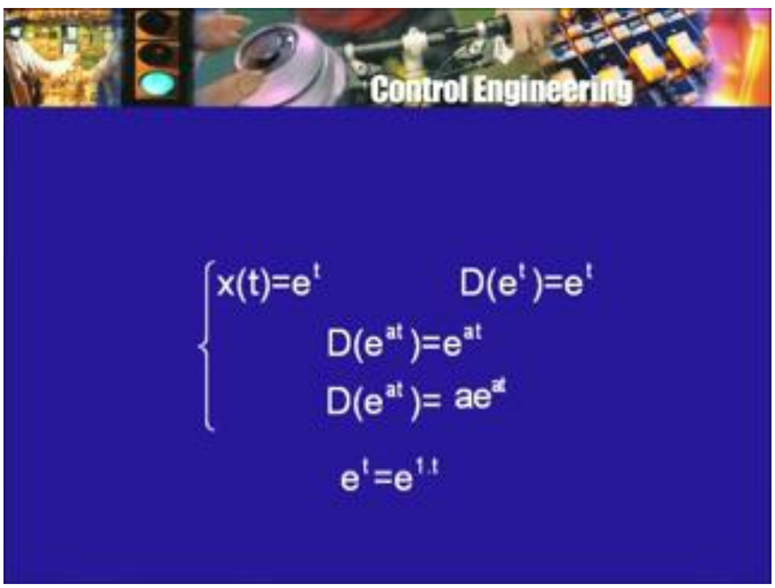
Of course, the text book sort of give you the solution readymade, they do not even let you think. But, if I am facing this differential equations for the first time and I am familiar with the exponential function e raise to t, let us say as the simplest of the exponential function. Then, I will say, why not try this? So let us try e raise to t, what will happen? Now this is where, one can make use of a simple property of the exponential function which is just going 1 step beyond the earlier property, what was the property of the exponential function? The derivative of examination raise to t was examination raise to t itself or the derivative of e raise to a t was a times e raise to a t that is to say, the differentiation is equivalent to multiplying the function by this coefficient a that multiplies t, all right. The special case of course e raise to t I can think of as e raise to 1 into t that is e raise to a t with a equal to 1, all right. So this be in the case and the action of d is simply to multiply by a, if the function is e raise to a t or in the function is e raise to t it is simply to multiplied by 1.

Now, look at the left hand side of the differential equation D square plus a 1 D plus a 0 acting on X. So what is the action of D acting on X, if the X is e raise to t just multiplied by 1 that means do nothing. So D as if D was replaced by 1 D X is simply 1 into X, if X is equal to e raise to t what about D square X? Well, the D of e raise to t is e raise to t D of D of that is still e raise to t. Once again, as if D was replaced by 1 and so, I have the following result that D square plus a 1 D plus a 0 acting on, e raise to a t the function is e raise to t is simply 1 square plus a 1 into 1 plus a 0 multiplying, e raise to t. Now that is equal to 0 that is my question, when will it be 0, it will be 0 only when this expression on the left hand side is coefficient is 0 and the coefficient is 1 plus a 1 plus a 0.

(Refer Slide Time: 16:42)

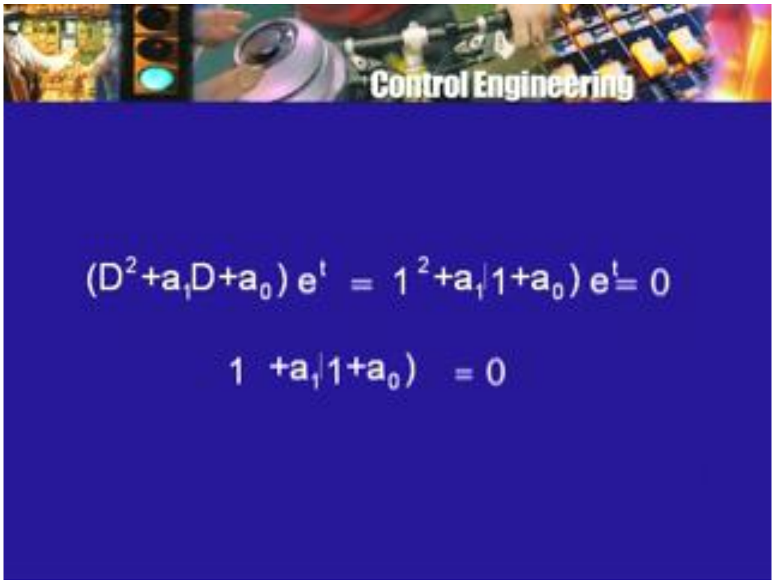


(Refer Slide Time: 17:02)

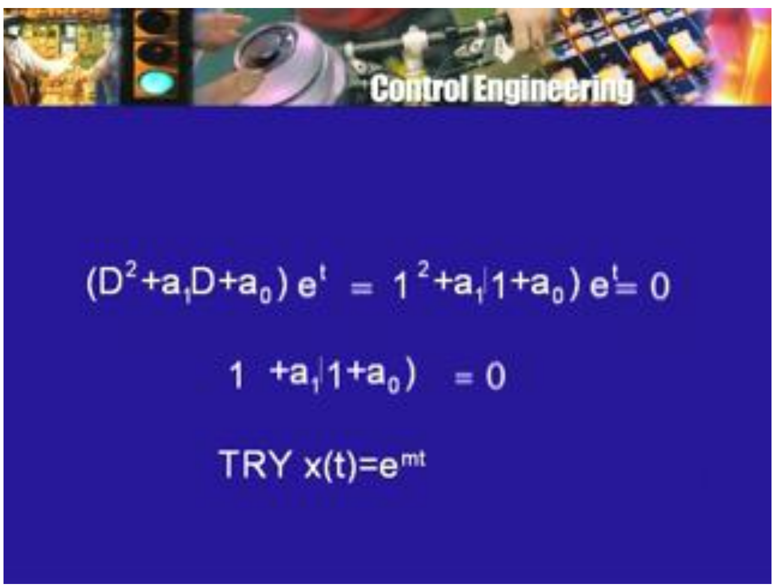


So, if that is 0 then only e raise to a t can be a solution of the differential equation that we are looking for. So e raise to t is not going to work. So let us try the slightly more general exponential function. Now, here it is customary in mathematics text books to use instead of a the letter m and its good for us because I have already using the letter a to denote the coefficient, in this case a 0 and a 1 are the 2 coefficients that are involved in the differential equation. Remember, linear differential equation with constant coefficients. So, let us try to see whether, e raise to m t is a solution of the differential equation that is whether, x t equal to e raise to m t will work.

(Refer Slide Time: 18:25)

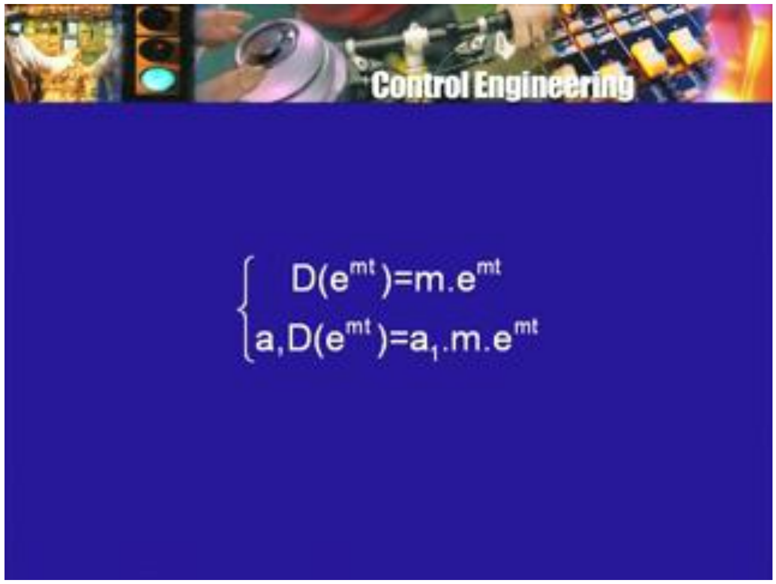


(Refer Slide Time: 19:16)

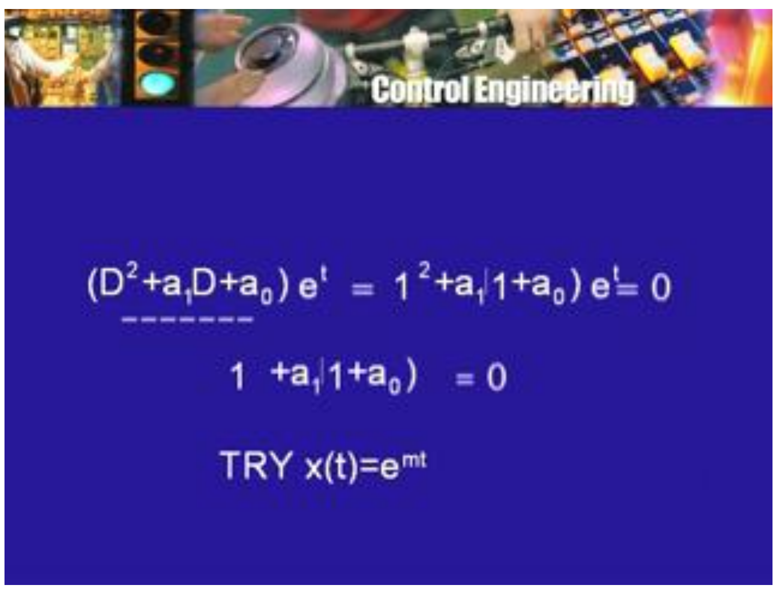


Now, what do I have to do we simply have to act on e raise to m t by this operator or as 1 says in mathematics, apply this operator to this function e raise to m t and see what happens. Well, as we saw earlier derivative of e raise to a t is a times e raise to a t. So derive differentiation simply multiplies the function by this coefficient of t. So d of e raise to m t will be m times e raise to t, so it is as if a 1 d, e raise to m t is simply a 1 m, e raise to m t that is replace d by m. So d of e raise to m t is simply replacing d by m times e raise to m t and if, I have m is the coefficient of t if I had a 1 d, e raise to m t then, that should be replaced by and that is equal to therefore it is as if this d is replaced m and a 1 into m, e raise to m t.

(Refer Slide Time: 19:58)



(Refer Slide Time: 20:59)



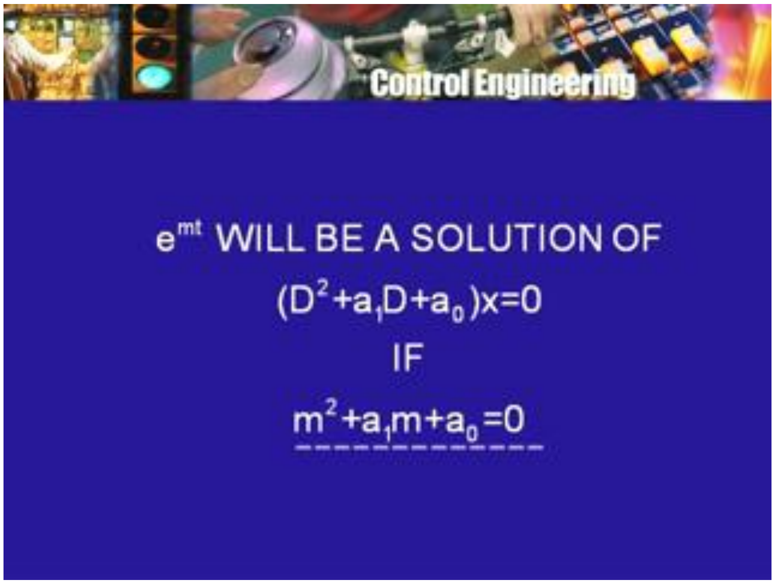
So this rule that when the differentiation operator acts on the exponential function, it is as if d can be replaced by the multiple of t and that is sometimes referred to as the exponent, in the general exponential function. Although, m t is really the exponent of e as you know 10 raise to minus 10 or 10 raise to 20, we referred to what you put at the top as the exponent. So e raise to m t m t is the exponent but for our work, the coefficient that multiplies t is conveniently thought of as the exponent of the exponential function.

So, if I go back and try e raise to m t as a solution of D square plus a 1 D plus a 0 equal to 0, the result I will get on the left hand side will be m square plus a 1 m plus a 0 multiplying e raise to m t and when is that equal to 0 or can it that, can that equal to 0 while e raise to m t is never 0, as we saw earlier. So this expression in the bracket, if it is 0 then this whole thing will be true and therefore e raise to m t will be a solution, if m satisfies the condition m square plus a 1 m plus a 0 equal to 0, m satisfies this condition. This condition in mathematics text is sometimes known as the indicial equation, meaning it is an expo, it is an equation for the index or the exponent and the indicial equation is very easily obtained as the operator D square plus a 1 D plus a 0 corresponding to that I have the indicial expression m square plus a 1 m plus a 0 and I have to solve the problem of finding out a number m, such that m square plus a 1 m plus a 0 equals 0 equal to the number 0.

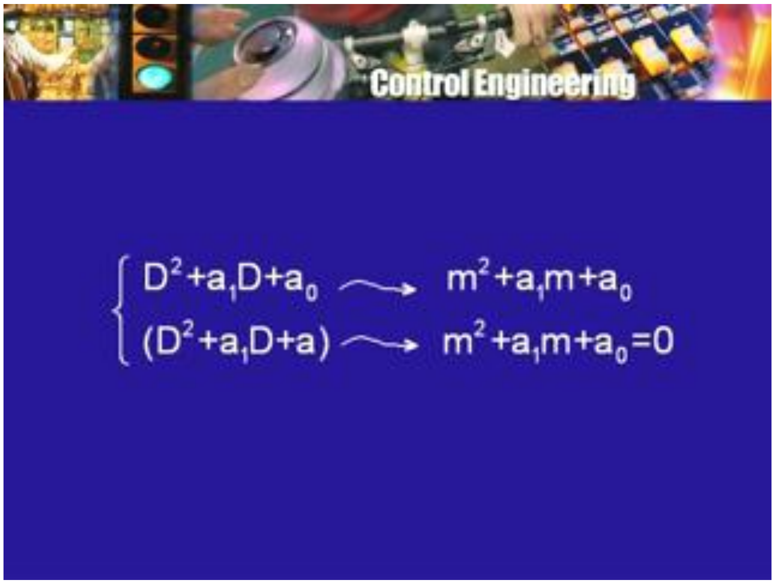
On the left hand side here, I have operators only or I may have an operator acting on a function such as X. Here, this is just an ordinary equation in which m is a number to be found out, it is a polynomial expression in m once again, it almost, it is looks likes the polynomial in D, except D is replaced by m but this is now, a polynomial equation involving a number, unknown number m or a variable m which is just a number, where as in D square plus a 1 D plus a 0 D was an operator not a number. It was something that acts on a function to change it into another function, there is a big difference between these 2, but from this we easily get this.

Now, again the solution of a what is this equation called? This is a quadratic equation the solution of a quadratic equation is something which no electrical engineer can afford to forget and I hope, you have not forgotten yet. The solution is important even the simple quadratic equation for an electrical engineer in particular and somebody whose studies systems like this electromechanical systems, so even for mechanical engineers and chemical engineers and so on even the simple looking quadratic equation is very important.

(Refer Slide Time: 21:35)



(Refer Slide Time: 22:05)



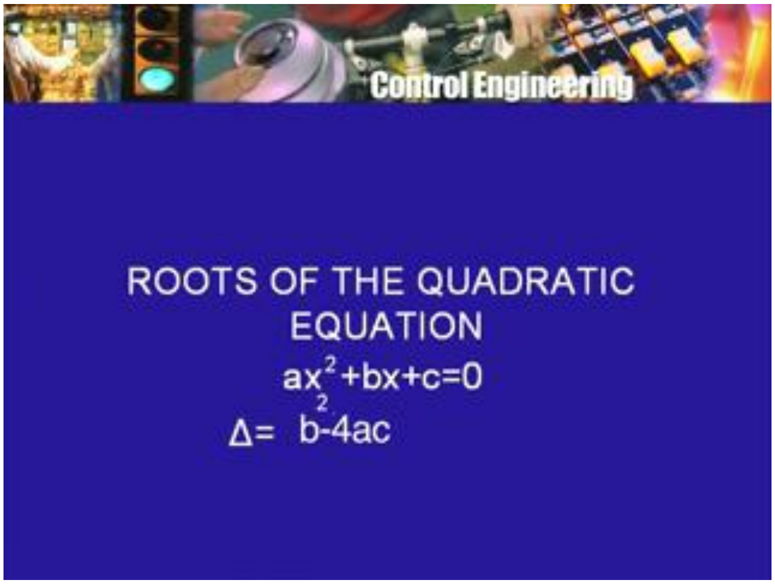
You may of course, remember the formula of the solution but what is even more important is that, there are several cases, there are several cases which have to be considered. The formula is one but then, from that formula depending on what the numbers a 1 and a 0 are, you will get different cases I would like you to go back to your algebra text and have a look very quickly at the 3 cases or 4 cases depending on whether, one of the cases is looked upon or split into 2 cases or not. Remember, these number a 1 and a 0 in our electrical example or electromechanical example, are simply some real numbers. They may be positive or negative but they are real numbers like 5 minus 20, what have you.

They are not complex number, they are real number and ultimately they will come from a resistance value being 100 Ohms or a EMF constant being such and such and so on or X of t itself is a real number because it may be voltage or current or torque or speed at any time t and so on. Now, the cases of the quadratic equation solution depend on what is called the discriminant, the discriminant again because the letter D 1 uses the Greek letter delta and the discriminant of this in this case is, what, you may remember it of course, as a X squared plus b X plus c equal to 0 likes the quadratic equation with the coefficient of X squared which is not 1, in that case it discriminant is b squared minus 4 a c.

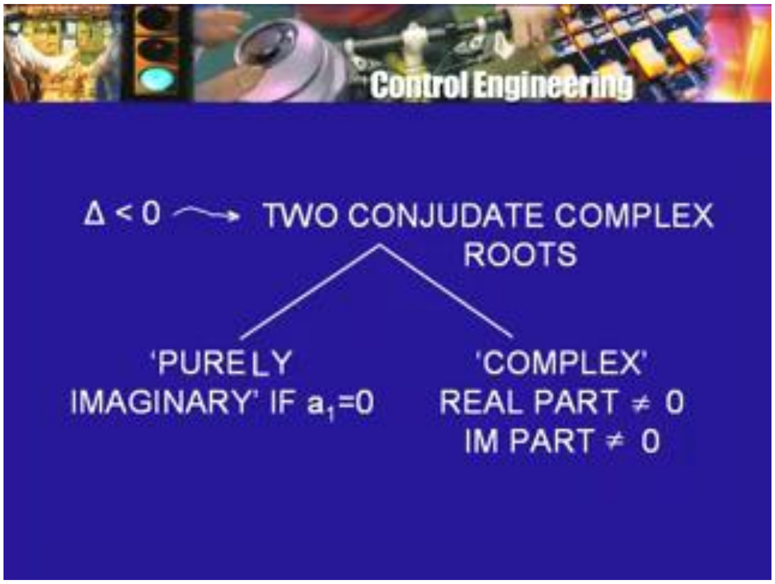
This number is called the discriminant of the quadratic equation and so, in this case because the coefficient of X squared is 1 our discriminant will be a 1 square minus 4 a 0. This is the discriminant of the quadratic equation and what are the cases to be considered, the cases to be considered are, this discriminant may be equal to 0, it may be positive that is greater than 0 or it may be negative that is less than 0. If the discriminant is 0, which means of course that a 1 square equal to 4 a 0 the numbers a 1 and a 0 are such that this is true then, we get one case and what does this case correspond to the discriminant is 0.

It corresponds to the case when the quadratic is actually a perfect square of a linear expression therefore, it has only 1 root or one 0 as it is sometimes called or if you wish, you can say that it has 2 roots but the 2 roots coincide. Actually that way of talking about 2 roots coinciding, when you studied in school and I studied in school really did not make any sense because, if 2 things coincided then, are they 2 or are they 1 but there is a good mathematical reason, for using this expression 2 roots which coincide rather than saying that it has only 1 root.

(Refer Slide Time: 25:28)

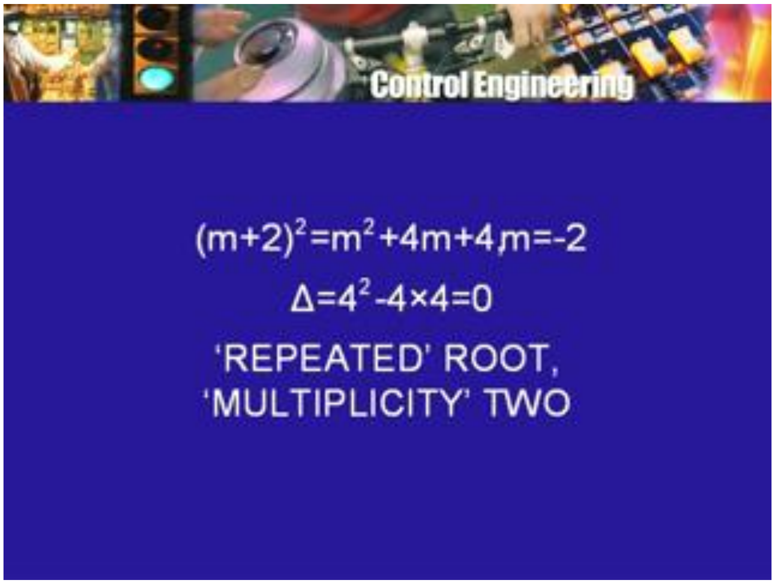


(Refer Slide Time: 27:46)



So accordingly then if delta is equal to 0 there is only 1 root or if you wish 2 coincident roots that is one case, the discriminant equals 0. Then there are the 2 cases, the discriminant is positive and discriminant is negative, if the discriminant is positive then, there are distinct roots that is there are 2 roots to speak about and they are both real. So we have 2 real distinct roots whereas, if the discriminant is negative then, there are 2 distinct roots but they are complex and there may be the 4 th case that I mentioned that is this case of complex roots, may be split into 2 cases, one case when the root is purely imaginary number and the other case is, when the root is complex but not purely imaginary that is, it has real part which is not equal to 0. But it also has an imaginary part which is not equal to 0. So in this way there can be 4 cases of the solution of the quadratic, the roots are 2 coincident that is only one root. In this case the root is real, so there is 1 real root then, there are 2 real distinct roots then, there are 2 distinct complex roots but as 2 sub cases, the 2 roots may be purely imaginary or the 2 roots may be complex with the both the real part and imaginary part not equal to 0.

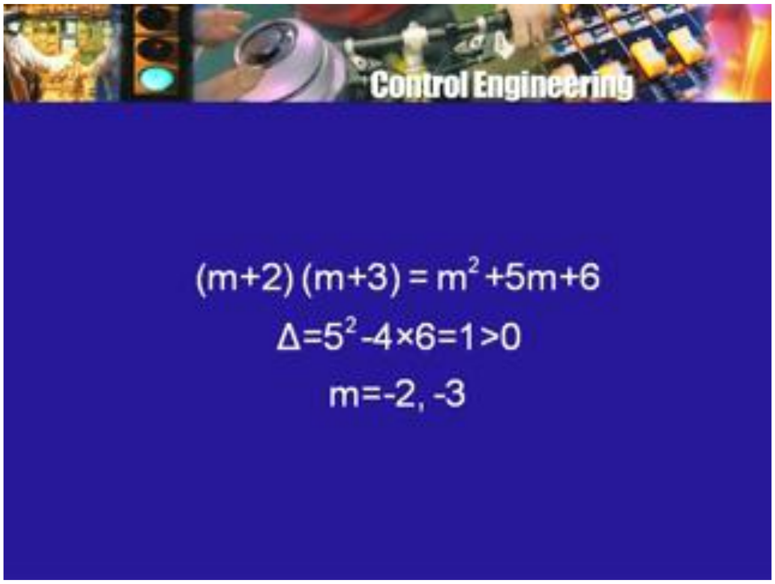
(Refer Slide Time: 29:31)



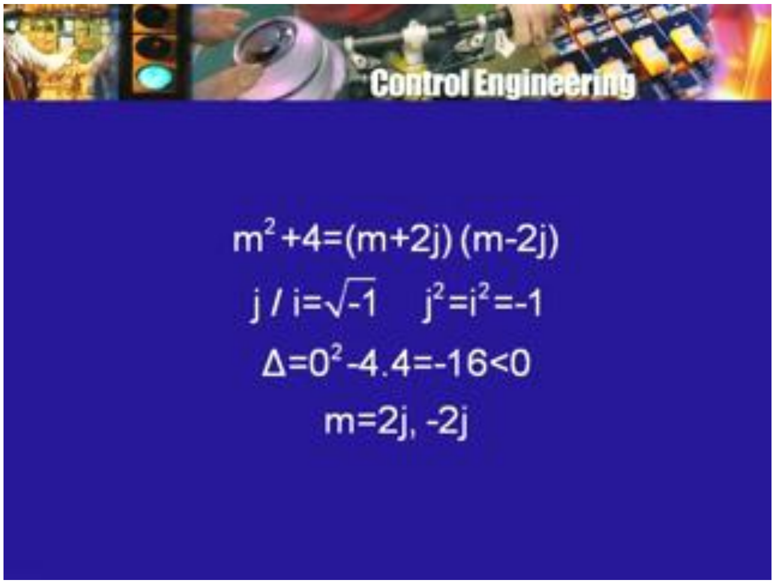
Now, I will just quickly give you few examples of all the cases and some of the cases will come across later on then, we look at our motor control problem and I can go backwards to construct the quadratic, whose roots will be of this kind. So for example, take the quadratic m plus 2 square, this is m square plus 4 m plus 4. Now the way have occurred obtained it, it is m plus 2 square, so its cleared that the root is only 1 root minus 2 or there are 2 coincident roots each minus 2. If that is the way you want to look at it and you can check that the discriminant is 0 delta is equal to 4 square minus 4 into 4 and that is 0.

So that is the case of one single root or its also then called repeated root and all these terms, if they are understood well enough but if you are not can cause a lot of confusion or a root of multiplicity to, again another expression or 2 coincident roots. Then, let us look at the situation delta greater than 0, so that there are 2 real distinct roots and again I can construct an example, m plus 2 into m plus 3. If I take that and work it out as the product, it will be m square plus 5 m plus 6 and because of the way I have obtained it, I know the roots are, what are the roots m equals to minus 2 is 1 root and minus 3 is the other root? Both are real and they are distinct minus 2 is not the same as minus 3, what is the discriminant delta here delta is 5 square minus 4 times 6, that is 1 and which is positive. So discriminant is positive the roots are real and distinct.

(Refer Slide Time: 30:12)



(Refer Slide Time: 30:57)

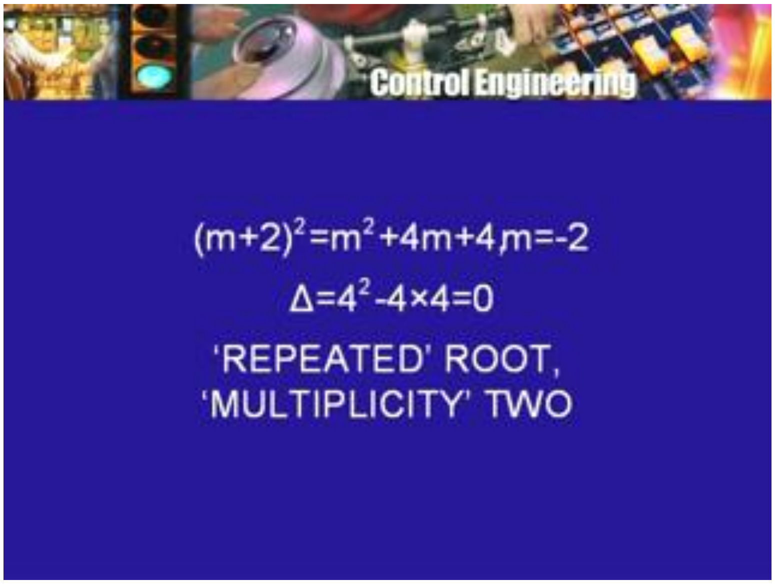


Let us take the case of, 2 imaginary roots and because the coefficients a 1, a 0 are real, we will see that the imaginary roots will be just the negative of each other. So how they are constructed, I can construct it like this say m plus 2 j into m minus 2 j, what if that out as a product, what is j here, j is the electrical engineer symbol for the complex number which is the positive square root of minus 1 that is the square root whose argument is phi by 2. In mathematics one may use the symbol i in place of j and therefore, we have the property j square equal to minus 1 or i square equal to minus 1.

So m plus 2 j into m minus 2 j is what, it is m square plus 4 as you can verify. Now what is the discriminant is there, delta equal to the m term is missing. So 0 square minus 4 into 4 and that is equal to minus 16 that is less than 0. So the roots are going to be complex but in this case, not just complex, they are purely imaginary. The roots are 2 j and minus 2 j, so they are purely imaginary roots and then, we will have the last case that is the roots are complex but not purely imaginary and therefore, also not purely real and I can construct that also as follows. I will write down m plus 1 plus j into m plus 1 minus j and if I take that product then, you can find out that it is m square plus 2 m plus 2, what is the discriminant here? Delta equal to this coefficient square so2 square minus 4 into this coefficient, so 2, so it is minus 4 less than 0.

When does correspond to the case, when the roots are complex but what are the roots here, the roots are m equal to minus 1 minus j and minus 1 plus j. So the roots are complex, distinct, they are different from one another but they are not purely imaginary. If this minus 1 was not there the roots would have been plus j and minus j as in the earlier case, there would be purely imaginary but they are not they are really complex, you can see. Now, will mention here something which will come across little later and in fact quite a bit of it and namely, when you come across roots or equations like this which are real numbers or complex numbers, they are shown on what is called the complex plane.

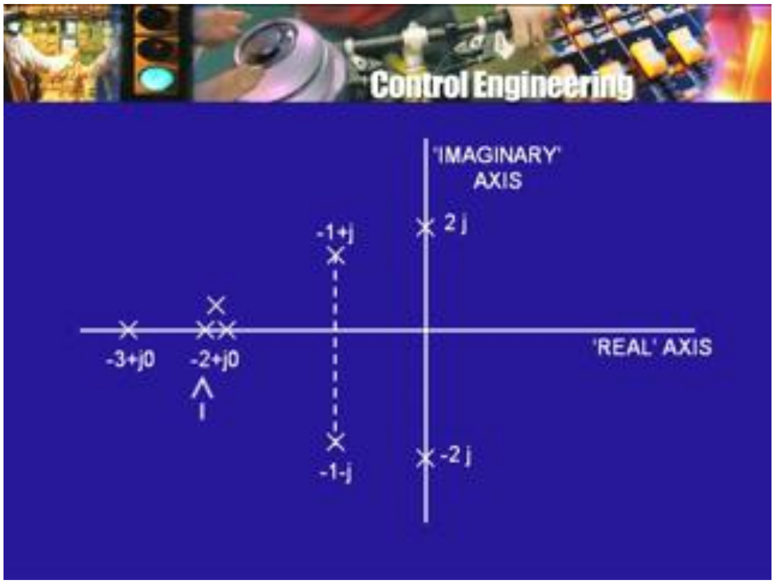
(Refer Slide Time: 34:22)



So this is something which one should learn and again one of the most important simple thing that an electrical engineer is required to know, a mathematician or a mathematics student may not really plot such things because for him, they are not particularly important, there are so many other things but for electrical engineers, these things are extremely important. So, let me look at the various examples that I took and I will show you, how we show it on the complex plane, what is the complex plane also called, it is sometimes called the Argand diagram named after the French mathematician, Argand, Argand diagram.

So we had the first case of m plus 2 square or m square plus 4 m plus 4 equal to 0. So the 2 roots where minus 2 and minus 2, so I will show them in this diagram by putting a cross here and I will put the 2 crosses very close to each other, suggesting that they are really the same and I will put down here minus 2 or because in general, a point here anywhere represents the complex number, I will write this as minus 2 plus j 0. So this is the quadratic or the roots of the quadratic or zeroes of the quadratic m square plus 4 m plus 4 or the solutions of the equation m square plus 4 m plus 4 equal to 0. They are minus 2 only one solution or 1 solution repeated 2 times or multiple serial 2 or 2 coincident root.

(Refer Slide Time: 35:00)



So this was one case the second case was real and distinct roots. So there I had1 root which was minus 2 and the other root which was minus 3. So I will show it different color of course now, I have to show it on the same diagram. So, I will show it here right to z on the top as minus 2 and so there is another root here minus 3. So, I will have minus 2 and minus 3 as the 2 real distinct roots of the quadratic m square plus 5 m plus 6 or the roots of the equation m square plus 5 m plus 6 equal to 0, 2 real distinct roots. In this case, both are negative then, we had the third case where we are roots were purely imaginary. So, where would they be on this diagram they would be on this, so called imaginary axis and the 2 roots where, what did I have them as plus minus 2 j plus minus 2 j. So drawing roughly to the same scale, here is 2 j and here is minus 2 j. So these are the 2 purely imaginary roots they are complex but they are purely imaginary, the real part is 0.

As you know of this horizontal axis, so to speak is called the real axis and the vertical perpendicular axis is called the imaginary axis and such a diagram is called the complex plane representation of a, of the representation of a number complex number in the complex plane. So this is the third case of purely imaginary roots and now, will have the 4th case where the 2 roots where complex but distinct and therefore, they were like this. Here was the root minus 1 plus j and here was the other root minus 1 minus j. So we have 2 complex roots which are distinct but they are not purely not imaginary. Of course, they are complex they are not purely easy unlike minus 2 and in this case, they happen to be what are called conjugates of one another in fact 2 j and minus 2 j are also conjugates of one another 2 complex numbers are said to be conjugate when, what happen ?

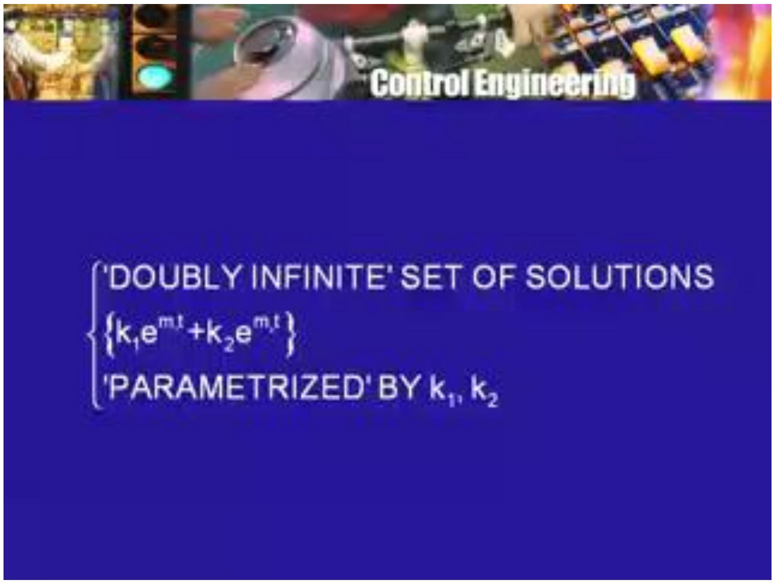
When they have the same real part but they have imaginary parts which have opposite signs. So we have complex conjugate roots and therefore, going back to our statement depending on the value and sign of the discriminant, when delta is equal to 0, the roots are real but repeated or there is only one real root, when delta is positive, there are 2 real distinct roots, when delta is negative, there are 2 complex roots which are conjugates of one another. However, in one special case the roots may be simply purely imaginary, in the other case the roots are complex and not purely imaginary.

So, even as simple thing as a quadratic then has this variety of situation associated with it and will see that this has important consequences. In our study, I am looking for a solution of a second order differential equation and it seems like e raise to m t will be a solution, if m is a root of the corresponding quadratic of the indicial equation, in our case m square plus a 1 m plus a 0 equal to 0. This e raise to m t will be a solution of the differential equation. So it is easy to show that if e raise to m t is a solution then, if I multiplied by any number k what so ever, it will be a solution and is if e raise to m t is a solution, 10 times e raise to m t will be a solution, 10 raise to minus3 times e raise to m t will be a solution and so on and so it is not just 1 solution that we have, when we are found this m but any multiple of it is a solution. So there is an infinite number of solution that we have found out this way but that is not all because this is a quadratic. So in general it will have 2 distinct root and therefore, there will be 2 values of m and its traditional to call them m 1 and m 2.

So e raise to m 1 t will be a solution that multiplied by any coefficient k 1 is also a solution, e raise to m 2 t is a solution that multiplied by a coefficient k 2 and k 2 because it may not be the same as k 1. This also is a solution but one can go 1 step further not only are these 2 also solutions but their sum is also a solution of the same differential equation. Now all this is a consequence or is related to the fact that the differential equation that we have is homogeneous that is right hand is 0 and the left hand side is linear.

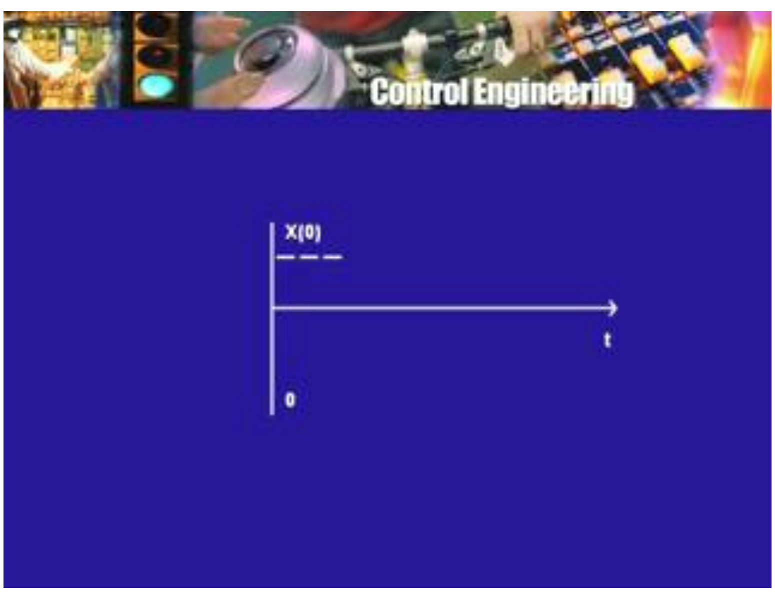
I told you that the word linear has some deep meaning associated with it and we will come to that the deep meaning a little later. But you should perhaps try to think about it that is why is the differential equation said to be linear and one of the, this what we have just now looked at it a sort of a hint that e raise to m 1 t is a solution then, k 1 e raise to m 1 t is also a solution, e raise to m 2 t is a solution then, k 2 times e raise to m 2 t is also a solution and when these 2 are solutions, k 1 e raise to m t is a solution, k 2 e raise to m t is solution, given this their sum is also a solution. This is connected with the fact or with the statement that the differential equation is linear and so now, what do we have we have a set of solution of the second order homogeneous equation k 1 e raise to m1 t plus k 2 e raise to m 2 t where, m 1 and m 2 are the 2 roots of the quadratic indicial equation and k 1 and k 2 are any 2 arbitrary number. So each choice of k 1, k 2 you will get one solution.

(Refer Slide Time: 40:59)



So we will get not only an infinite family but we will get a infinite family which is said to be doubly infinite, in the sense there is a k 1 that I can choose so as any real number. So 1 choice think of real numbers as put in some bag then that I infinitely many real numbers I pick 1 of them and then I pick another number k 2 from the same bag, may be same number, may be a different number. So I have 2 choices that is why it is said to be a doubly infinite set of solution or you can think of k 1 and k 2 as 2 parameters of the set of solutions. So for each value of k 1, k 2, you get one solution. So, we get solutions which have parameterized by 2 numbers k 1, k 2 and there are 2 parameters, so parameterized by k 1, k 2.

(Refer Slide Time: 43:09)



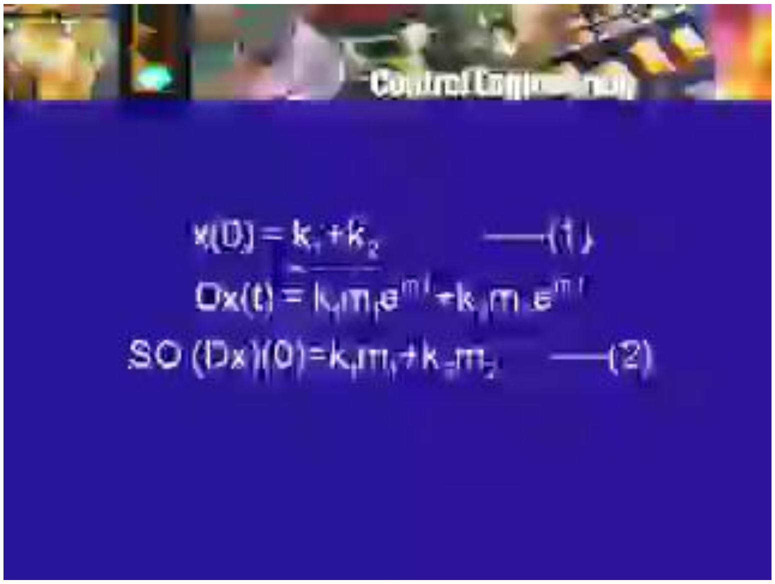
Now, of course in a given situation I am given actual numbers a 1, a 0, so I find the roots of the quadratic, so I will get m 1 and m 2 but what about k 1, k 2, how do I get these numbers k1 and k 2 because I will not solving just the differential equation as I told you, but there are also what are called initial values, initial conditions or boundary conditions. In other words, we have an infinite number of solution but my actual physical system has only one particular behavior says, the voltage varies this way or the speed varies this way and so on. It does not show infinitely many behaviors at one end the same time. So which one of these is what I am, I want to find. Now for that, I need the initial values and boundary values or boundary conditions.

Now, just as for the first order differential equation, it was enough to know what X 0 is, the value of the solution at time t equal to 0 because in control theory our action begins at some moment of time which you call 0, what was the system variable X at that moment. If I know that is 0 and if X satisfies a first order homogeneous equation then, I know the solution uniquely. This is what we saw earlier. Now, we have a second order differential equation and so, I need 2 initial values but 2 initial values of what, X is only one function. In turns out that I need to know the value of X at 0 and I need to know the value of derivative of X at 0, not only, if X is some position, the position of the particle at time t equal to 0 but then, D x is the velocity I need to know the velocity of the particle at time t equal to 0.

So, if my differential equation say describes the motion of a particle under the action of no external force perhaps 0 input or free motion, mass spring that is spot system in the beginning remember that, that is one example. Then, I need to know the initial position of the mass and the initial velocity of the mass to determine the position X of t of the mass for any future time t greater than 0. So I need to know these 2 initial conditions X 0 and D x 0, knowing them I can determine these 2 parameters k 1 and k 2 uniquely. Of course, all of this is familiar to you but I am revising it because we cannot afford to forget it and because it is going to play a very important part in our analysis and remember, we are only looking at some simple control systems, we are not really doing anything very complex and sophisticated, this is just the beginning course in control engineering or control system theory. So these are really very simple things there is much more to be studied.

So, let us not get stuck here but then, if you do not know this then there is no point in going further. So given the initial values X 0 and D x 0, how do I determine the coefficient k 1 and k 2 are the parameters k 1 and k 2. In other words but I am saying it for the solutions I have x of t equal to k 1, e raise to m 1 t plus k 2, e raise to m 2 t, this is x of t I have found out what m 1 and m 2 are by solving the quadratic but I do not know what these 2 parameters are coefficients k 1 and k 2 are and I want to determine that. So I set x 0 perhaps will help, so what is x 0 put t equal to 0, what do we get exponential of 0 is 1, so I get k 1 exponential of m 2 into 0 is exponential 0 it is 1, so I get k 1 plus k 2.

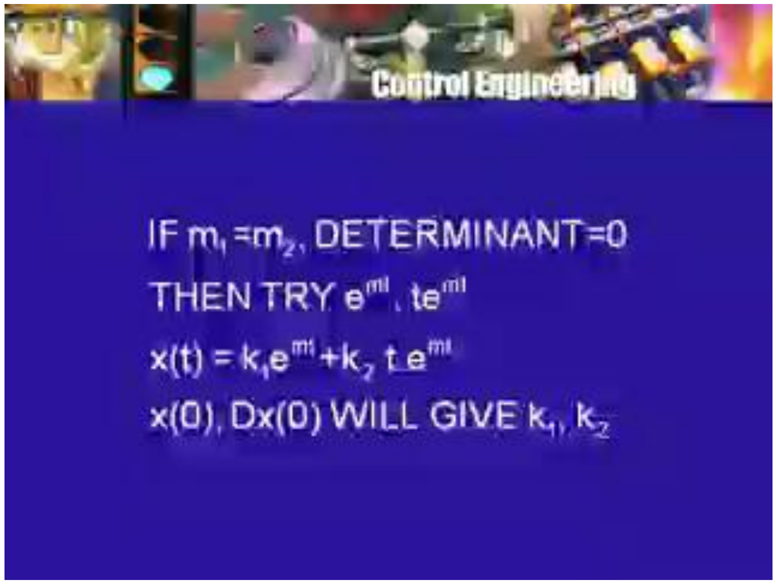
(Refer Slide Time: 46:05)



So, I get an equation k 1 plus k 2 equal to X 0 and X 0 supposed to be known then, I know k 1 plus k 2 but there are 2 of them not one. So I need one more equation. So I look at the derivative of X. Now, what is the derivative of X, if X is given by this expression D of x t will be equal to D of this and what is D of this, it is the m 1 times then, so I will get k 1, m 1, e raise to m 1 t and the second term becomes now, the derivative k 2, m 2, e raise to m 2 t. Now, if I evaluate the derivative of x at 0 and I am writing it as D x as 0 then, what is it, k 1, m 1 plus k 2, m 2. So, one equation here, k 1 plus k 2 equals X 0. Now, another equation k 1, m 1 plus k 2, m 2 equals D x at 0, the derivative of x are 0. I have 2 simple equations linear equations ordinary simultaneous linear equations for the 2 unknown coefficients or parameters k 1, k 2 and in this case, I can solve them and get k 1 and k 2 uniquely that is there is only one pair of number k 1 and k 2, which will do the job, k 1 will multiply e raise to m 1 t, k 2 will multiply e raise to m 2 t and the 2 values of k 1 and k 2 can be easily calculated and I said uniquely because I assumed without telling you that m 1 and m 2 are distinct that is I have 2 distinct roots of the quadratic. So it is not just m 1 and m 2 and that m 2 is just equal to m 1 but they.

I have made the assumption that m 1 and m 2 are different that is they are really 2 distinct roots of the quadratic because if m 1, m 2 are really not distinct that is they are the same number then, what is the point then writing e raise to m 1 t and e raise to m 2, they are the same function. I might as well I have combined 2 functions into one single function. So in other words, we are assuming that the discriminant delta is not equal to 0. So if the discriminant delta is not equal to 0 then, if we know the initial conditions X 0 and D x 0 then, we have found out the unique solution of the differential equation.

(Refer Slide Time: 48:58)



So we have to consider the case when the discriminant is equal to 0, what do we do in that case. So, I hope you recall from your maths course that in this case e raise to m t is 1 solution where, the m is the root of the quadratic or 2 coincident roots for that is really 1 root but there is also one more solution and what is that solution, it is t e raise to m t so e raise to m t is a solution. So also the e raise to m t and because of that once again, I can wrist X of t or I can say that the following also a solution what function X of t given by k 1, e raise to m t plus k 2 into t into e raise to m t.

So once again, we have 2 parameters k 1 and k 2 appearing and the general solution of the differential equation and once again, we have the problem of k 1 and k 2 and we can determine them in exactly the same way as we did earlier, namely by knowing X 0 and D x 0, we get 2 equations for k 1 and k 2 which can be solved uniquely and so, this is the special case that has to be treated separately of 2 coincident or repeated root with multiplicity 2. So as I told using earlier even the quadratic equations sum to such a simple thing, when it occurs in the solution of differential equations, when it occurs in the descriptions of systems that we are going to look at.

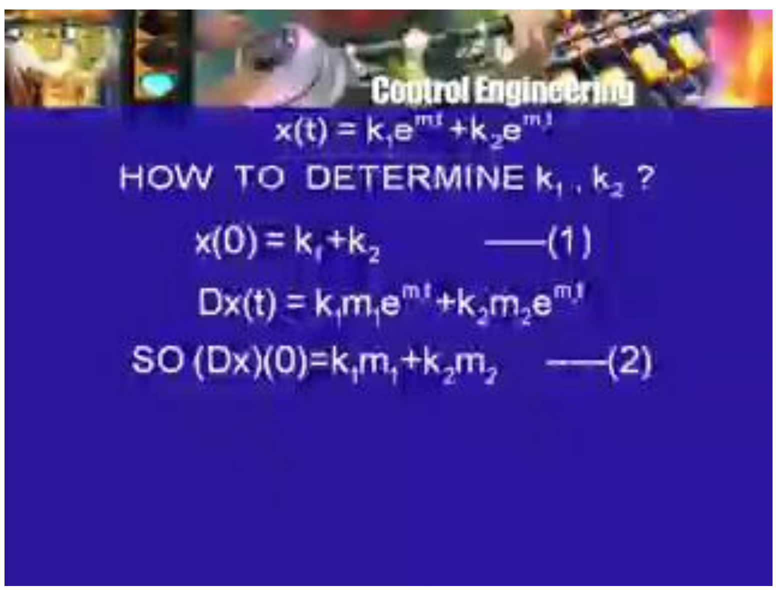
You have to handle these various cases, we have to consider various alternative situations and there is a big difference between a solution of the kind e raise to m 1 t plus e raise to m 2 t with appropriate multipliers where, m 1 and m 2 are real numbers and a solution of this kind k 1, e raise to m t plus k 2 into t into e raise to m t where there is only one exponent m and in1 of the functions we have a multiplier t there is a difference as we will see later on.

Of course, what about the other cases that I talked about where, I have simply said m 1 and m 2 are 2 roots, I did not ask whether the roots were purely imaginary or they were real or they were complex. I only said that they should be distinct but for us, It will be important to distinguish the cases m 1 and m 2 are both real, m 1, m 2 are both purely imaginary or m 1 and m 2 are both complex with real part not equal to 0 and imaginary part not equal to 0 and that is because, how the function X of t, the solution X of t will behave with time that is, what will happen to it as time passes depends on the nature of the solution whether, it corresponds to the exponent which is purely real or purely imaginary or complex as will see later on.

So these 4 cases that I talked about this distinction between the 4 cases and what happens to the solution in each one of the 4 cases, it is mix them very important to know it although it is very simple. When there is actually one more assumption that I have made and I just gone ahead with it and some of you perhaps will be a little uncomfortable because I have not stated that assumption, I said that if e raise to m t is a solution or if, I substitute then m must satisfy certain condition and therefore, if m satisfies that condition and be the indicial equation then, e raise to m t will be a solution.

So we are left to the quadratic equation its 2 roots or 1 root etcetera, etcetera. So I have found one set of solutions of the differential equation but can there be other solution. I know the exponential function, so I choose e raise to m t try it see, if it work, it works if m is a root of the indicial equation then I multiplied by k 1, I get again another solution, I get m 2, e raise to m 2 t, I multiplied it add. So I get the whole family of solutions k 1, e raise to m1 t plus k 2, e raise to m 2 t as we saw earlier, this one here k 1, e raise to m 1 t plus k 2, e raise to m 2 t. The question is can there be other solutions, other than this that our this the only solution the answer to that is yes but it is not easy to prove that these are the only solutions.

(Refer Slide Time: 53:03)



So that the 2 parameter family gives you all the solutions of the second order differential equation is a result which is true but it is not easy to prove it. It requires some good bit of mathematics to prove but it can be proved and therefore, we have the complete set of solutions of the second order differential equation available to us. We do not have to look for anything more secondly, we can also show that knowing the initial values X 0 and D x 0, we can uniquely determine k 1 and k 2. So given the initial values the solution can be found out among that infinite set, we have the infinite set, w infinite set of solutions which one is actually going to the case can be found out, if we know X 0, D x 0.

So knowing the initial values the solution can be determined uniquely. So this is the situation for the second order or second degree linear differential equation with constant coefficient which is homogeneous with 0 input or free system. Even for such a simple thing the considerations are fairly involved, what happens if the differential equation is of higher order or higher degree? Well, was can expect that there will be more work and there will be more complexity to the problem and we will have to look at it because an practice, you will just do not come across only second order systems, you come across higher order systems of much higher order and systems which have not described simply by an ordinary LDEWCC and systems which are homogeneous. We have to consider the effect of inputs or forcing functions, so we have to get into greater depth as we will do very soon.

But as you will realize this method of finding the solution is really too simple. It is trying to make the exponential function do the job. Fortunately, it does the job all right. In fact, we will see that for higher order differential equation also the exponential function will do the job. But, whenever there is an input present then, this exponential function by itself may not be enough and so one may have to look at other method of solving the differential equation and one of them I mentioned earlier, namely the method that uses the Laplace transformation technique or the Laplace transformation method of solving such differential equation. So to that we will turn next.