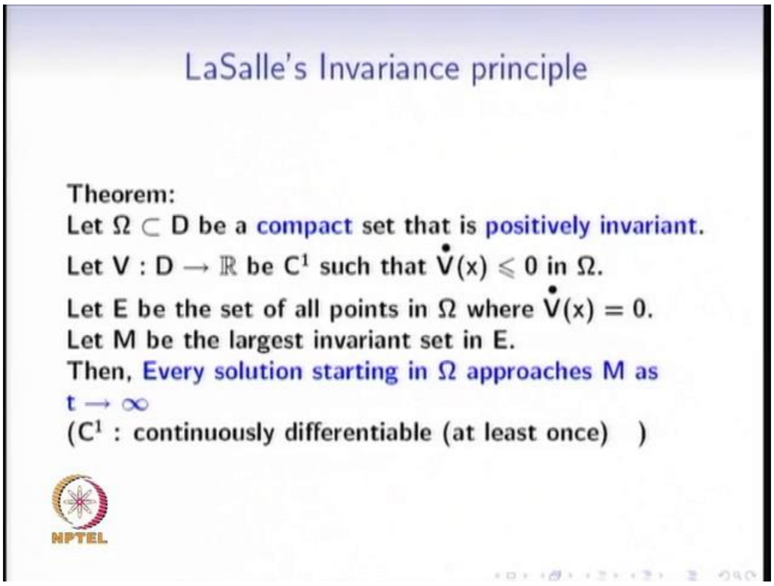
Nonlinear Dynamical Systems

Prof. Madhu N. Belur and Prof. Harish K. Pillai
Department of Electrical Engineering
Indian Institute of Technology, Bombay

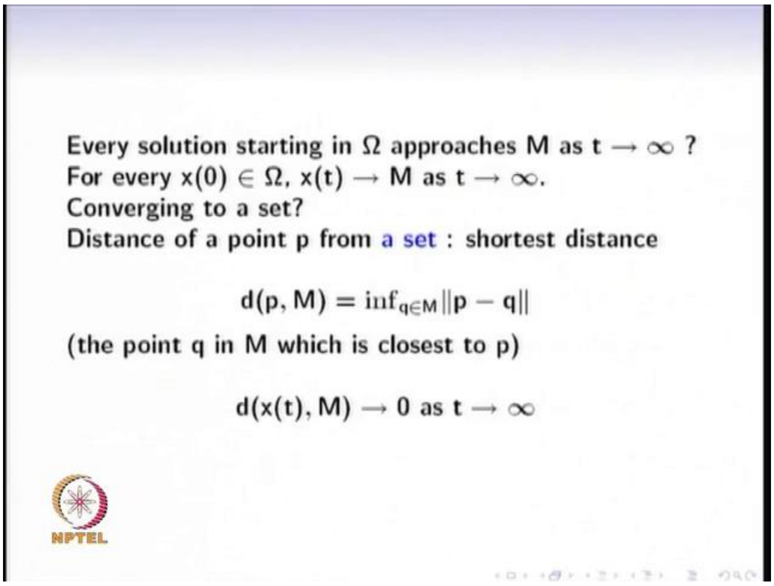
Lecture - 9
LaSalle's Invariance Principle, Barbashin and Krasovski Theorems, Periodic Orbits

(Refer Slide Time: 00:25)



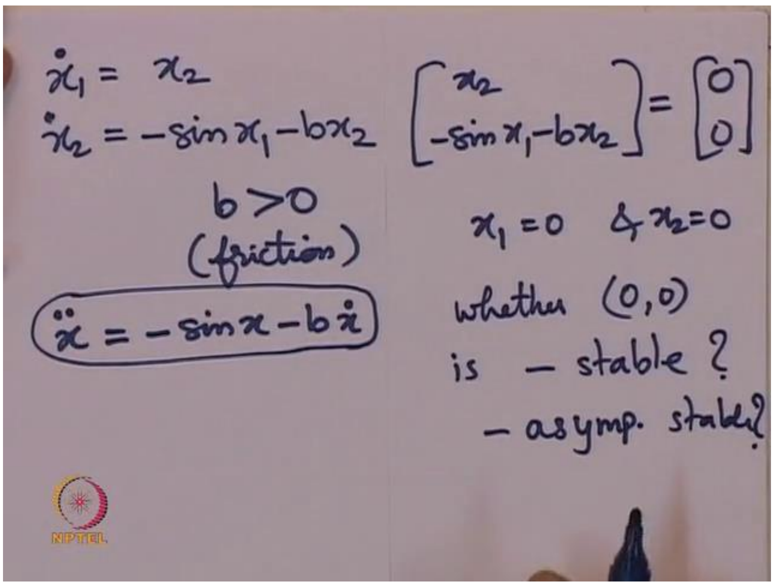
Welcome everyone to lecture number 9 on non-linear dynamical systems. So, we had started with LaSalle's invariance principle the last lecture. So, let us just quickly review this, suppose omega is a compact set that is positively invariant. Suppose we have found a function V that is C 1, C 1 means differentiable and the derivative's continuous, such that this V satisfies V dot is less than or equal to 0 on the set omega. For this V we will now find a set e such that V dot equal to 0 on set e. Let M be the largest invariant setting e, largest invariant setting e means, it is invariant under the dynamics of this dynamical system. It is containing e and it is the largest such set. In other words any other subset of e that satisfies these properties is also contained in M. If these conditions are satisfied then every solution starting in omega approaches this set M as t tends to infinity.

(Refer Slide Time: 01:32)



So, approach a set we had seen this as a definition. So, what the LaSalle's invariance principle says that every starting in the set approaches min other words for every initial condition the trajectory converges to M, what is the meaning of converging to a set. It is the distance of a. P from a set, is defined as the distance of P to different points in M and the shortest such distance. So, this is definition of distance of. P from the set M. Now, as x evolves as a function of time x of t is a point. We look at the distance of x of t from the set M and this distance decreases as t tends to infinity that is the statement of the LaSalle's invariance principle.

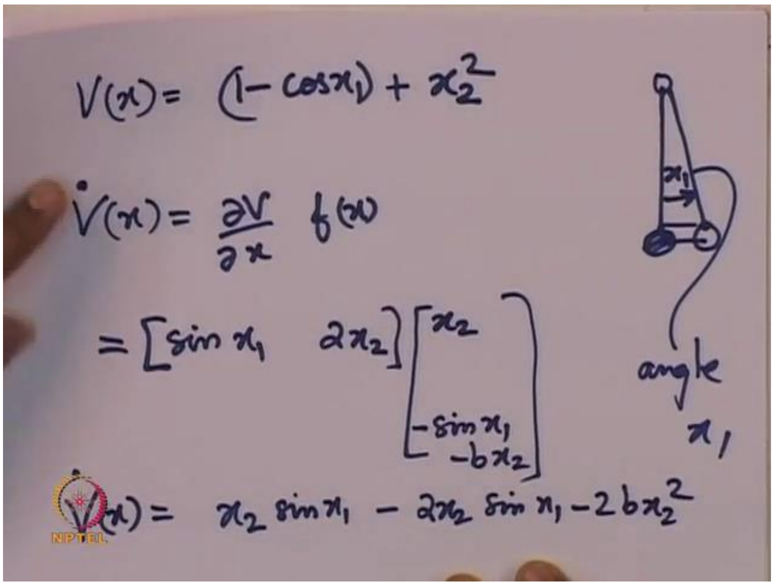
(Refer Slide Time: 02:18)



So, we had already encountered the situation of the pendulum example, in which the natural energy function to take did not satisfy strictly less than zero. So, let us do this example again. So, this is what we call as friction, this is the situation of a pendulum the original differential equation was of second order which was equal to like this.

This differential equation of second order we converted to a first order differential equation, 2 first order differential equations by introducing x 2. So, x 1 is same as x and x 2 is a derivative of x 1 and these 2 first order differential equations we will. Now, study the equilibrium point and the stability properties of the equilibrium point. So, for this dynamical system, we will now see x 2 and minus sin x 1 minus b x 2 this equal to 0 zero. This gives us x 1 equal to 0 and x 2 equal to 0 as 1 of the equilibrium point points. We can have x 1 equal to pie also that corresponds to the pendulum standing upwards which we know as unstable. We can also obtain that as a conclusion by linearizing about that point. Checking that the Eigen values are at least 1 of them is in the open right of plane that we will keep as an exercise what we will. Now, check is whether this equilibrium. Whether 0, 0 is stable asymptotically stable, this is what we will check.

(Refer Slide Time: 04:30)

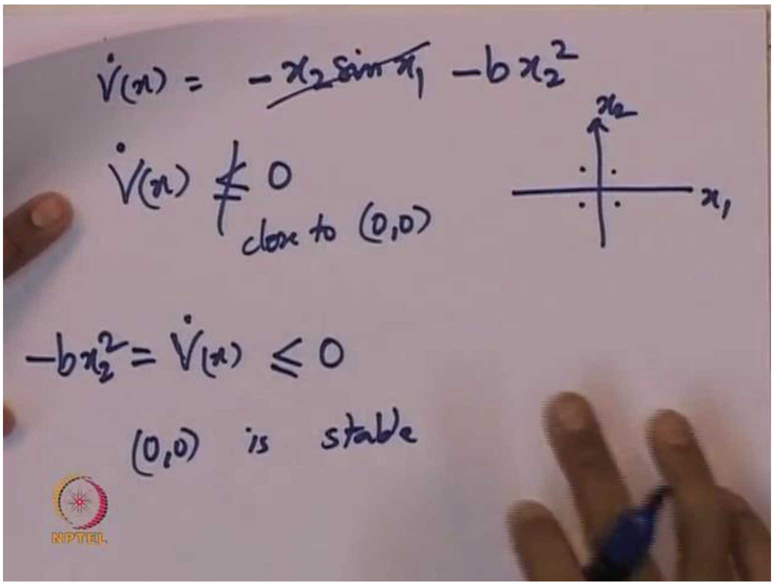


Now, for this purpose we will take the function coming as the energy of the system. So, take V of x equal to 1 minus cos of x 1, why because this is the potential energy, what was our x 1 variable. This is our pendulum when it undergoes a deviation of angle x 1 this is an angle. The time how much does it get raised the amount by which it gets raised is the potential energy accumulated into the system and that turns out to be 1 minus cos x 1 of course, multiply it by the mass and the gravitational acceleration g, but we have considered a model where those parameters are not arising.

This can be considered as normalization of the equations or as normalization of mass to 1. This is only the potential energy the other energy term is actually x 2 square by 2. Let us check what happens if we take just x 2 square. This is not really energy because this is not kinetic energy, second term is not kinetic energy, but twice the kinetic energy. So, let us check what happens to V dot of x. So, this turns out to be del V by del x, times f of x.

When we evaluate this del V by del x is a row vector in which the first component here is a derivative of this with respect to x 1, which is exactly sin of x 1. The derivative for this with respect to x 2, that is 2 x 2 times f of x what was f of x the first component was x 2 second component was minus sin x 1 minus b x 2. When we multiply this product, this is an inner product we get x 2 sin x 1 minus 2 times x 2 sin x 1 minus 2 b x 2 square.

(Refer Slide Time: 06:59)

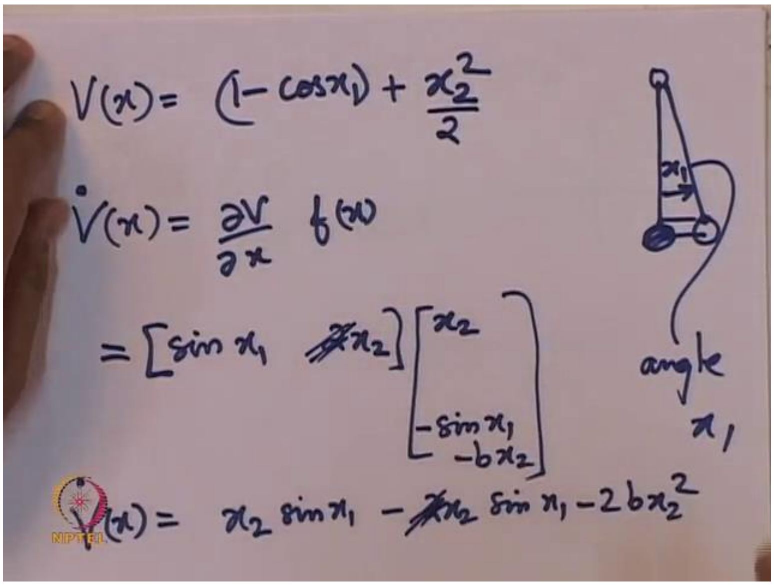


So, this is what we get V dot of x. So, is this quantity positive or negative that is the next thing we will investigate. We got V dot of x was equal to this term is well behaved and the other term is, this is our x 1 this is our x 2. So, one term of course does not change sign it is always negative or equal to zero, but the other term x 1 and x 2 sin of x 1 has a same sin as x 1 close to x 1 equal to 0, but x 2 times this can change it sign depending on which quadrant it is for small values of x 1 x 2.

In other words close to the origin we are not able to say that V dot of x is less than or equal to 0. It is not satisfied close to 0, 0. This one can check oneself, in order to check oneself 1 could first ignore this particular term, why we can ignore this term because this is to put D equal to 0. Just means that we have a pendulum without friction and for the pendulum without friction we know that the system is stable by intuition.

We want to obtain that as a conclusion for the lyapunov theorem of stability. For that purpose this particular quantity certainly changes sign, it will have different signs depending on x 1 x 2 one, one each of these 4 quadrants. Hence this does not satisfy less than or equal to zero. So, this is not a valid lyapunov function why because it is a lyapunov C it is positive definite, but it is not decreasing it is non-increasing around the origin. So, let us got back to our lyapunov function and make a small change here. Now, divide this by 2 this perhaps we have already verified once.

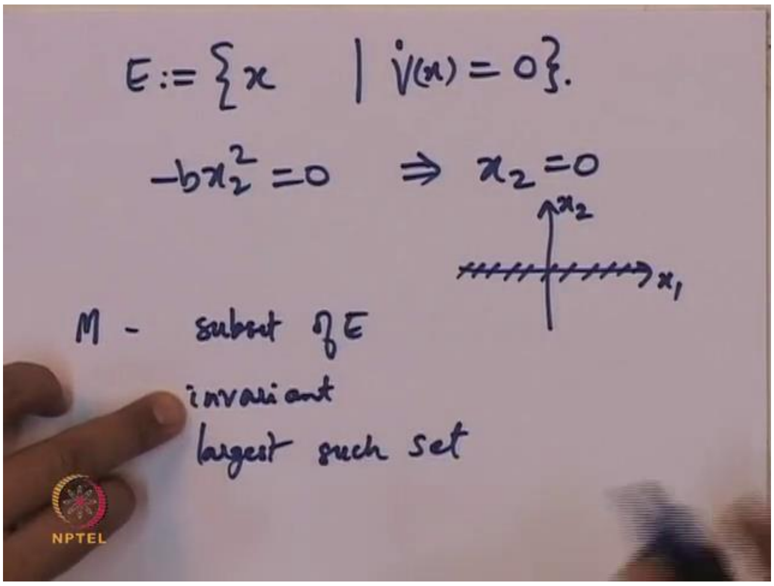
(Refer Slide Time: 08:58)



So, this now is indeed the kinetic energy. So, by doing this we do not get this 2 term here. We remove this also here because of it this term. Now, cancels out and we have V dot of x. Now, it is indeed less than or equal to 0 why because V dot of x was equal to minus b x 2 square. So, this at least proves that 0, 0 is stable however this has not helped us to prove that the origin is asymptoting the stable.

Even though by intuition we know that equilibrium point is in fact asymptoting, this table because we have friction which continuously dissipates off the energy. So, how do we obtain that this particular function lyapunov Candace will not help us. However we can use LaSalle's invariance principle for the same lyapunov function. So, construct the set e set of all x such that V dot of x was equal to 0 in other words minus b x 2 square equal to 0 it gives us set of all points that x 2 is equal to 0.

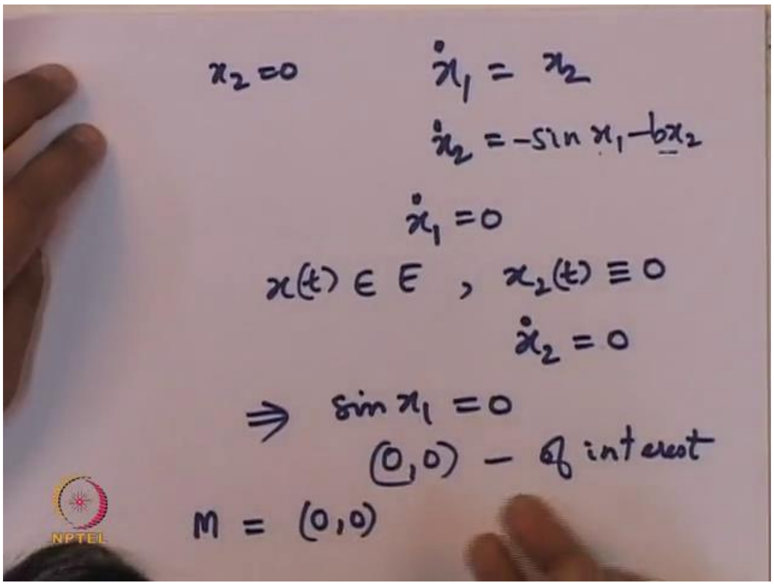
(Refer Slide Time: 10:11)



So, this x 2 equal to 0 is nothing but the x 1 axis is a set of all points where the rate of change of lyapunov function is equal to 0. So, this is our set e, now we want to look at the set M which is subset of e invariant, invariant under the dynamics of the system and largest set largest such set largest, such set which satisfies, that it is a subset of e and it is invariant under the dynamics of the system. So, how will we find the largest such, set we will look for what values of x 1 and x 2 are subset of e and are also invariant.

When we try to do this we will automatically get the set of all x 1 x 2 points that are invariant and containing d. Hence it will be the largest such set. So, x 2 equal to 0 is a requirement that the set M is contained in e. Now, we will put this in x 1 dot equal to x 2 and x 2 dot equal to minus sin x 1 minus b x 2 and we put x 2 equal to 0.

(Refer Slide Time: 11:34)

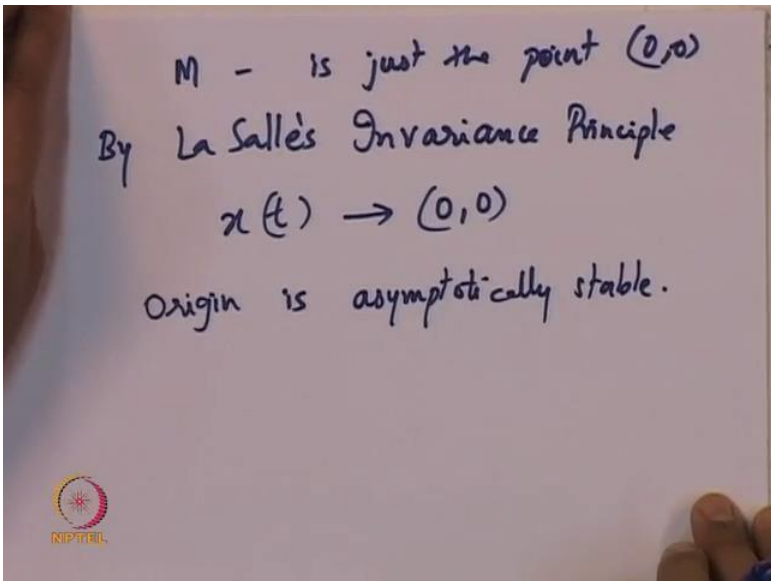


We get x 1 dot equal to 0 we also put x 2 sox of t is contained in set e which means x 2 of t is equal to 0 always is uniformly equal to 0 equivalently equal to 0, identically equal to 0. These are the different ways we interpret this symbol this equation. So, if some particular value of x as a function of time always equal to 0, it is like a constant function which automatically means that x 2 dot is also equal to 0 identically. So, when we put that x 2 dot equal to 0, then we also get sin of x 1 equal to 0. This implies that sin of x 1 equal to 0 which of course, we know happens at either the vertically down position which is x 1 equal to 0 or the vertically up position, which is x 1 equal to pie. Since you are interested about the stability properties of the 0, 0 we get this is of interest.

Since the interest at this we get that M is just the 0,0 when we ask the question inside the set e which are all those points, which are invariant under the dynamics of, f the time to the set e, which means we put the equation x 2 equal to 0. We studied invariance by invariance for invariance, we put the fact that x 2 is always equal to 0. which means that x 2 dot equal to 0. We substituted that back here, we got sin of x 1 also equal to 0. This term was already 0 this term, we now got equal to 0 because of which we obtained sin of x 1 equal to 0, which means that we this can happen at pie, 0 also the first component is equal to 0. That is one of the interest which is vertically down position.

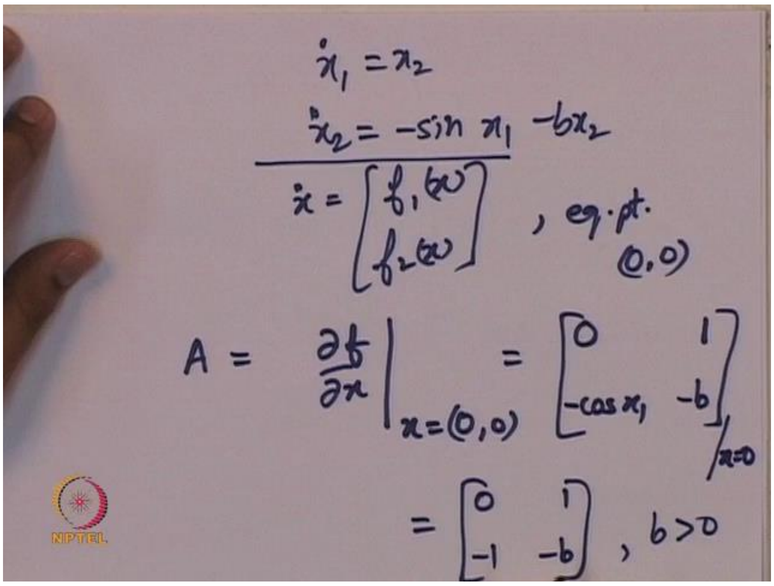
So, we have obtained the set of all invariance points that satisfies the property that it is invariant and subset of e gives us only this point. This is the largest set any other set would not satisfy the equations. We looked at looked for all the points that satisfy the equation and got this point.

(Refer Slide Time: 14:07)



In other words we have obtained that M is just the set just the point 0, 0 what is the meaning, by LaSalle's invariance principle x of t converges to the set M which is just 0, 0. So, this in other words proves that origin is not just stable, which we already concluded for the lyapunov stability, but we in fact got that the origin is asymptotically stable. So, this concludes proof of the statement that the origin is asymptotically stable, by using what principle not by using lyapunov theorem of asymptoting stability, but by using LaSalle's invariance principle, which we use to conclude that the set M has just 1 point the origin and by LaSalle's invariance principle the trajectory x of t converts to M. Hence, the origin is asymptotically stable.

(Refer Slide Time: 15:31)

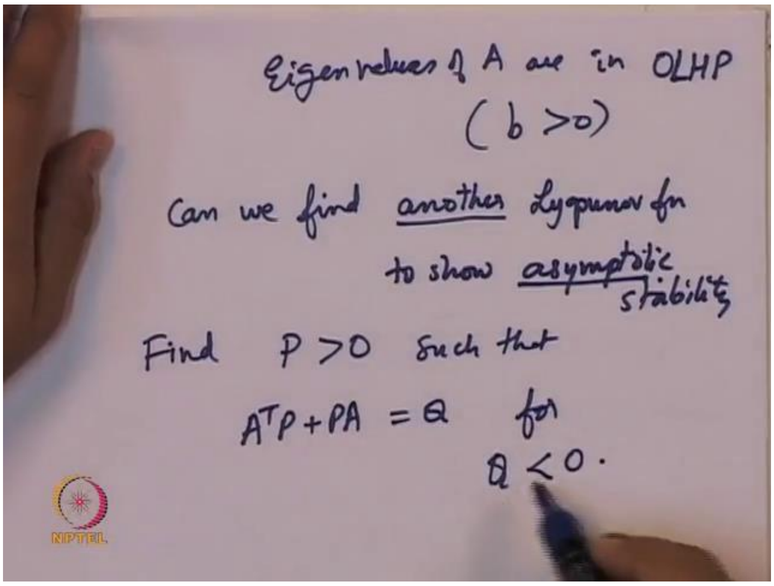


Now, we will investigate whether the linearized system at this point is also asymptotically stable. So, consider again x 1 dot equal to x 2 and x 2 dot equal to minus sin x 1 minus b x 2. So, x dot is equal to this can be written as f 1 of x and f 2 of x as a vector. So, what is a linearization, we have already checked that the equilibrium point of interest is 0, 0. Let us check that this is an equilibrium point. Now, what is this linearized system, it is this particular matrix evaluated t x equal to 0 comma 0.

So, let us find what this matrix is the term that comes here is the derivative of f 1 with respect to x 1. So, in f 1 which is this equation x 1 does not come at all in other words the derivative of f 1 with respect to x 1 is 0. What is the derivative of f 1 with respect to x 2 that is the term that comes here that is precisely equal to 1 why because f 1 of x is equal to x 2. So, derivative of, f 1 of x with respect to x 2 is equal to 1, what is the derivative of f 2 with respect to x 1. So, where all does x 1 appear in this equation it appears only here. In other words derivative of minus sign x 1 with respect to x 1 that is minus cos x 1. What is the derivative of this term with respect to x 2 x 2, does not come here it appears only here.

So, we put minus b here this as excepted is a matrix, but it depends on x 1 and x 2 it depends only on x 2 in this case. So, we are now required to evaluate this at x equal to 0 at the origin. So, which means that in these in the first 2 entries x 1 x 2 do not appear it appears only here, when you put x 1 equal to 0, we get minus 1 and of course, b is greater than 0. Let us check how the Eigen values of this matrix look. So, upon checking one can do the calculations and check that Eigen value.

(Refer Slide Time: 18:02)

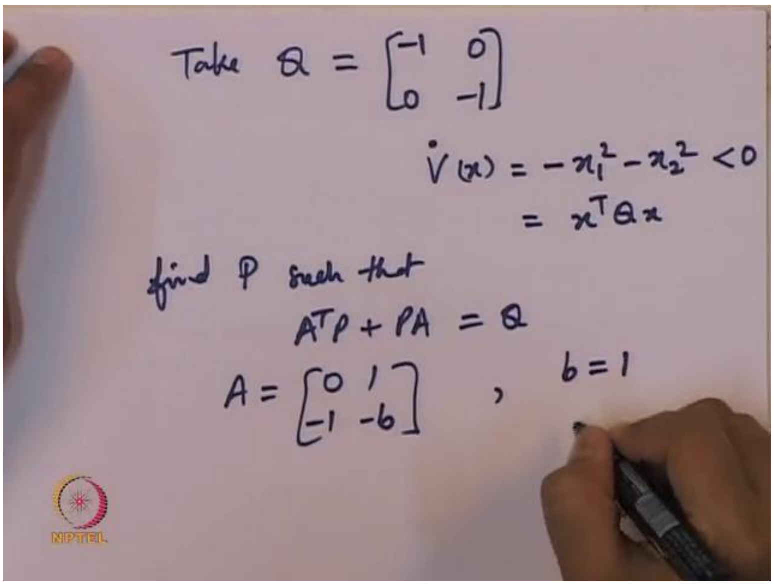


Eigen values of a are in the open left half complex plane, one can check that by using the fact that this greater than 0, Eigen values of that particular matrix we wrote are both in the open left half plane, which means that the origin is asymptotically stable. For a linearized system if a, is if the origin of the linearized system has all Eigen values in the open left half plane. Then we know that the non-linear systems equilibrium point is also asymptotically stable. However that lyapunov function could not help us with that.

So, can we find another lyapunov function, after all the lyapunov theorem was only a sufficient condition for stability and asymptotic stability. Since, we already know that the equilibrium point is asymptotically stable, can we find another lyapunov function to prove to show asymptotic stability the energy function, already helped us to prove stability, but we want to prove asymptotic stability. So, we will consider lyapunov function for the linearized system.

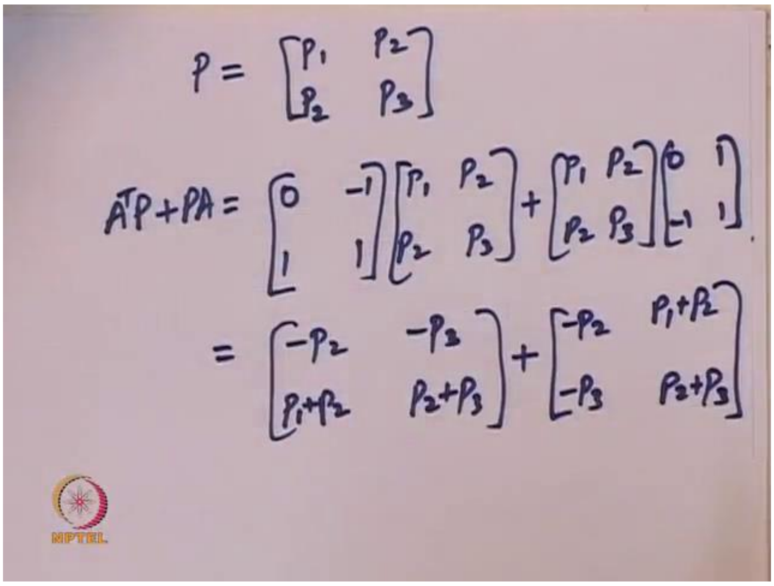
In other words find P greater than 0, such that a transpose P plus P a, is equal to Q for Q a negative definite matrix, this is the problem that we will solve. Now, why because this particular lyapunov function for the linearized system will also help as a lyapunov function for the non-linear system. So, we can in fact choose for linear systems because a, is for any Q we will be able to find such a p. So, take Q equal to minus 1 0 0 minus 1.

(Refer Slide Time: 20:36)



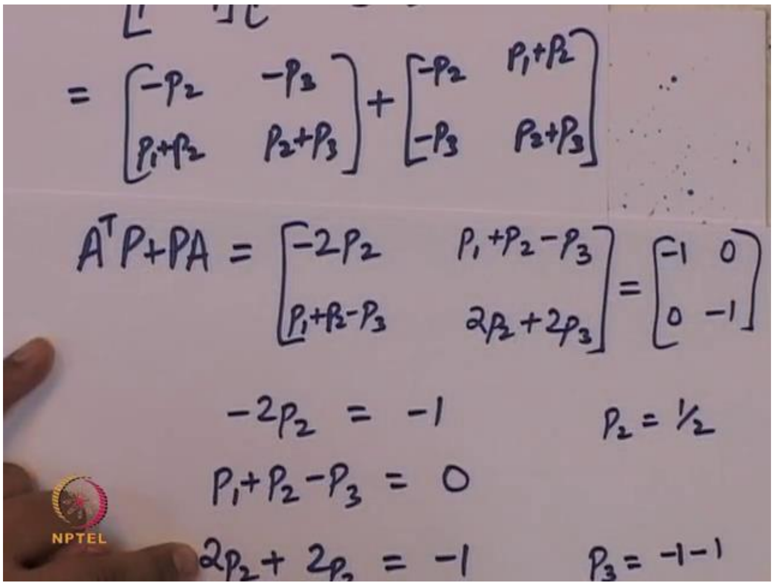
In other words, this will Q will correspond to our V dot of x. So, the corresponding V dot of x will turn out to be equal to minus x 1 square minus x 2 square, why because V dot of x is nothing but x transpose Q x on linear systems. So, when we put this particular Q we will get precisely this. This we know is negative definite, it is strictly less than 0 for all x 1 x 2 except, of course x 1 equal to 0 and x 2 equal to 0. So, for this particular Q we will now look for a p, such that find P such that a transpose P plus P a, is equal to this Q because that particular a is harvitz the P, that we will obtain from this equation, will turn out to be positive definite matrix this is equation that we will solve. Now, notice that, A was equal to 0 1 minus 1 minus b. For the purpose of solving we could take b equal to 1, this is the rate at which energy decreases due to friction and this is required to be positive. So, we have taken b equal to 1.

(Refer Slide Time: 21:20)



What do we get by solving for solving for P. We will assume P as these entries P 1 P 2 P is a symmetric matrix, hence this entry is also equal to P 2 and P 3. So, when we do A transpose P plus P A, that time we get this to be equal to this is A transpose, what is written here is a transpose hence P 1 P 2 P 2 P 3 plus same matrix P 1 P 2 P 2 P 3 times a, which was equal to 0 1 minus 1 1. So, we will now evaluate this, this is equal to minus P 2 minus P 3 this is P 1 plus P 2. Here we have P 2 plus P 3, this is minus P 2 minus P 3 and P 1 plus P 2 P 2 plus P 3. Now, we will this to Q and while doing that, so we can add this 2 matrices to get finally, a transpose P plus P a, is equal to minus 2 P 2 P 1 plus P 2 minus P 3.

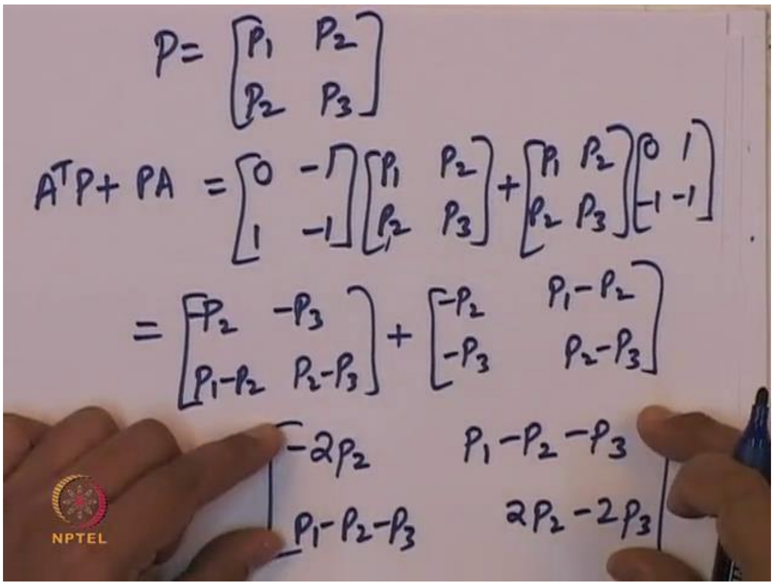
(Refer Slide Time: 23:00)



So, we get the same thing P 1 plus P 2 minus P 3. Here we get 2 P 2 plus 2 P 3. So, since P was symmetric, we have got this particular matrix to be symmetric. That is a reason that we should be choosing Q also to be symmetric. We have chosen Q to be equal to let us find values P 1 P 2 P 3 a particular theorem. We already saw claims that this system of equations is solvable. So, there are only 3 entries to 3 equations 1 2 and 3, why because this entry equal to this is a same equation as this entry equal to 0. So, let us put minus 2 P 2 equal to minus 1 P 1 plus P 2 minus P 3 is equal to 0. Finally, 2 P 2 plus 2 P 3 equal to minus 1.

So, the first equation just tells that P 2 is equal to 1 by 2, which when we substitute in the last equation we get P 3 was equal to minus 1 minus 2 P 2 minus 2 times P 2 is nothing but minus 1, again which gives us P 3 equal to. So, we have taken A equal to 0 1. Let us go back to this equation, we have got A equal to 0 1 minus b and we have put b equal to 1. Let us now take P equal to P 1 P 2 P 2 P 3 we have taken P to be symmetric.

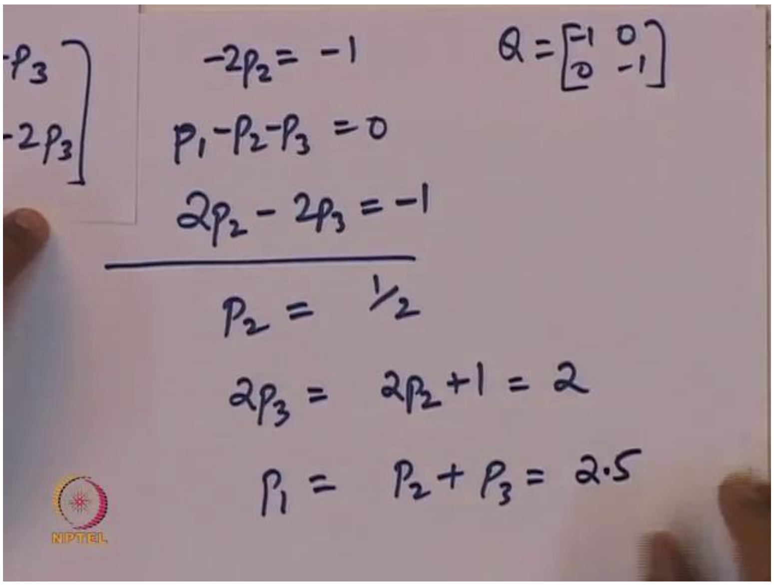
(Refer Slide Time: 24:44)



That is why we have taken a same entry here. So, let us solve for A transpose P plus P A This gives us for A transpose. We will write 0 minus 1 1 minus 1 times P 1 P 2 P 3 plus the same matrix P 1 P 2 P 2 P 3 times A which is equal to 0 1 minus 1 minus 1. Then we solve this. So, this when we do we get minus P 2 minus P 3 P 1 minus P 2 and P 2 minus P 3 plus this particular matrix product. When we evaluate we get minus P 2 minus P 3 P 1 minus P 2 P 2 minus P 3. When we add these 2 matrices we get minus 2 P 2 P 1 minus P 2 minus P 3 P 1 minus P 2 minus P 3 and finally, 2 P 2 minus 2 P 3.

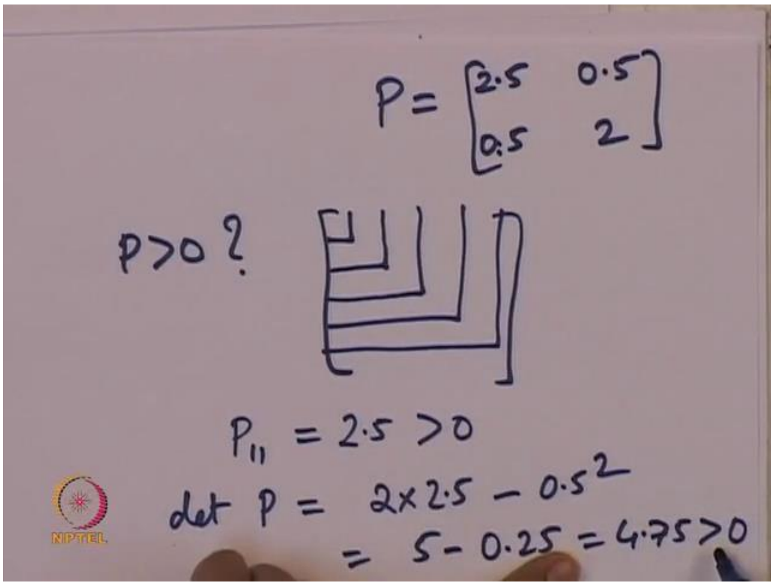
So, this matrix we have finally, got is nothing but a transpose P plus P times A. Now, we will equate this to Q. So, we had already taken notice that, this matrix is symmetric because we have taken P to be symmetric. This matrix has been obtained to be symmetric. Hence it is important that this matrix b equal to A Q, which also should be assumed to be symmetric. So, we have taken Q to be equal to minus 1 0 0 minus 1. So, when we equate this matrix to Q, then we have it appears like 4 equations.

(Refer Slide Time: 27:00)



This entry equal to minus 1, this entry equal to 0, this entry equal to 0 is again the same equation. So, it is not really 4 equations, but 3 what is the last equation, this entry equal to minus 1. So, these 4 equations now we will write here. So, we have minus 2 P 2 equal to minus 1 P 1 minus P 2 minus P 3 equal to 0 and 2 P 2 minus 2 P 3 equal to minus 1. So, the first equation gives us 1 by 2, which when we substitute into the last equation we get 2 P 3 is equal to 2 P 2 plus 1 which was equal to 2, when we put P 2 equal to half here we get 2 and these P 1 P 2 and P 3. When we substitute to the second equation we get P 1 equal to P 2 plus P 3 which was equal to 2.5. So, what is our matrix P as a result of this the matrix P was equal to P 1 P 2 P 3.

(Refer Slide Time: 28:12)

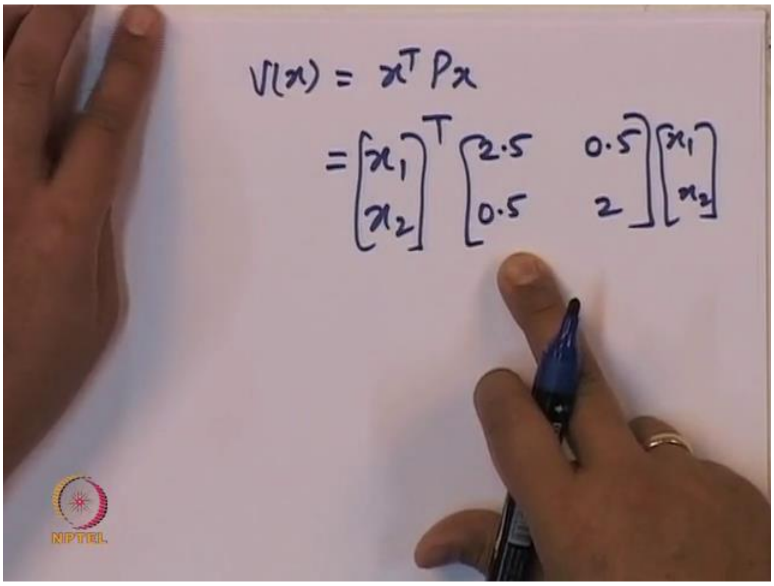


So, P 1 has been obtained to be equal to 2.5, P 2 was equal to 0.5 which is a same entry here. Finally, P 3, which was equal to 2. So, the claim is that this matrix P can be obtained is positive definite because A was Horvitz and Q was negative definite. So, we can check this P greater than 0. How will we check this, one way to check that a matrix is positive definite is that, all the principle minors all the leading principle minors. So, for a square symmetric matrix this is a 1 by 1 minor.

It is a principle minor because it has symmetric rows and columns take to construct that matrix and only the leading ones. So, we take only this now and every such matrix we take and look at the determinant and each of these determinants should be a positive number that is a necessary and sufficient function for this matrix P to be a positive definite matrix. So, let us do the check for this here we have to take only 2 determinants the first 1 by 1 determinant is nothing but 2.5 P 1, 1 the first sub matrix is equal to 2.5 that is greater than 0.

What about the determinant of the while matrix the next leading principle minor is nothing but the whole matrix P the determinant of the whole matrix is 2 into 2.5 minus 0.5 square which is equal to 5 minus 0.25 which is equal to 4.75 that is positive. So, because both the first leading 1 by 1 minor and the second leading 2 by 2 minor are both positive determinant. This means that the matrix P is positive definite. So, if we had taken the lyapunov function coming from this P for the linearized system.

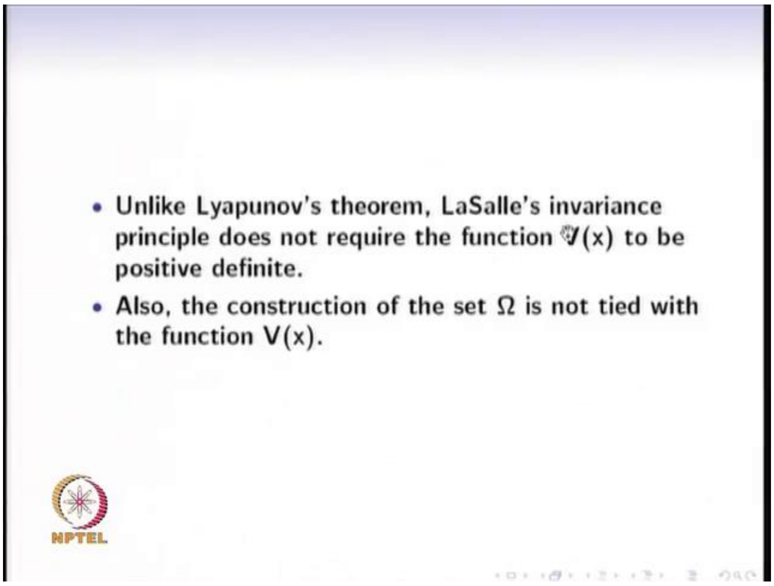
(Refer Slide Time: 30:19)



So, what is the lyapunov function it is x transpose P x in which P was in which matrix P was the 1 we just now obtained, if we take this a lyapunov function the origin turns out to be asymptotically stable by lyapunov theorem of asymptotic stability. The same lyapunov function will also help us in proving asymptotic stability of the non-linear systems equilibrium. which happens to be the origin again, but if we had started with this lyapunov function, then we would not have required LaSalle's invariance principle because the lyapunov function theorem would have claimed stated, that the equilibrium. is asymptotically stable. Unlike the energy function, unlike the physical energy that we have taken which helped us to prove only stability lyapunov theorem of stability.

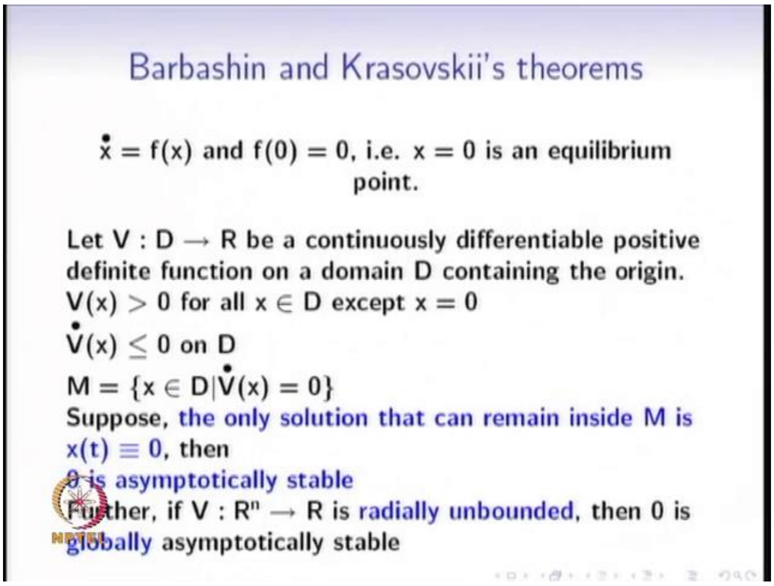
So, this completes lyapunov analysis we have seen some solved examples also. We can have another set of problems, which we will use as exercises. We will now move on to the next topic, which is about periodic orbits, why because periodic orbits are an important part in the context of building oscillators. So, before we move to that topic there is one other slide that we had skipped.

(Refer Slide Time: 31:47)



So, the LaSalle's invariance principle which we saw in detail also saw an example, is different form lyapunov theorem in two ways. The first way is, unlike the lyapunov theorem the LaSalle's invariance principle does not require the function V to be positive definite. Notice that we did not assume that V was positive definite, second the positive invariance at that we have constructed in the proof of the lyapunov theorem, that set omega was constructed using the lyapunov function V. Here we are assuming that we already have a positive invariant set. In fact it is that is the reason, that we are not assuming we as positive definite because on a compact set omega, we are V always achieves its minimum. We can subtract that minimum form the function V.

(Refer Slide Time: 32:41)



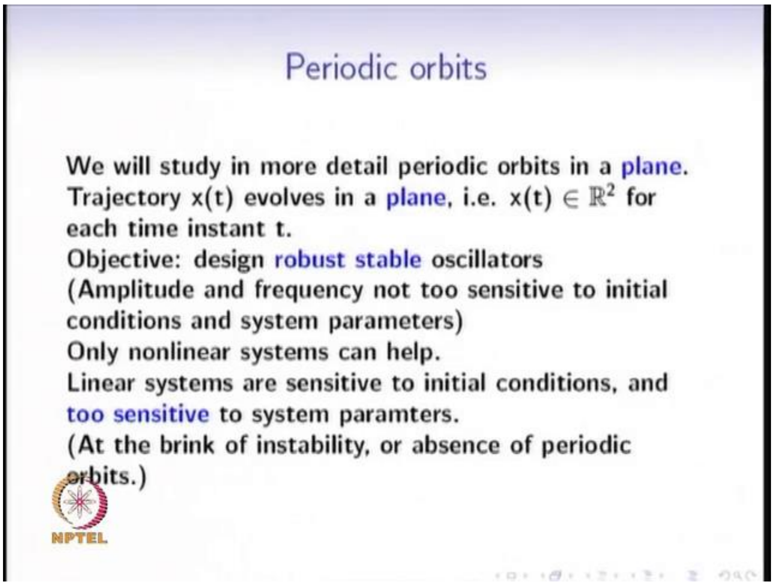
Here by which, we can always obtain another function V that indeed is positive definite. We will also see an application of LaSalle's invariance principle. So, that are well known results that turn out to be a special case of the LaSalle's invariance principle. So, one of them is Barbashin Krasovskii's theorems, what is the statement of the theorem.

So, suppose x not is equal to f of x id a system in which x can have many components x is an element of R n. Suppose the origin is in equilibrium, suppose there exists a function V from a domain D to R, which is continuously differentiable and suppose V is positive definite function in other words V of x is greater than 0. For all except origin x equal to 0 and also V satisfies that it is less than or equal to 0, on the domain D construct the set M that is made up of all the points where V dot is equal to 0.

Suppose, this particular M has a property that the only solution that can remain inside is x t identical equal to 0. Then the origin is asymptotically stable notice that, this is precisely the situation that happened for the pendulum example with friction. So, the Barbashin Krasovskii's theorem is a more general statement to this effect. What can we speak say about global asymptotic stability of that equilibrium. We just now claimed to be asymptotically stable if V is radially unbounded. Further in addition to the above assumptions, if V is also radially unbounded, then the origin is in fact globally asymptotically stable.

So, this particular theorem we already saw for the case of a pendulum as far the asymptotic stability is concerned of course, the pendulum example it is not globally asymptotically stable simple because there are other equilibrium point points. However the Barbashin Krasovskii's theorem says that, if V where radially unbounded, then the origin is in fact globally asymptotically stable. One can check that the lyapunov function V we had used for the case of the pendulum example with friction, that is not going to be radially unbounded, otherwise the origin there would have been globally asymptotically stable according to this theorem.

(Refer Slide Time: 34:55)

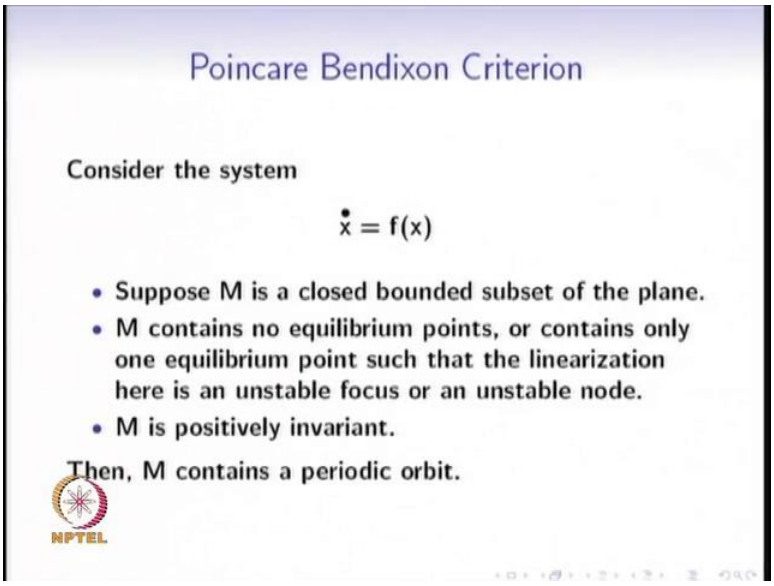


Now we come to the other topic about periodic orbits for this purpose. We will study periodic orbits in more detail for a plane where the trajectory evolves in a plane. In other words at each time instant at each time instant x of t is an element of R 2, it has 2 components only. So, what is the objective the objective is to design robust and stable oscillators. So, what is robust about this and what is stable about this, we want that the amplitude and the frequency of the oscillations are not too sensitive to the initial conditions and are not too sensitive to the system parameters.

So, as we had noted at the beginning of this of this lectures only non-linear systems can help why is that because linear system first of all are very sensitive to are sensitive to initial conditions. In other words if we start with a different initial condition, then the amplitude is different of course, for linear systems of frequency remains the same, but the amplitude is different more over the fact that there are periodic orbits is extremely sensitive to system parameters.

In other words Eigen values are on the imaginary accesses for small changes in the system parameters, Eigen values could be in the right half plane or in the left half plane, which means that we might have either no periodic orbits and all trajectories go to 0 or the trajectories could become unbounded. There is again absence of periodic orbits in other words linear systems are at the brink of instability and hence periodic orbits are extremely sensitive to the system parameters. So, for non-linear system the question arises how to even claim that there are periodic orbits for this system of equation. So, one extremely important result in this context is the Poincare Bendixon Criterion.

(Refer Slide Time: 36:51)

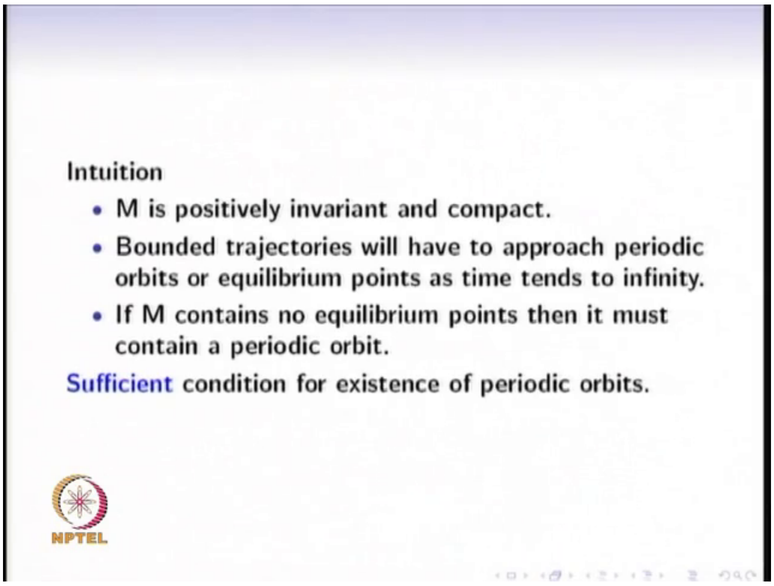


So, what does the criteria tell. So, consider the system x dot is equal to f of x in which note that x has only 2 x here has only 2 components. Suppose, the set M suppose there is a set M which is compact set it is a close end bounded set of the plane. Suppose M has a property that M contains no equilibrium point points or M could contain an equilibrium point such that that equilibrium point is either unstable focus or an unstable or an unstable node. So, there are 2 situations for the second bullet. The first case is M contains no equilibrium point points the second situation is that when we linearized at that equilibrium. If there is equilibrium, then at most 1 equilibrium point is allowed and at when we linearize at that equilibrium, then the linearized system has an unstable focus or an unstable node.

In other words both the Eigen values of the matrix a, which we get by linearizing at this equilibrium point are instable. Both Eigen values are in the open right half complex plane. So, suppose M has this property further suppose M is also positively invariant. If M satisfies these 3 conditions that it is a compact set its positively invariant and either M has no equilibrium point points or at most which is unstable. These 3 conditions are sufficient to ensure that M contains a periodic orbit.

Under these assumptions the Poincare Bendixon criterion claims that, there is M is guaranteed to contain a periodic orbit. So, what is intuition behind this. So, M is positively invariant and compact, in other words trajectories that start inside M remain inside M for all future time.

(Refer Slide Time: 38:50)

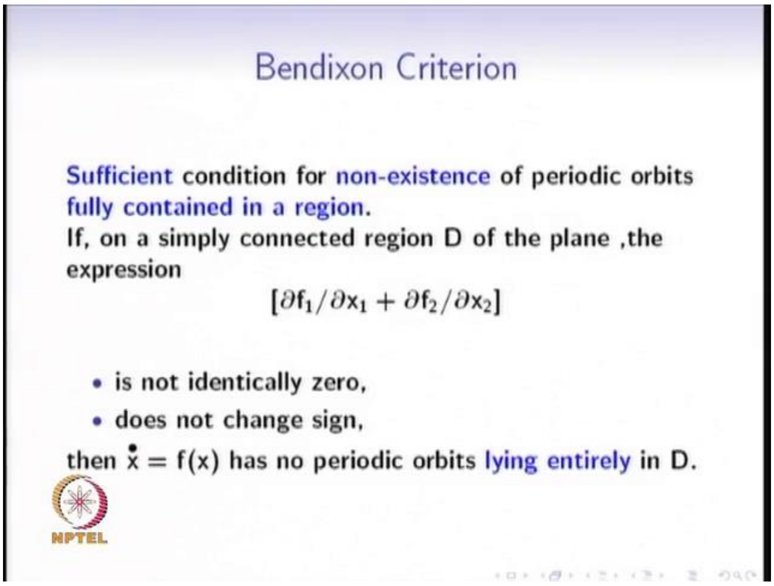


Since M is compact these trajectories are all bounded they cannot become unbounded because they do not even leave M and M is bounded. Further these bounded trajectories will have to approach periodic orbits or they can approach equilibrium points as t tends to infinity what happens to all these bounded trajectories they either approach equilibrium point or they approach periodic orbit. These are the only 2 possibilities why because the trajectories are all bounded.

They exist for all future times, now if we rule out existence of any equilibrium point in side M then we are forced to have a periodic orbit this is what Poincare Bendixon criteria say, secondly even if M had a periodic even. If M had an equilibrium point, but if that where unstable the trajectories could not be converging into them or trajectories could not be converging to the equilibrium point. So, we would have a periodic orbits even if M had an equilibrium point which was unstable in that case. So, these 3 conditions on M ensure that there is a periodic orbit. So, please note that this is only a sufficient condition for existence of a periodic orbit.

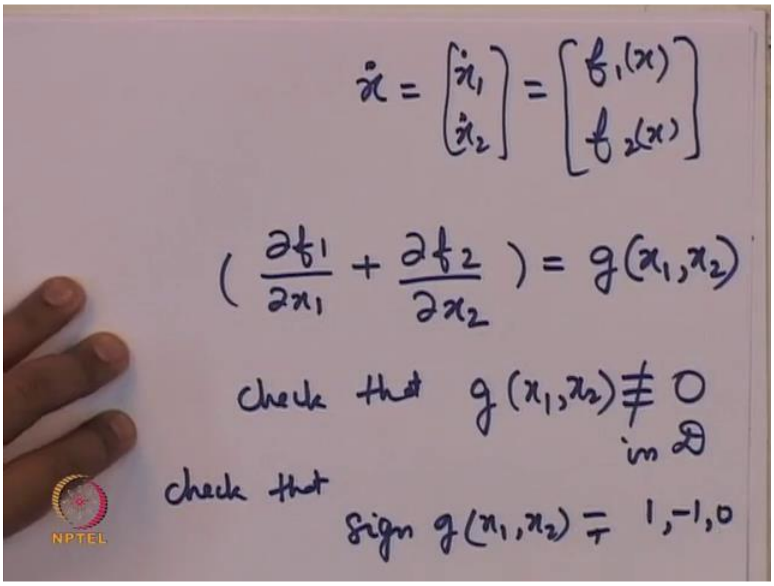
Of course there can also be a continuum there can be non-unique periodic orbits there can also be a continuum of periodic orbits, which we will see very soon another important criteria in this situation is the, so called Bendixon criteria. So, what does this criteria say it is a sufficient condition please note which conditions are necessary which conditions are sufficient the Bendixon criteria is a sufficient condition for non-existence of periodic order.

(Refer Slide Time: 40:43)



It is a sufficient condition for non-existence of periodic orbits that are fully contained inside the region. So, what is the criterion if one is simply connected region D of the plane. So, we will quickly see what simply connected region is. So, on a simply connected region of the plane if this particular expression here satisfies the condition, that it is not identically 0 and it does not change it is time, if this conditions are satisfied then the system of equation x dot equal to f of x has no periodic orbits lying entirely in D. So, inside the region D we check that this quantity is not always equal to 0 and it also does not change sign inside D if those 2 properties are satisfied by this particular function. Then they cannot be any periodic orbits lying completely in d.

(Refer Slide Time: 41:35)

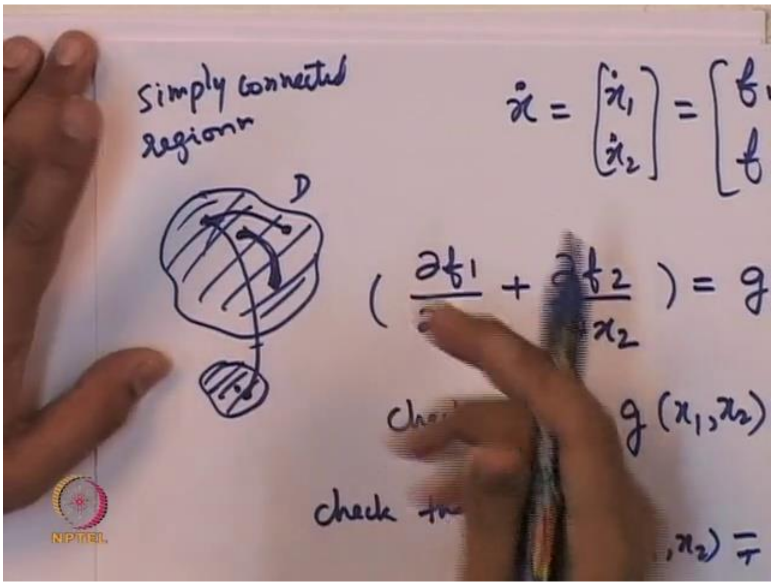


So, please note that it is only non-existence fully contained inside D of periodic orbits that is being guaranteed by the criteria. So, x dot is equal to x 1 dot x 2 dot which is equal to f 1 of x f 2 of x. So, this is our dynamical system, as I said we are considering evolution of trajectories in a plane. So, x has only 2 components x 1 and x 2 and hence its differential equation has only 2 equations inside this. Now, we differentiate the first 1 we differentiate f 1 with respect to x 1 and to that we add the partial derivative of f 2 with respect to x 2. Note that f 1 depends on x, which is x 1 and x 2. So, f 1 can depend on x 1 also and x 2 also and similarly, x 2 can depend on x 1 and x 2.

Hence we have partial derivatives of, f 1 and f 2 here partial derivative of f 1 with respect to x 1 partial derivative of f 2 with respect to x 2. This particular quantity is some function you know suppose that function is called g it depends on both x 1 and x 2. So, what does the Bendixon criteria say that this particular quantity, you check that g of x 1 comma x 2 is not identically 0 in d. So, D is a region inside this region this quantity is not identically 0. In other words there is at least 1, x 1 x 2 where this is not equal to 0 as soon as this is not equal to 0 at least at single. It means it is not identically 0 in D it is allowed to be 0 at a few points at several points.

However it should not be equal to 0 in at all the points in d, that is a statement that this is not identically 0 also check that the sign of g can be equal to be either 1 or minus 1 or 0. This is in general possible, but we require that the sign of this should not change in other words it should not go from minus 1 to 1 from minus 1 to 1 or 1 to minus 1. As long as it is always 1 or always minus 1 may be at some it becomes equal to 0, if the sign of this does not change. When you check for different x 1 and x 2 points in this region d, if these 2 conditions are satisfied then the Bendixon criteria says, then D cannot fully contain a periodic orbit there is no periodic orbit that lies entirely in D.

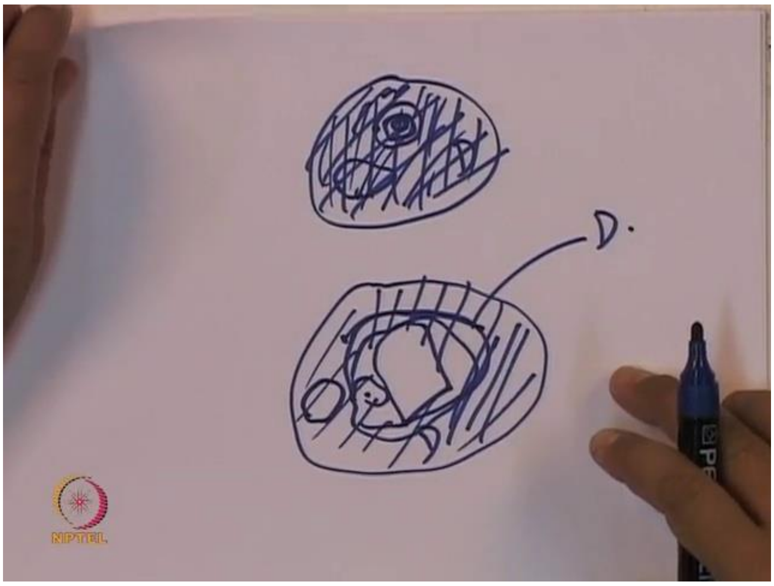
(Refer Slide Time: 44:27)



So, another assumption that we had made was, the D is simply connected region what is a simply connected region. So, in the plane x 1 x 2 suppose this is region D we will call this region D simply connected of course, means that D should not be made up of 2 such parts this is in D and this is in d. So, this is not connected right because to say that it is connected we take any 2 points.

There should be a curve there should be a path between these 2 points, and the path the points on the path also should lie in d. This should be possible for every 2 points in d, if D had 2 components while certain points are connected by a path lying in D every 2 points are not connected and look at a. There and a here a path from there to here is forced to go outside d. So, this such a set would not be connected even. So, for a region that is connected what do we mean by simply connected. Now, we take this region D.

(Refer Slide Time: 45:43)

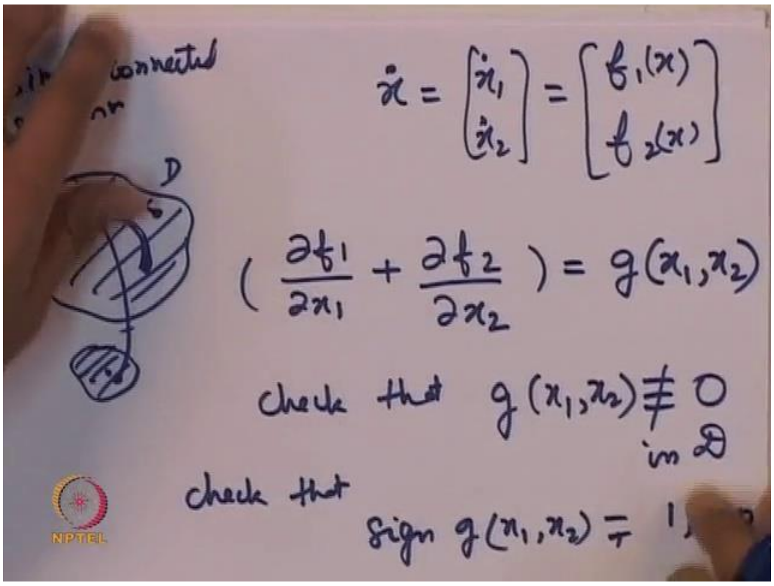


So, we take a closed curve inside this and this is just a close curve and this close curve can be shrunk to a. We can take a smaller curve slightly smaller curve. So, and this shrinking eventually lead to a. In the process of shrinking to a, inside at no situation does the curve have to leave the set d. In other words every closed curve can be shrunk to a. While being inside d, if that is the situation that is the property that D has, then we will say D is simply connected. So, all these regions that we usually think off are indeed simply connected. So, an example of set D that is connected, but not simply connected is a set which has holes.

So, take this and we rule out this particular case. So, what is our D our D is this shaded region and this shaded region without this particular place without this hole. So, this D with a hole this is our D this is an example for a region that is connected why it is connected because we can take any 2 points on d. There is a path that connects these 2 and all points in the path are also d, but what about a closed curve notice that this closed curve cannot be shrunk cannot be made smaller than smaller such, that all the whole curve is in in D why because it cannot be shrunk to a.

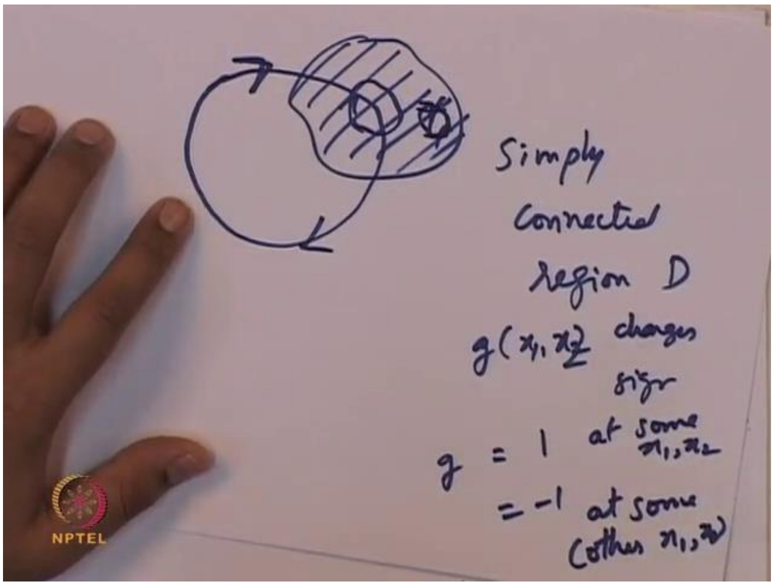
So, in the process of shrinking this curve to this to any, it turns out that this hole because it is inside the curve, but it is not inside D inside the region D we are n to able to shrink this closed curve to a. We might be able to shrink other closed curves to a point, but for it for the region to be simply connected every closed curve we should be able to shrink to a point. So, there are curves here which we cannot shrink to a. Hence this D is not simply connected, but this D is simply connected so. The Bendixon criteria requires that the region D for which we are checking is simply is a simply connected region. So, on the simply connected region we check whether the two functions, whether this function g whether the function g here which is obtained from f 1 f 2 by doing this partial derivative operation.

(Refer Slide Time: 48:22)



This g should not be identical to 0 on this region. It should also not change signs from 1 to minus 1 it is allowed to be 0 at a few points in which case the sign is equal to 0 that is not of concern, but it should not become from 1 to minus 1 or minus 1 to 1. If D satisfies these 2 properties at all points in D g satisfies this properties, then the Bendixon criteria says that there is no periodic orbit lying entirely in D what the Bendixon criteria does not say is that.

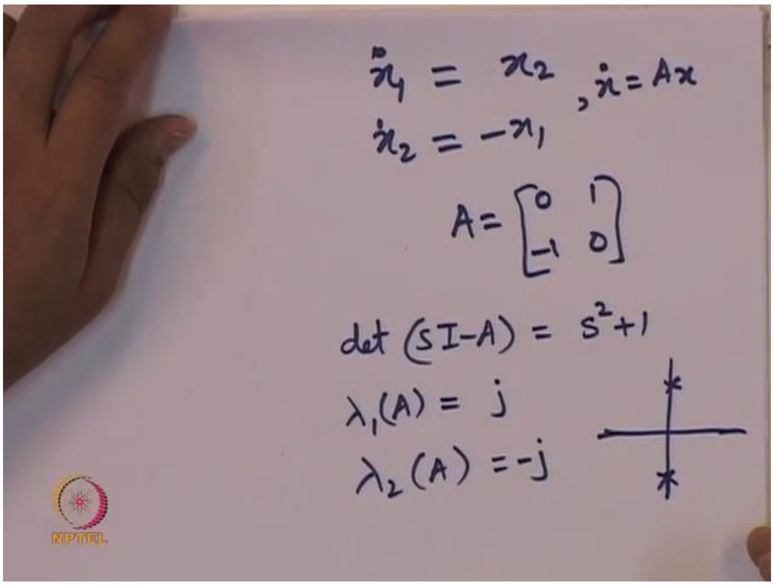
(Refer Slide Time: 49:04)



Suppose this is a region g, it is simply connected simply connected region d. Suppose g of the previous slide changes sign means what when we take different points x 1 and x 2, then it is equal to 1 at certain x 1 x 2 points. It is equal to minus 1 at some x 1, x 2 it is equal to minus 1 at certain other points at some at some other points. If g is changing it is sign from 1 and minus then the Bendixon criteria only says is what it says is that, there is no such periodic orbit.

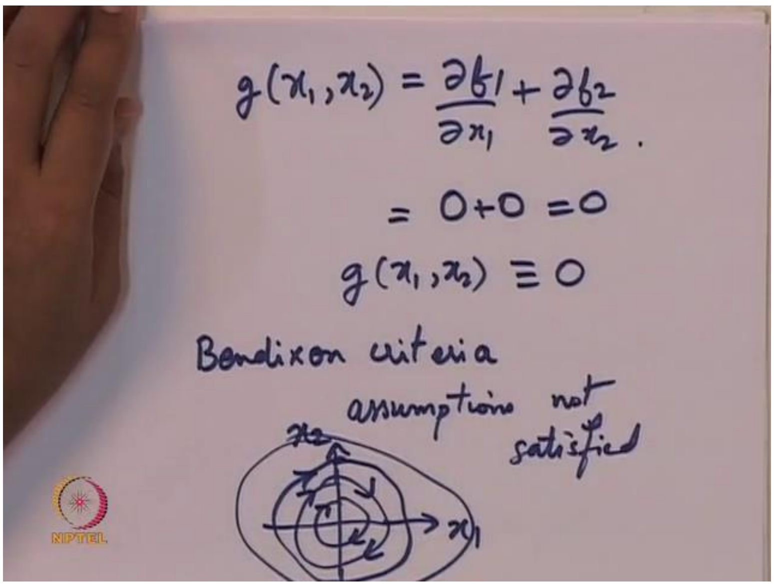
There cannot be such a periodic orbit, but there could be a periodic orbit that is not lying entirely in d, such a periodic orbit could be there which partly is inside D and partly outside d. Such a periodic orbit could still exist the Bendixon criteria does not rule out such periodic orbits existence. It only rules out any periodic orbit that lies entirely in D this is ruled out. So, please note that there is a certain difference in lying entirely in D and passing through D and the criteria only says that even g does not change, it sign while being checked in D, then there is no periodic orbit that is contained in d.

(Refer Slide Time: 50:46)



So, let us take an example of a linear system. So, the corresponding matrix a, this can be also written as x dot is equal to A x where A is equal to 0 1 minus 1 0. So, let us check the Eigen values for this matrix a. So, determinant of s i minus a, is equal to s square plus 1. So, please check that the determinant of this the characteristic polynomial turns out to be this. So, the Eigen values of A plus and minus j 1, Eigen value of a is equal to plus j. The other one is minus j in other words there are 2 Eigen values both on the imaginary axis which suggests that there are periodic orbits. The equilibrium point 0, 0 is a center for this particular A. So, let us see what happens to the Bendixon criteria.

(Refer Slide Time: 51:59)



So, what is our g which we have defined as del f 1 by del x 1 plus del f 2 by del x 2. So, for this particular case this is our f 1 and this is f 2. So, derivative of f 1 with respect to x 1 is equal to 0 derivative of f 2 x does not even appear in f 2 only x 1 appears. So, derivative of f 2 with respect to x 2 is also again 0, this is 0. So, no matter which region you take no matter which simply connected region you take g of x 1 x 2 is identically equal to 0. It is equal to 0 without even having to specify at which x 1 x 2 we are checking this. So, this is a situation where Bendixon criteria is not applicable you know.

So, Bendixon criteria assumptions not satisfied the assumptions are not satisfied. Does that mean that there are no periodic orbits lying entirely inside, that simply connected region D no, it does not mean that it only means that, because the assumptions of the Bendixon criteria are not satisfied we cannot go ahead and conclude anything because Bendixon criteria is not valid the statement is not valid. However we know in this particular case that it is identically 0 and there are periodic orbits indeed. In fact these are all the periodic orbits.

So, from any initial condition on this plane, there is a periodic orbit passing through that in other words for every simply connected region that contains the origin as long as this region is some region like this. There are plenty of periodic orbits, however Bendixon criteria does not tell us that why because Bendixon criteria assumes that, this g is not identically 0. That situation is not satisfied here for the case of linear system periodic orbits and hence we are able to use Bendixon criteria. Here we will see some more examples of where Bendixon criteria is applicable in the next lecture.

Thank you.