Nonlinear Dynamical Systems
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Lecture - 8
Extension of Lyapunov's Theorem in different contexts

Welcome everyone to the lecture number 8 of non-linear dynamical systems, so we will see some further extension of the Lyapunov's theorem in different contexts.

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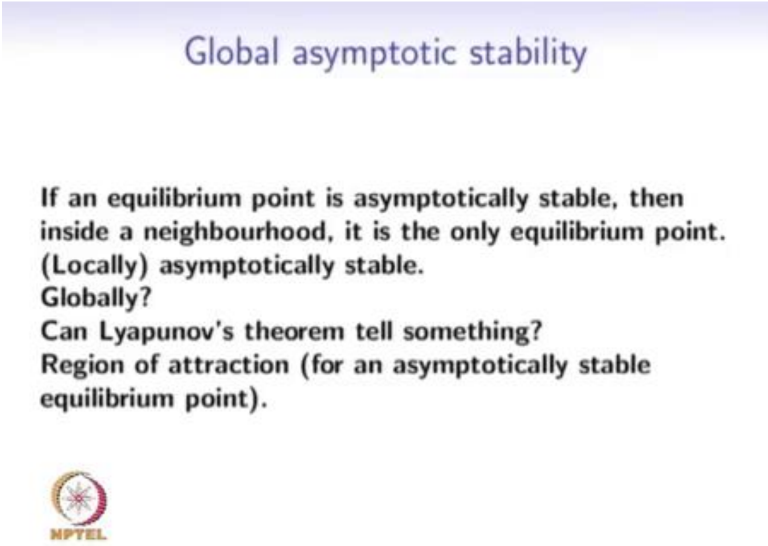

So, we will begin with global asymptotic stability, so what is global about the asymptotic stability of an equilibrium point. If an equilibrium point is asymptotically stable then we know that inside a small neighborhood, inside some neighborhood. It is in fact the only equilibrium point we cannot make such a statement for stable equilibrium points. But, for asymptotically stable equilibrium points we know that there is a neighborhood such that all trajectories beginning within that neighborhood all of these converge to that equilibrium point.

Hence, that equilibrium point is the only equilibrium point inside a neighborhood, hence this is also locally asymptotically stable. But, we are curious if this equilibrium point is a globally asymptotically stable equilibrium point, in other words no matter where the initial condition is all these trajectories starting from arbitrary. So, initial conditions of all them do they converge to the same equilibrium point asymptotically, this would make it a globally asymptotically stable equilibrium point. So, can Lyapunov's theorem tell us something about this, so for this purpose we need to see what is the region of attraction?

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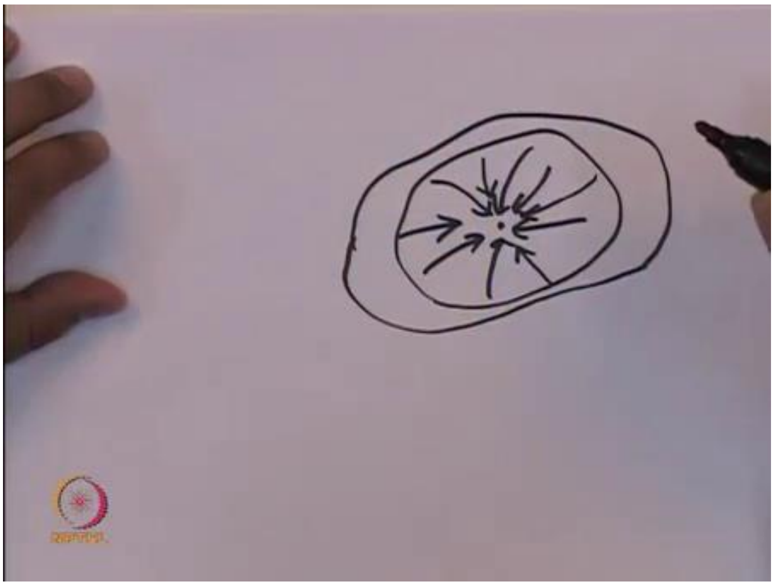

So, suppose this is an equilibrium point and trajectories are coming to this inside some neighborhood we would like to make this neighborhood larger and larger in the context of equilibrium. Now, in the context of asymptotically stable equilibrium point, we can speak about a region of attraction, no matter where the initial condition starts inside this region this equilibrium point is attractive.

Now, it attracts all these initial condition, so that the trajectories converge to this equilibrium point, so we would like to look for a larger set. So, that is such that all initial conditions inside this set converge to the equilibrium point, so we can speak of the region of attraction which is the largest set of all initial conditions such that starting from this set.

Now, starting anywhere from this set, trajectories converge to the equilibrium point, so the region of attraction being the whole space is what makes that equilibrium point globally asymptotically stable equilibrium point. Now, of course this rules out any other equilibrium point, so of course when there are some more equilibrium points then the equilibrium point cannot be a globally asymptotically stable. So, equilibrium points that time the region of attraction is something genuinely difficult to calculate. But, Lyapunov's theorem can give us some estimates, so let us go back to that particular proof of the Lyapunov's theorem.

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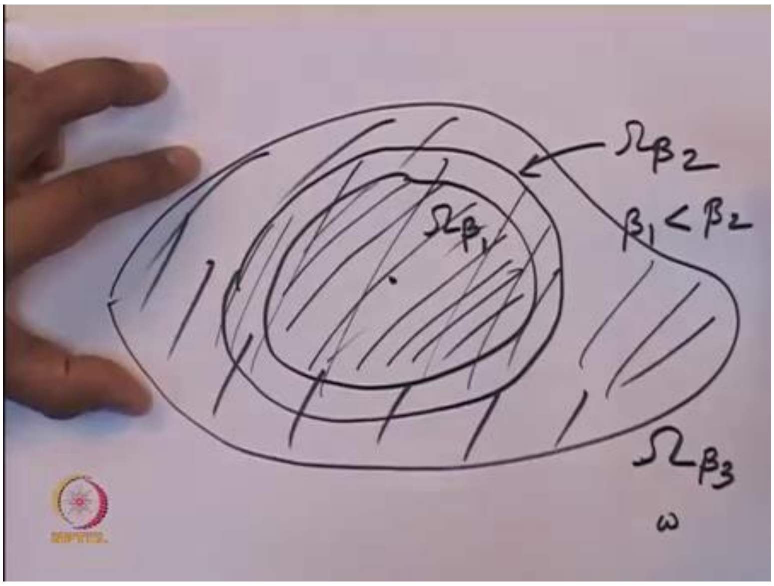

This was our equilibrium point and we had taken this set omega beta we had taken a value of beta and we had taken all those particular points whose values is less than or equal to beta where all those points were. So, the Lyapunov function has value at most beta, of course we also saw that if we take a larger beta value. Then this omega beta will be larger because it of course contains this lower value beta set. So, this is omega beta 2 and earlier one was omega beta 1 because beta 1 is less than beta 2, this omega beta 2 set will contain omega beta 1 set. Now, we also saw that this omega beta set is positively invariant and hence starting inside these trajectories will remain inside the set.

Moreover, if the Lyapunov function is strictly decreasing at all points except the equilibrium point then anywhere it starts inside this omega beta set it converges. So, we could consider taking larger and larger values of beta and consider the corresponding omega beta set this particular Greek alphabet is omega. So, it is capital omega, unlike the smaller omega we have seen the capital omega is what we also use for ohms, see if we take a beta 3 value that is very large. Then the corresponding omega beta three set could look something like this which is itself another estimate of the region of attraction.

So, can we take this beta set larger and larger and construct this omega beta set that is the next question, so this cannot be done for all Lyapunov functions unfortunately. But, whenever we can find the beta value such that the omega beta set is the bounded set it is automatically closed by the definition. But, it is also bounded set this is not guaranteed for large values of beta if it is a bounded set, then that particular set is guaranteed to be a region of attraction. So, let us see what can happen for large values of beta, why would this not be a bounded set for large values also.

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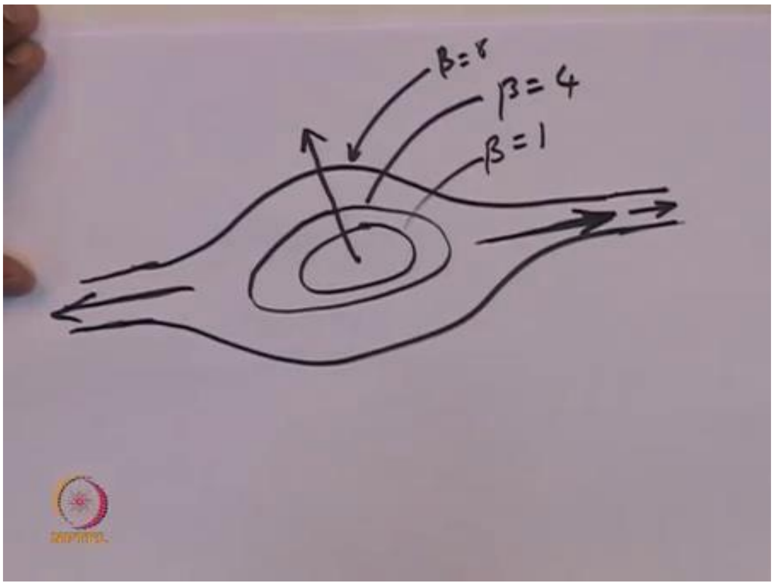

So, this particular, this is the equilibrium point, its interior and this particular contour is itself what we call omega beta set consider only the contour. Now, a larger value of beta will have a contour that encircles this that encloses this, but for this beta value a little larger it is possible that this contour does not close it could go and become very large. So, we will see some problems about how this beta for this contour set corresponding to larger values may not close, so this is some possibility that can happen because of which our region of attraction estimate cannot be obtained from the omega beta set.

So, this was suppose for beta equal to 1, this one was for beta equal to 4, while for beta equal to 8 for beta equal to 8. So, suppose it turns out that this particular contour does not close in this direction nor in this direction, so if it is not a closed contour then that omega beta would not be a compact set and we will not be able to use our result that this compact set is positively infinite. Hence, the solution exists for all future times, due to these difficulties we are not able to get a good bound for the region of good estimate for the region of attraction.

So, what do we do in this case can we can we have a Lyapunov function that rules out such contour sets, that is the next question we will ask. So, what is the problem before we formulate the property for such Lyapunov functions, so notice that when we take when we go along this direction. So, as we go further and further from this equilibrium point the value of beta is not increasing, for example if this contour does not close for beta equal to 8 then no matter how far you go along this direction beta equal to eight value is never reached.

Similarly, in this direction beta equal to 8 value is not reached when we go arbitrarily far along this direction here, however when we go along this direction along this particular radial direction the beta value does increase. So, it could become arbitrarily large it appears than on this director, but not in this direction this is what motivates us to define a property called radially unbounded, so as I said level sets of the Lyapunov function.

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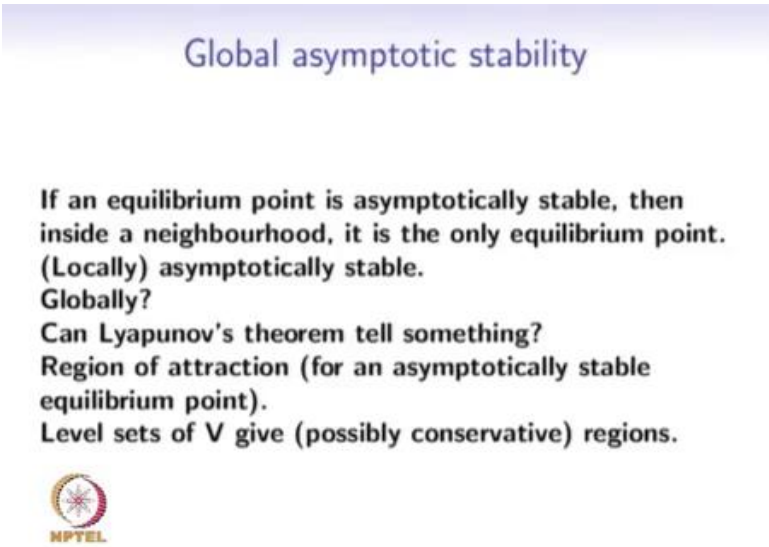

Give us some estimates of the region of attraction, this could be possibly conservative, so in the sense that the region of attraction might be larger, but this level set gives us only a conservative estimate gives us a smaller estimate this is possible.

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Some property of V ensures global asymptotic stability?
Is V 'radially unbounded'?
V is called radially unbounded if along any radial direction, V becomes unbounded. i.e.

$$\|x\| \to \infty \quad \Rightarrow \quad V(x) \to \infty$$

Not-closed contours for any level set is ruled out.

So, the next question is can we ensure that the Lyapunov function has some property such that these closed sets these contour sets will be automatically closed will be bounded. So, this is what we will call radially unbounded, so we will call a Lyapunov function V radially unbounded if along any radial direction V becomes unbounded. In other words, whenever the norm of x becomes very large which means that we are going very far from the origin.

So, if we are going far from the origin then we are going very far from any other finite point also, whenever we are going very far then it implies that the Lyapunov function value also becomes arbitrarily large. So, this is what with this property that the norm becoming large implies that the Lyapunov function also becomes large is called radially unbounded. So, it is unbounded along every radial direction this is precisely the problem we had seen in the previous contour set because of which contours were not closing.

However, if we ensure that this Lyapunov function is radially unbounded then it is ruled out that some contour will not close will not be a closed curve. So, not closed contours for any level set are all ruled out by ensuring that this Lyapunov function is radially unbounded.

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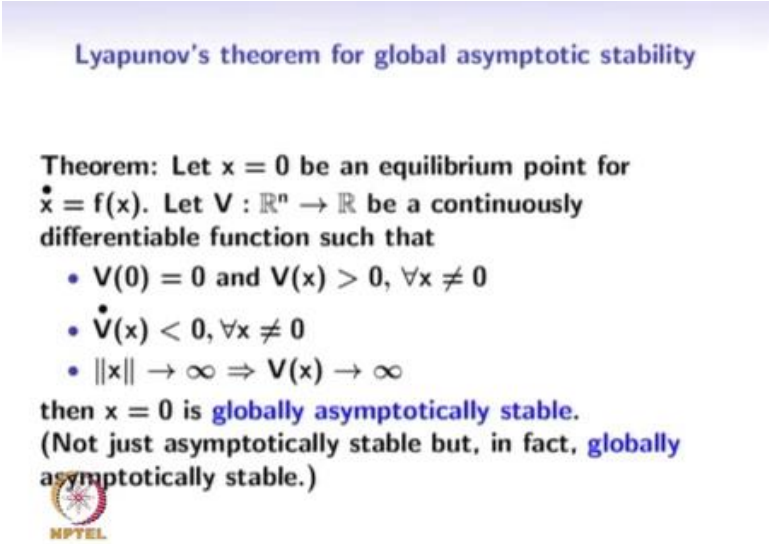

So, what is Lyapunov theorem on global asymptotic stability, this is also again sufficient condition, let x equal to 0 be an equilibrium point. So, for this differential equation x dot is equal to f of x and let V be a function R n to R which is continuously differentiable. Now, continuously differentiable functions are also called c 1, why because they are differentiable ones and their derivative is continuous. Now, suppose there exists a function V that is continuously differentiable such that V of 0 is equal to 0 and V of x is greater than 0 for all non zero x.

So, of course this is like we saw before V dot of x is strictly less than 0 for all non equilibrium, for all points other than 0 and this important third property that the Lyapunov function V is also radially unbounded. Then the point x equal to 0 is globally asymptotically stable, it is asymptotically stable was already implied by the first two of these three conditions by including the radially unbounded condition. It is also globally asymptotically stable, so it is not just asymptotically stable, but in fact globally asymptotically stable.

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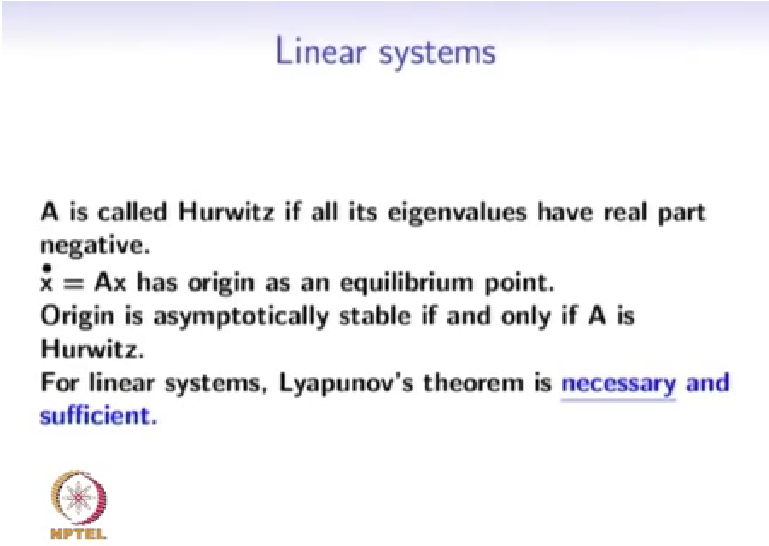

So, let us now come to the case of linear system why because this is one situation where the asymptotically stable equilibrium point is in fact globally asymptotically stable also this is a situation where the condition is Lyapunov theorem is not just sufficient. But, also necessary, so matrix A, a square matrix A is called Hurwitz if all its Eigen values have their real part negative. So, if all the real parts, if all the Eigen values are in the open left half complex plane then that matrix a is called Hurwitz.

So, x dot is equal to A of x A, x has it has the origin as an equilibrium point certainly this origin is an asymptotically stable equilibrium point if and only if the matrix A is Hurwitz. So, this is a standard result for linear systems for linear systems what happens to the Lyapunov Lyapunov's theorem we will very soon state a theorem that says that this Lyapunov's theorem is necessary and sufficient.

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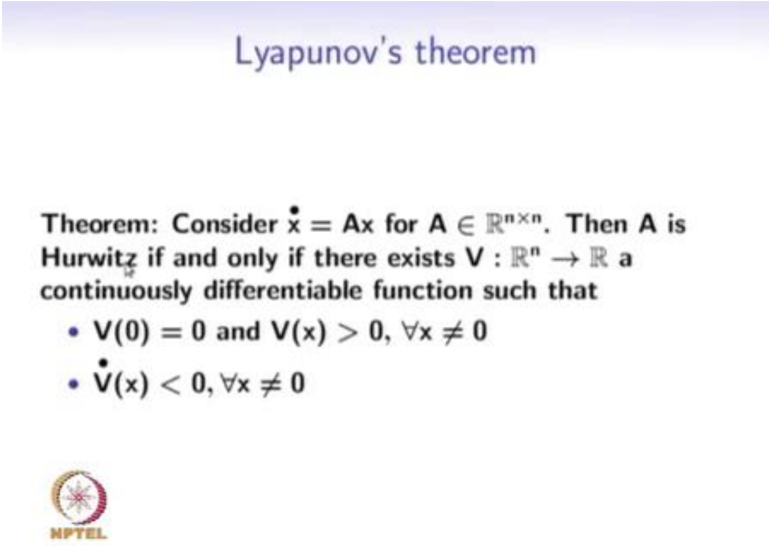

So, what is the result consider the system x dot is equal to A x where A is the n by n matrix then the matrix A is Hurwitz if and only if that exists a function V from R n to R which is continuously differentiable such that V, V is 0 at the equilibrium point. So, it is positive everywhere else and its rate of change if strictly decreasing its rate of change is strictly negative. In other words, Lyapunov function is strictly decreasing along every trajectory, so please note that we have if and only if here which means that the equilibrium point is asymptotically stable if and only if the Lyapunov function exists.

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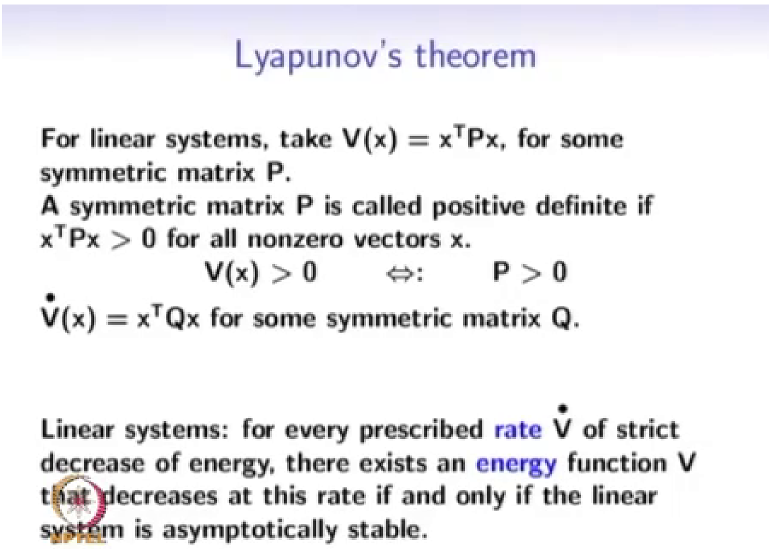

In fact, the Lyapunov’s theorem can be utilized in a more nicer way where the rate of change rate of decrease can be specified for Lyapunov for linear system take V of x equal to x transpose P x for some symmetric matrix P. So, a symmetric matrix P is called positive definite if x transpose P x is greater than 0 for all non zero vectors x. So, V of x is a positive definite function if and only if the matrix P is the positive definite matrix. So, what is the rate of change rate of change also term for linear systems where the Lyapunov function V of x is a quadratic function coming from a symmetric matrix.

So, it turns out that the rate of change is also again coming from such a symmetric from another symmetric matrix x transpose Q x. Now, this we will verify very quickly, so it turns out that for linear systems for every prescribed rate of decrease for every prescribed rate V dot of strict decrease of energy the impact. So, there exists an energy function V that decreases at this prescribed rate this is possible such a, such a Lyapunov function V exist that decreases at a prescribed rate if and only if the linear system is asymptotically stable.

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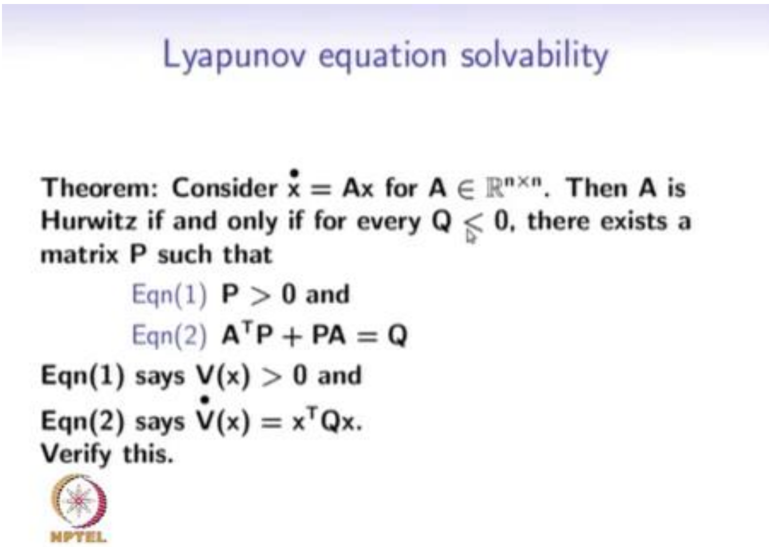

So, this is something about solvability of A, so called Lyapunov equation, so consider the system x dot is equal to f x for a square matrix n by n then A is Hurwitz if and only if for every symmetric matrix Q less than 0. So, what was less than 0 we will strictly review the definition of less than 0 less than 0 is just the negative of greater than 0. So, if Q, Q being negative definite negative definite just means that minus Q is positive definite, so A is Hurwitz if and only if for every Q that is negative definite that exists some matrix P such that P is symmetric and positive definite.

So, please note that we have used the word positive definite only for symmetric matrices that is, this is the main convention if and only if there exists a P such that P is greater than 0. So, A transpose P plus P, A is equal to Q, so equation one says that the Lyapunov function is positive which is, which is required for a Lyapunov function. Now, the second condition says that the rate of change of the Lyapunov function is equal to x transpose Q x, x which is guaranteed to be negative for all non zero x because Q was a negative definite matrix, so this is something we will quickly verify.

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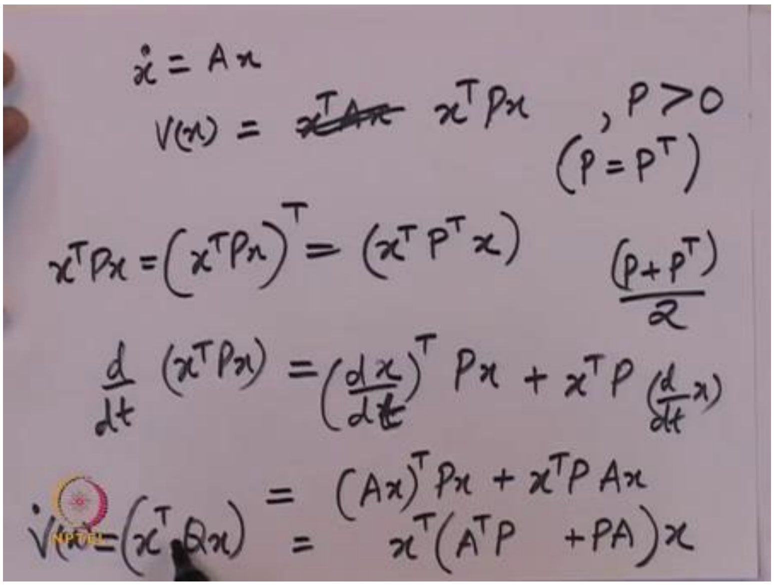

So, consider the system x dot is equal to A of x, x in which the Lyapunov function is defined as x transpose A x, sorry x it was defined as x transpose P x where P was positive definite. But, we have already assumed that P was symmetric which means P is equal to P transpose, so what is the meaning of P is equal to P transpose. So, in the context of this function because this is a scalar this is just a real number x transpose P x transpose of this is same as the number itself.

So, what happens when we take transpose of such a product we can take transpose of all the individual elements and also reverse the order. Now, first we will write this vector x and put a transpose then we will write this matrix here and put a transpose and, finally we will write this particular vector x here transpose again transpose. So, that leaves only x, so because P was symmetric even if P were not symmetric it appears like as far as the effect on x transpose P x is concerned whether you take P or P transpose it is the same.

Hence, we could as well assume without loss of generality that P was symmetric otherwise we could have considered it is so called symmetric part. So, this is our Lyapunov function we have already assumed P to be symmetric, in this case let us see what happens when you differentiate the Lyapunov function with respect to time.

Here, we get by the rule when we have a function depending on x on a product of such functions we will differentiate this. So, this is nothing but d by d t of x whole transpose times P of x plus x transpose P times d by d t of x. So, in this particular term we have differentiated this x in this particular term we have differentiated this x whether you take the derivative first and then take the transpose or you take the transpose and then differentiate it is the same.

So, this d by d t of x is nothing but a times x this is our dynamical system, so this is nothing but A x whole transpose P x plus x transpose P of d by d t of x again is nothing but A x. So, we see that this can be simplified as x transpose something in the middle and x we can pull out this x, this particular term evaluates to x transpose a transpose P x. Now, this is nothing but the x here has gone here and this x transpose has come here and as I said this is nothing but x a transpose P so the rate of change of this Lyapunov function.

But, we have prescribed as equal to x transpose Q x, this is what we want what is that Q it is precisely a transpose P plus P A if P was symmetric. So, you can check that this particular matrix inside the bracket here is also symmetric we can take transpose of this whole thing and see that you get back the same two terms. So, it is guaranteed to be symmetric and it is precisely equal to V dot of x, that is equal to x transpose Q x, so now in other words.

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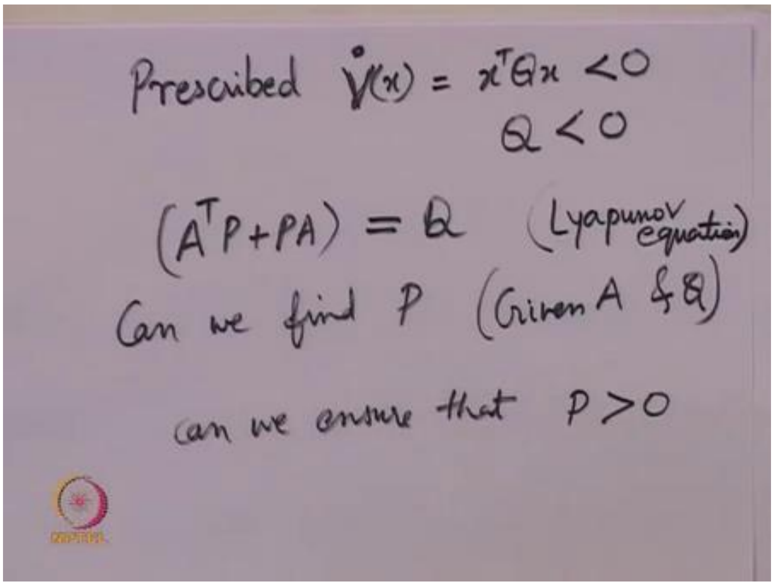

Prescribed V dot of x equal to x transpose Q x, our rate of change of the Lyapunov function is negative for all non zero x which is nothing but the definition of Q less than 0, Q negative definite matrix if this is prescribed. So, it means that a transpose P plus P, A is prescribed to be equal to Q of course the dynamical system A is already given. So, we can find P such that this equation is satisfied after we are given with A and Q to be given with, A is just to be given the dynamical system.

But, while to be given Q is to be given a prescribed rate of decrease of the Lyapunov function, so given A and Q in this matrix equation each of these three matrices a P and Q are n by n matrices in which P and Q are symmetric matrices. Now, if A and Q are given, can we find a P such this equation is satisfied that is the first question is this particular equation is called the Lyapunov equation. So, this is called motivated by Lyapunov's theorem, of course Lyapunov equation it is the matrix equation, so we are interested in solvability of this equation, given A and Q find the P.

Moreover, we are also interested can we ensure, can we ensure that this P that we have found also satisfies P positive definite that is what we want from Lyapunov functions. So, this should also be positive definite functions, so to this particular result it says that you can find such a P if and only if the matrix A was Hurwitz, so let us look at this particular equation slide again.

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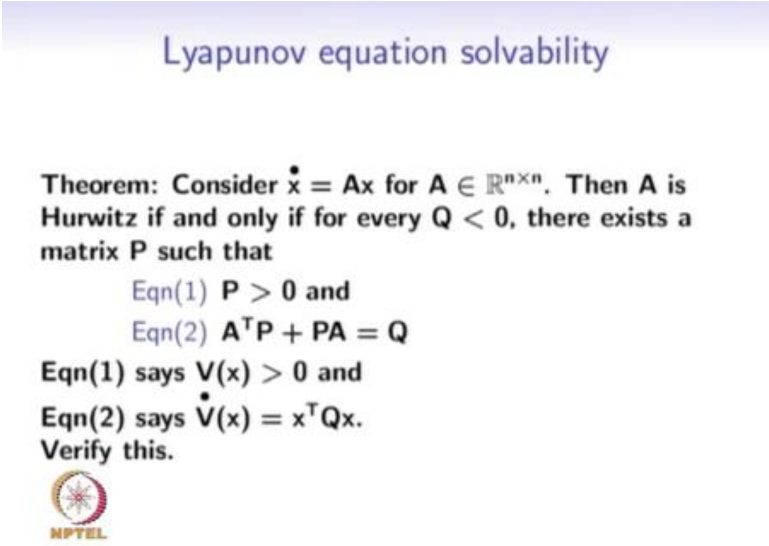

So, consider the system x dot equal to A of x for A an n by n matrix then A is Hurwitz which means that all its Eigen values are in the open left half plane if and only if for every prescribed V dot. In other words, for every Q negative definite there exists a matrix P such that P is positive definite and P satisfies the Lyapunov equation, in other words a transpose P plus P, A is equal to Q.

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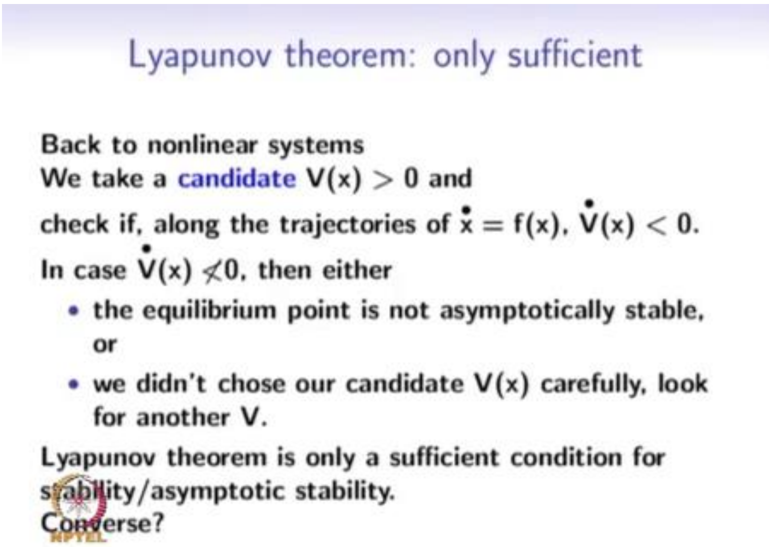

So, we are now back to non linear systems, we have seen what happens about Lyapunov theorem for linear systems there it was necessary and sufficient. So, we are to the situation of non linear systems again, so here it was only sufficient, so we take a candid view of x greater than 0. Now, we check if along the trajectories of x dot is equal to f of x, whether V dot of x is less than 0, in case V dot of x is not less than 0 then there are two possibilities. Now, the equilibrium point is not asymptotically stable, this is one possibility other possibility is that we did not chose candidate V of x carefully which means that we should look for another V.

So, we might have some physical reason some intuition why the differential equation equilibrium point is in fact asymptotically stable. But, our candidate Lyapunov function did not satisfy strictly less than 0, hence we should not go and conclude that it is not asymptotically stable because less than Lyapunov function not negative definite implies any one of these two, possibly both. So, whether it can be possibly both or not is a good exercise to think about, so because the Lyapunov's theorem is only a sufficient condition for stability or asymptotic stability, it can be any one of these two cases. So, an important question is what about the converse if we knew for sure that the system is asymptotically stable. Then can we say that there should exist a Lyapunov function for the case of linear system we have already seen.

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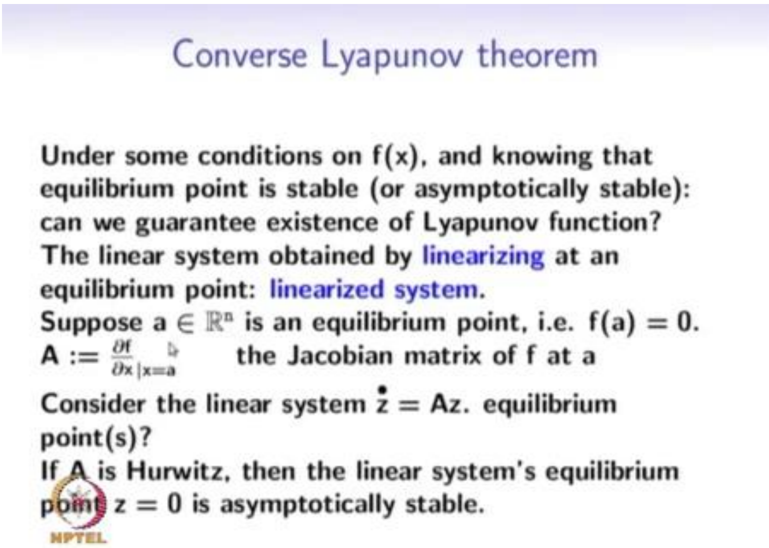

So, under some conditions of f of x and knowing that the equilibrium point is already stable by knowing by some other means whether it is possible that we know that the equilibrium point is stable or asymptotically stable. So, we guarantee the existence of a

Lyapunov function, so one way to do it is since we already know that we can find Lyapunov functions for linear systems. So, the linear system that is obtained by linearising the non linear system at an equilibrium point that linearised system also can be studied for studying the stability of the equilibrium point.

So, suppose a, suppose a is an equilibrium point which means that f evaluated at a is equal to the 0 vector define a of define the matrix a as the Jacobian matrix of f at the point a and the point x equal to a. So, we have we will very soon see a very big matrix definition of this del f by del x evaluated at x equal to a, so consider the linear system z dot is equal to A z. So, what does the equilibrium point for this z equal to 0 is certainly one equilibrium point, it is the only equilibrium point.

Now, we have seen if and only if a is a non singular matrix if A is Hurwitz then the linear system equilibrium point it has only one equilibrium point. So, if A is Hurwitz why because if A is Hurwitz all the Eigen values are in the open left half plane in particular no Eigen value can be at the origin. Hence, it is non singular then the linear system has only one equilibrium point z equal to 0 and, we have already seen that this equilibrium point is asymptotically stable if and only if A is Hurwitz.

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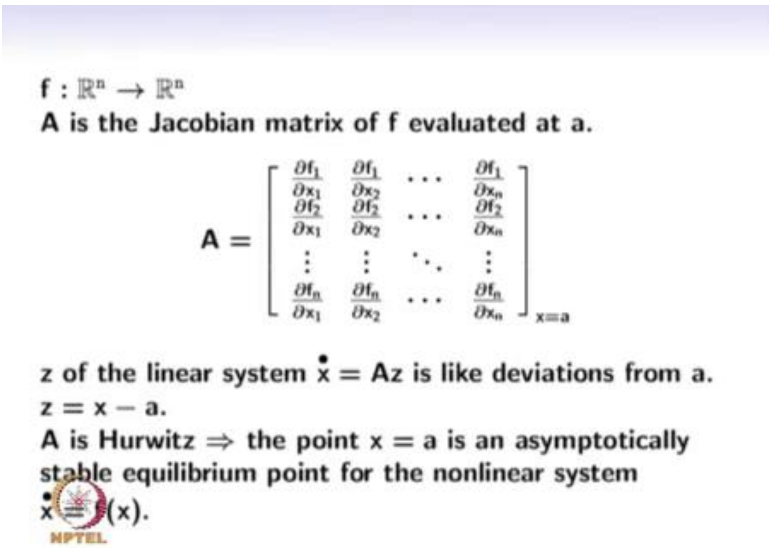

Then we can use the conclusion from a we can use the Lyapunov function from A for the non linear system also, let us see this in little more detail z was a map from R n to R n a is the so called Jacobian matrix of evaluated at a. So, what is the Jacobian matrix, the first row is to differentiate the first component of f first with respect to x 1 then x 2 etcetera.

So, second row of the matrix, of the matrix A is the derivative of f 2 with respect to x 1, x 2 etcetera, similarly we construct the last row of the matrix A as the derivative of f n because f was a map from R n to R n. But, f itself has n components we call them f 1, f 2, f n this Jacobian matrix itself is a function from x 1 up to x n, we evaluate that matrix at x equal to A then it becomes a constant matrix.

Now, for this constant matrix we consider its Eigen values and check whether they are Hurwitz, A is Hurwitz or not, so z of the linear system, I am sorry about this mistake here. So, we need here z dot is equal to a of z, z dot is equal to A z, so z of the linear system is like deviations from A. So, z equal to x minus A, so A is Hurwitz implies that the point x equal to A is an asymptotically stable equilibrium point for the non linear system x dot is equal to f of x.

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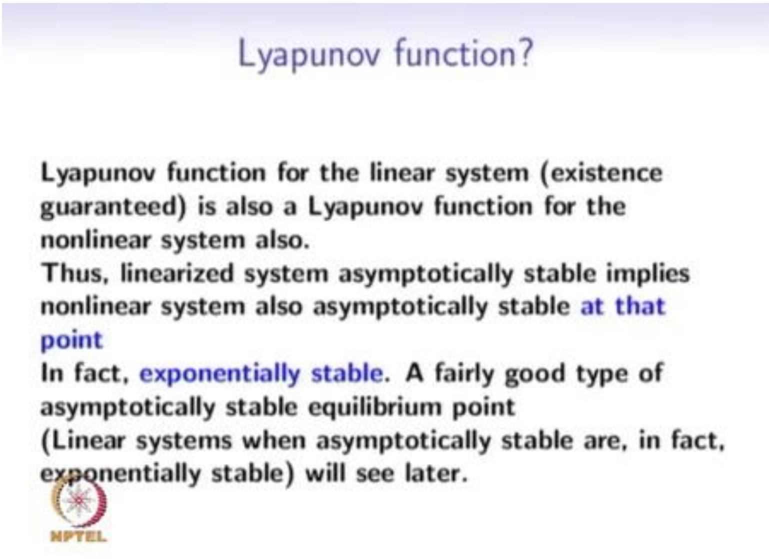

So, what about the Lyapunov function for the non linear system Lyapunov function, for the linear system already exists we know because for the linear system the Lyapunov theorem is necessary and sufficient. So, for the non linear system also we will consider using this Lyapunov function, so the Lyapunov function for the linear system whose existence was guaranteed is also the Lyapunov function for the non linear system.

So, this is great part about non linear system when we linearise, if the linearised system is asymptotically stable then the non linear system is also asymptotically stable. Now, we can in fact use the Lyapunov function for the linear function as the Lyapunov function, for the non linear system also does the linear system asymptotically.

So, stable imply that the non linear system is also stable at that equilibrium point another equilibrium point we might consider again linearising whose Eigen values may or may not be in the left half plane. So, as far as this equilibrium point is concerned, which when we linearise we get a matrix A that is Hurwitz that equilibrium point is what we will call is so called exponentially stable. This is some good type of asymptotically stable equilibrium point which is automatically satisfied for linear system. So, whenever the linear system is asymptotically stable then that equilibrium point is also exponentially stable we will see this in little more detail later.

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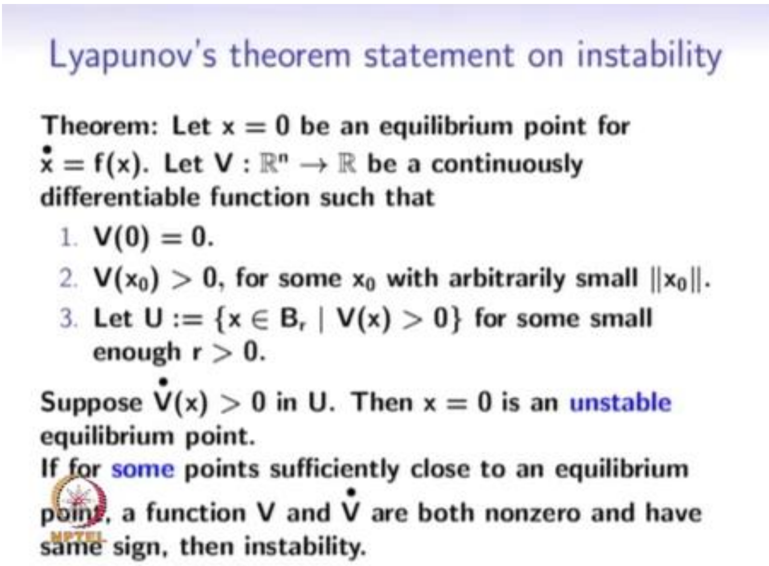

So, as I said this Lyapunov theorem is only a sufficient condition if one particular candidate fails to be less than or equal to 0 or strictly less than 0 then we might consider another Lyapunov candidate. But, after sometime when various candidates do not work we could consider trying to prove instability using one or more of these candidates.

So, here we have a sufficient theorem on for instability, so what should that function V satisfy. So, that we conclude instability after we have failed to conclude stability for some time we could consider instability proving since none of our candidates worked.

So, this is again a sufficient condition, so let x equal to 0 be an equilibrium point for the equation x dot is equal to f of x and let V be a map from R n to R.

So, suppose it is continuously differentiable and V of 0 is equal to 0, suppose V x naught is greater than 0 for some point x naught that is having arbitrarily small length. So, no matter what length no matter how smaller positive length somebody specifies, suppose we can find A x naught point whose length is smaller than that particular specification. So, that V of x naught is greater than 0 at that point x naught, construct the set of all U such that. Now, construct a set U which is the set of all points inside a ball V, where that function V is positive for some small enough R for some small enough radius R greater than 0.

So, suppose it turns out that this function V of x is increasing inside this set then this x equal to 0 is an unstable equilibrium point. So, if we can show inside some set U that comes arbitrarily close to the equilibrium point 0, if this Lyapunov function and also the rate of change of the Lyapunov function are both positive. Then this particular equilibrium point which we initially aimed for proving stable is now unstable, so this is the sufficient condition to show that a equilibrium point is unstable. So, the fact that V is positive and V dot is positive is nothing very important about positive both could be negative also, because then we could have taken minus V instead of plus V.

So, if for some point sufficiently close to an equilibrium point the function V and V dot have the same sign and are both non zero. Then that particular equilibrium point is unstable this is the same theorem rewarded in plane words, now that completes Lyapunov theorem and its various extensions. But, there is one important extension called the Lassalle’s invariance principle that we will now see before we go to this particular principle, we will see an example.

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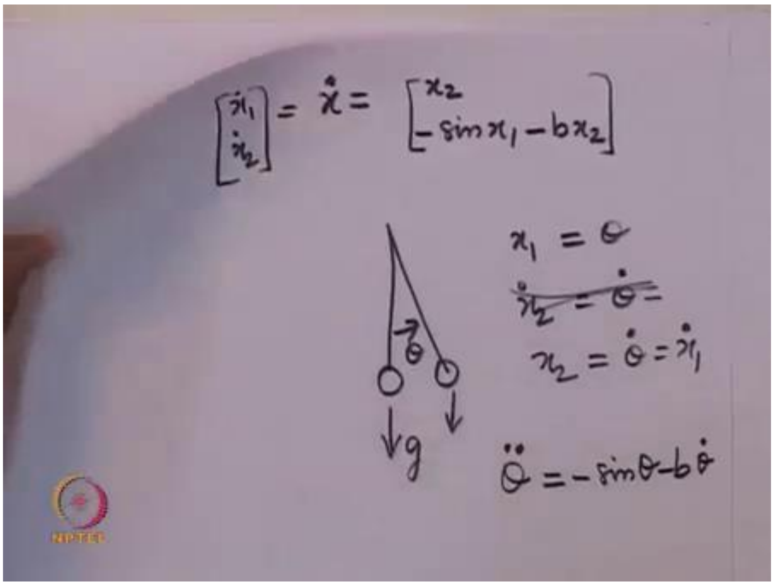

So, consider the system x dot is equal to this particular example, for example this example comes when studying the differential equation of a pendulum suppose there is this pendulum and there is this deflection theta and this is how gravitation acts here. So, then we have two states let us call it x 1 equal to theta x 2 equal to theta dot x 2 equal to theta dot is nothing but sorry is this x 2. Now, we want to call as theta dot which is nothing but x 1 dot, notice that this equation is precise to the first equation that is written here.

So, also this is the pendulum with some friction, in other words whenever it is moving at some velocity theta dot that time there is some rate of decrease of x 2 dot coming from some frictional force. So, this d times x 2 has a interpretation that there is some deceleration due to friction, in other words the second differential equation says that this is coming from this one second order differential equation. So, this particular differential equation is what we are going to study, now in the context of Lyapunov's theorem and also the next principle that we will see, so from physical principles we can consider.

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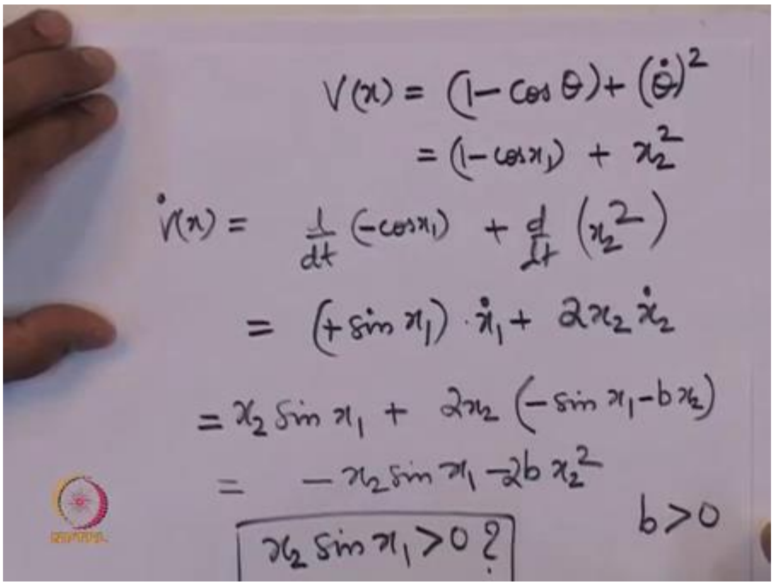

A Lyapunov function as, this is nothing but our potential energy plus the kinetic energy, so assume for now the kinetic energy is just taken as theta dot square whether this is good Lyapunov function or not we will decide soon. But, of course I have written theta and theta dot here we will write this back in terms of x 1 and x 2, 1 minus cos of x 1 plus x 2 square, so let us see what happens to V dot of x V dot of x is rate of change of this with respect to time. So, this is nothing but d by d t of minus cos of x 1 plus d by d t of x 2 square, so derivative of cos of something is minus sin of that.

So, this becomes plus sine x 1, but x 1 itself is a function of time and we were differentiating with respect to time, this time x 1 dot plus 2 times x 2 times x 2 dot. So, this is the rate derivative of this term, now we will use x 1 dot and x 2 dot from our dynamical system equations and put that here sine of x 1. Now, x 1 dot is nothing but x 2 plus 2 times x 2 and x 2 dot was equal to minus sin of x 1 minus b x 2, this was x 2 dot, so when we evaluate this it seems like sine x 1 x times x 2 is also here. But, with a factor 2 and it is not cancelling well, so we are not able to conclude, so this whole thing is equal to minus x 2 sine x 1 minus 2 b x 2 square.

Now, of course the frictional element b is such that b is greater than 0, so this term is helping us to show that this terms is less than equal to 0. So, fortunately this one is this guaranteed to be always less than 0 can we say that x 2 times sin x 1 is always positive, can we say that x 2 times sine of x 1 is always greater than 0. But, if we could say this then this would be always less than 0 and we would show that the pendulum is a stable has x one and x 2 equal to 0, 0 as a stable as a asymptotically stable equilibrium point in fact we are not able to see this.

So, of course that only means that this that could means that this Lyapunov function was not a good candidate. But, then we could also go back and see that look this was not really kinetic energy while this is potential energy, this is not kinetic energy we could consider dividing by 2.

So, let me make a change on the same slide with a different color pen, so here we have just divided by 2, we have this again divided by 2, this cancels off this 2 this cancels off this 2 also, because of which we perfectly have a cancellation here. Now, because of which this 2 is gone, but that is not a problem, now we have that this is equal to we will see the same example in a different context also, but at least here it appears that.

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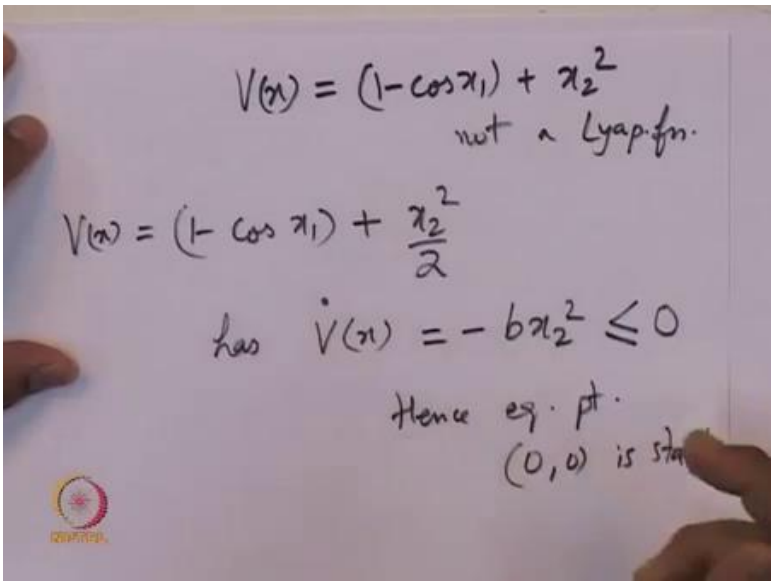

V of x equal to 1 minus cos of x 1 plus x 2 square, not a Lyapunov function, but V of x 1 minus cos of x 1 plus x 2 square by 2 has V dot of x equal to minus b x 2 square which is less than or equal to 0. Hence, equilibrium point 0, 0 is stable of course we should have checked at the equilibrium point whether 0, 0 is an equilibrium point. So, to begin with one can verify in the system of equations that 0, 0 is indeed an equilibrium point.

That is where we are trying to study whether this equilibrium point is stable unstable asymptotically stable. So, the pendulum with some friction has been shown to be a stable equilibrium point is this asymptotically stable, can we say that this is strictly less than 0 whenever x 1 and x 2 are both not equal to the 0, 0 point.

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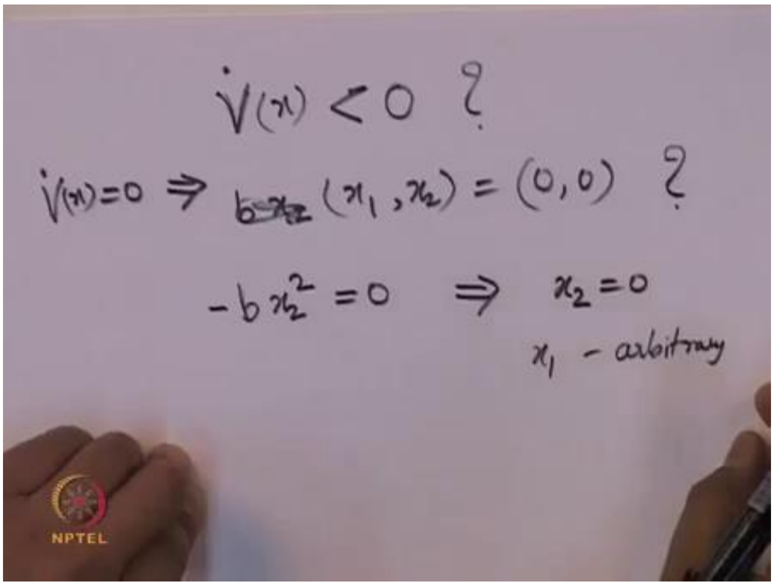

So, can we say strictly less than 0 for the second candidate, the first one was not a Lyapunov function for the second one. In other words, if we say we say if V dot of x equal to 0 does that imply that when we would, when we would say that this Lyapunov function is rate of change of the Lyapunov function. So, it is negative definite only if the only point where it was equal to 0 where the rate of change was equal to 0 happen at 0, 0 see if b x 2 square minus of that is equal to 0 is the question mark.

Here, sorry it only implies that x 2 is equal to 0 we cannot say anything about x 1, so x 1 can be arbitrary, in other words when the pendulum is not moving. So, when the pendulum stationary at all places Lyapunov function rate of change is equal to 0, the Lyapunov function is not strictly decreasing at those points. But, that does not imply that the x 1 component itself is also 0 1 is the x 2 component is 0, so there are various points where the rate of change of the Lyapunov function becomes equal to 0.

Now, it is not necessarily only the equilibrium point and hence this Lyapunov candidate does not help us to prove asymptotic stability. However, do we think that the equilibrium point is asymptotically stable we could consider linearising it at the equilibrium point and checking.

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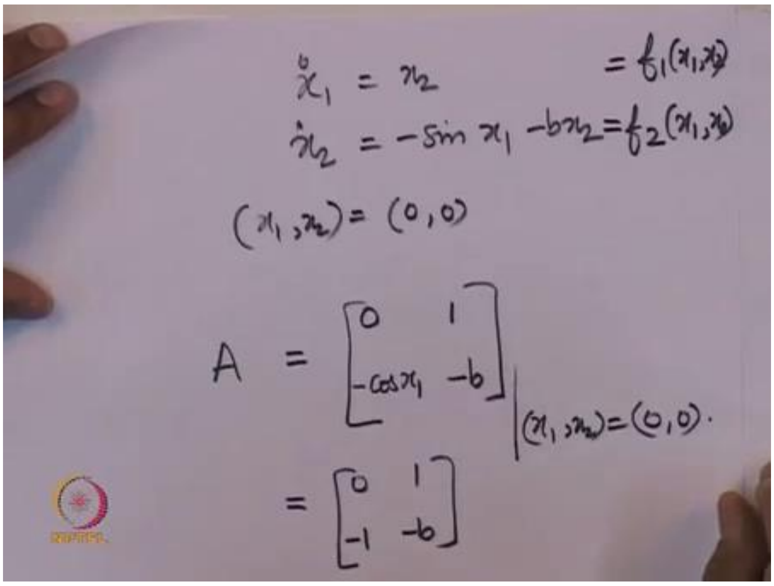

So, equilibrium point is x 1, x 2 equal to 0, 0, so we can differentiate this and consider a matrix A the first function is not a function of x 1. Hence, we have 0, here this is a function of x 2 rate of change of this partial derivative of this function with respect to x 2 is equal to 1, second function this is of f 2. So, this is equal to f 1 of x 1, x 2 and the second one is equal to f 2 of x 1, x 2 we are going to use our big matrix that we saw in the slide. Here, the component the value that comes here is derivative of f 2, partial derivative of f 2 with respect to x 1, in other words derivative of this with respect to x 1.

So, this is equal to minus cos of x 1 and what comes here is derivative of this f 2 with respect to x 2 which is equal to minus band, this is expected to be a function of x 1 and x 2. So, we have to be evaluating at x 1, x 2 equal to 0, 0, so when evaluated this point we get this equal to 0, 1 minus 1 minus b cos of 0 is equal to 0 cos of 0 is equal to 1, sorry. So, what about this matrix does, it have Eigen values in the open left half plane at least our Lyapunov function could not prove it is asymptotically stable and the linearised system, let us investigate the Eigen values of the linearised system.

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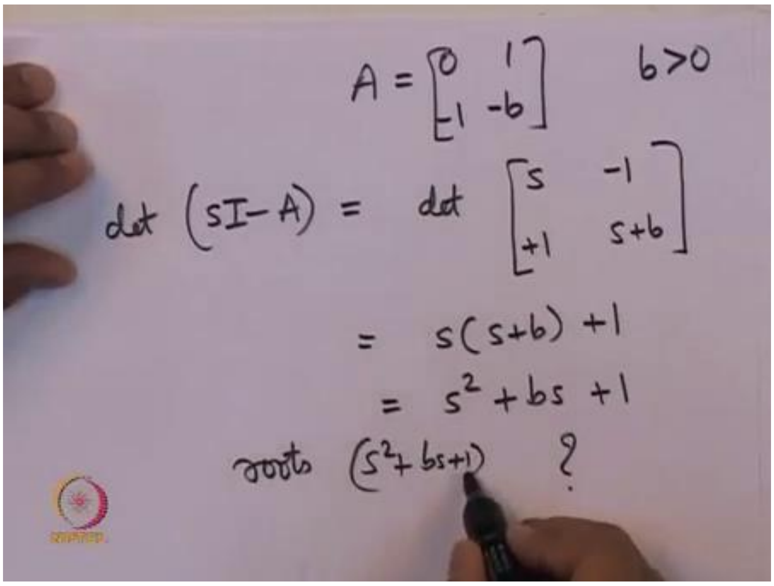

So, the Eigen values of A, in other words the Eigen values of this matrix assuming b itself the friction is caused by some b that is positive only then it is deceleration due to friction. So, Eigen values of a can be found by finding the determinant of s I minus A, in other words determinant of s minus 1 plus 1 f plus b.

Now, this is equal to s times s plus b plus 1 this is s square plus b s plus 1, so what can we say about the Eigen about the roots of this polynomial. Now, roots of this polynomial is precisely the Eigen values of this matrix A, we have constructed the characteristic polynomial and found the determinant is minus a matrix and its roots

So, roots of s square plus b s plus 1 of course we could use the quadratic equation and find the roots by using that b is positive it will turn out that the roots are in the left half complex plane. But, why the roots have product equal to plus 1 we could evaluate this, so that we do not need to discuss in detail why the roots are in the left half plane.

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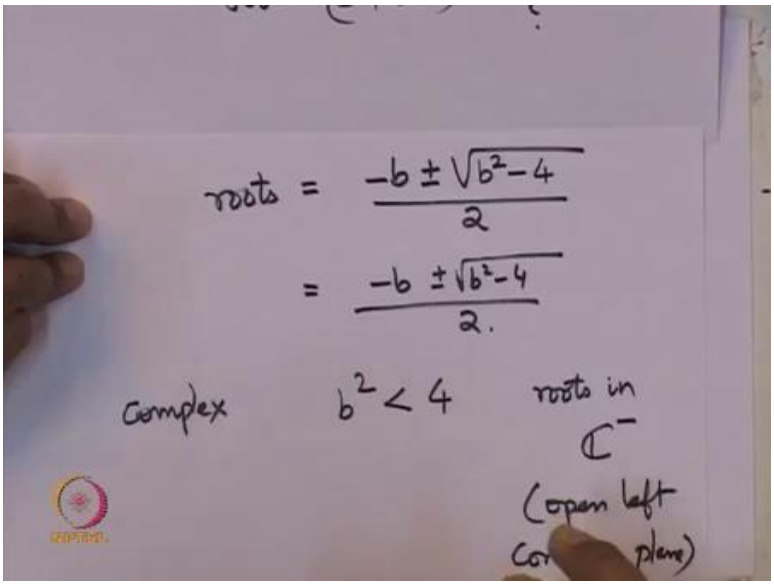

Now, roots are equal to minus b plus minus square root of b square minus 4 a c, so that is precisely b square minus a and c are both equal to 1 divided by 2. So, suppose the roots need not be real, roots could be either real or complex if the roots are complex which means b square is less than 4. Then because b itself is positive then roots in, we call this open left half complex plane if the roots are complex roots when we have complex roots if this discriminant is negative.

In other words, b square is less than 4 this, when this quantity within the square root bracket under the square root sign becomes negative in that case the imaginary part is positive for one negative for the other. But, the real part is minus b by 2 and you only assume that b is positive and hence clearly the roots are in the left half complex plane and they are complex. But, what about when the roots are real that means b square is greater than or equal to 4.

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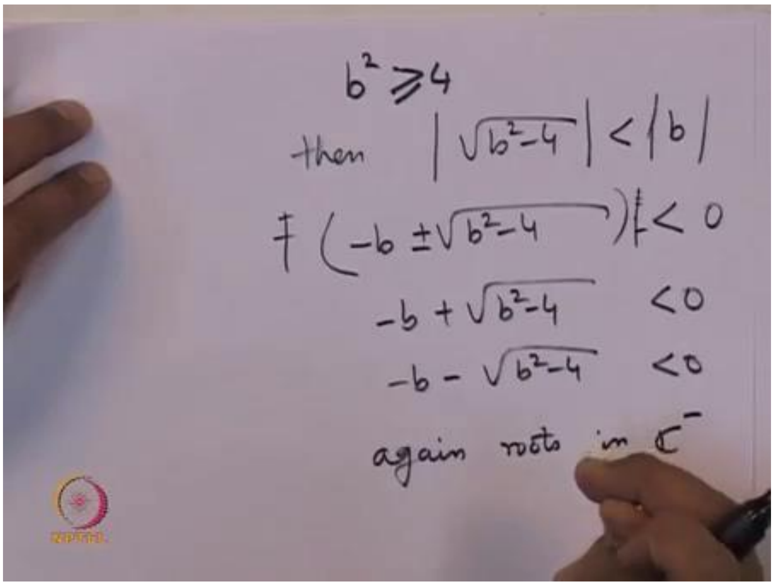

So, when b square is greater than or equal to 4 then when we say that square root of b square minus 4 in absolute value is less than, is less than is strictly less than b, strictly less than b. Now, again absolute value why because you see this b square is greater than 4 and this is some number from which subtracted 4 and then taking the square root, and it cannot be even equal to b it will be strictly less than b. In other words this minus b plus minus square root of b square minus 4, this quantity itself in absolute value will be less than less than 0 when b square minus 4 we take positive root.

Now, we add that to minus b, I am sorry without the absolute sign minus b plus b square minus 4 is also negative why because b square minus 4 quantity itself is less than b in absolute value. Now, of course when both are negative it will only be further negative, so whether as long as b is positive whether the roots are real or complex they are in the left half complex plane.

So, again roots in c minus, so we have investigated both the cases whether the roots are real or complex we know that the Eigen values of the matrix are in the open left half plane. So, open meaning it is not on the imaginary axis it is strictly in the left half complex plane, so what did we conclude from this linearised system.

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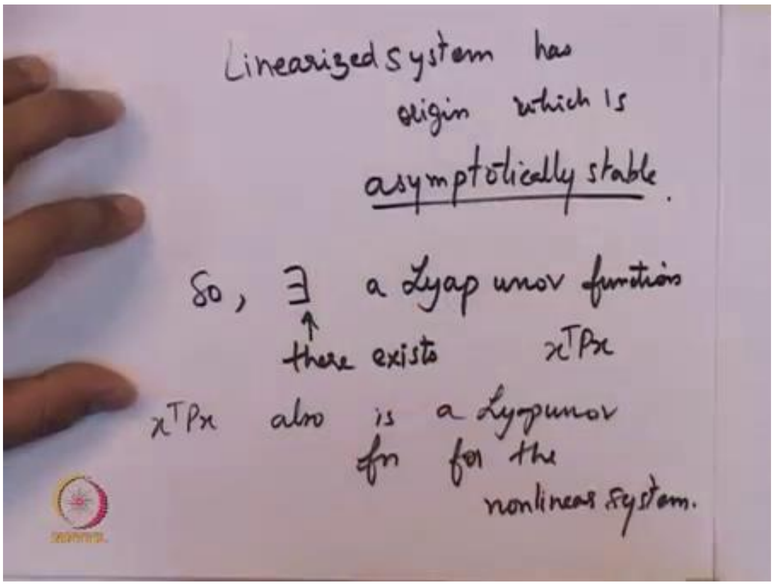

So, a linearised system for this differential equation has origin that is which is asymptotically stable, even though our Lyapunov function constructed from physical principles. Now, what physical principle we use the Lyapunov function as a notion of energy in which we added the potential energy and the kinetic energy. So, in the kinetic energy term, we ensured that the number 2 is important to divide by 2 divide, the velocity square by 2 was required.

But, in spite of constructing the energy function from physical principles we could not prove that the origin is an asymptotically stable equilibrium point we could only show that it is stable. But, the linearised system has Eigen values that are in the open left half complex plane because of which the origin for the linearised system is asymptotically stable. So, there should exist a Lyapunov function this particular symbol just says there exists as soon as the linear system is asymptotically stable that means if A was Hurwitz.

Hence, we already saw that there should exist a Lyapunov function because for linear system, the Lyapunov theorem is not just sufficient, but also necessary. So, if we know that the origin is an asymptotically stable equilibrium point then there is a guarantee that there is a Lyapunov function.

Now, this Lyapunov function we can in fact construct as x transpose P x U m, so x transpose P x also is a Lyapunov function for the non linear system. So, we know that there should exist some Lyapunov function that did not, could not be motivated from physical energy principles. Now, there should exist a Lyapunov function which can in fact prove asymptotic stability, so how do we actually find this is something that we can do as an exercise because we know that for linear systems we can prescribe the rate of decrease.

But, still be able to find the energy function that has precisely that decrease this is the energy function also guaranteed to be positive definite simply because the matrix A was Hurwitz. So, we will see an example of such a prescribed rate of decrease and solve the Lyapunov equation in an exercise. So, the next thing to do is we could consider using that same Lyapunov function which could prove only stability, but not asymptotic stability.

Now, to arrive at the conclusion of asymptotic stability by using A, so called Lassalle's invariance principle, so Lassalle's invariance principle helps us. Now, for example in this situation where the Lyapunov theorem could prove only stability, but not asymptotic stability it can prove at elsewhere places also we will see. So, for example there could be a situation where we want convergence though not an equilibrium point, but to a set.

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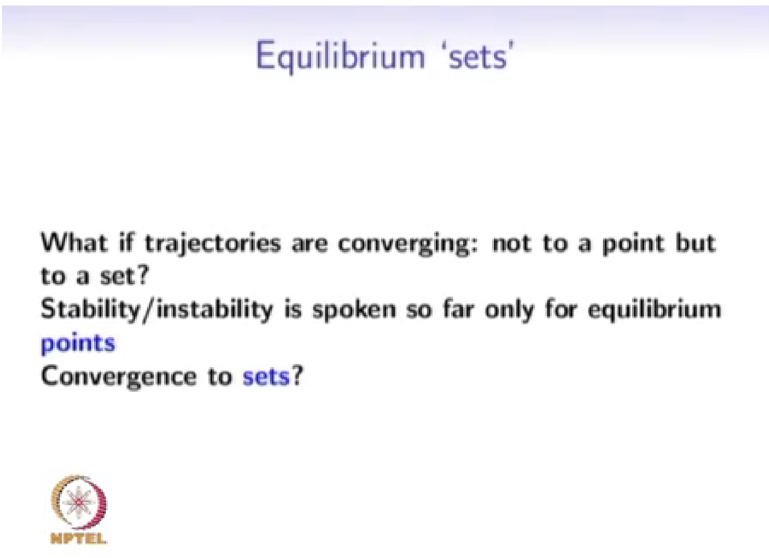

So, what are equilibrium sets what is the trajectories are converging not to point, but to a set we could speak about stability instability for sets also, so far we spoke only for equilibrium points, but we could also consider speaking for sets.

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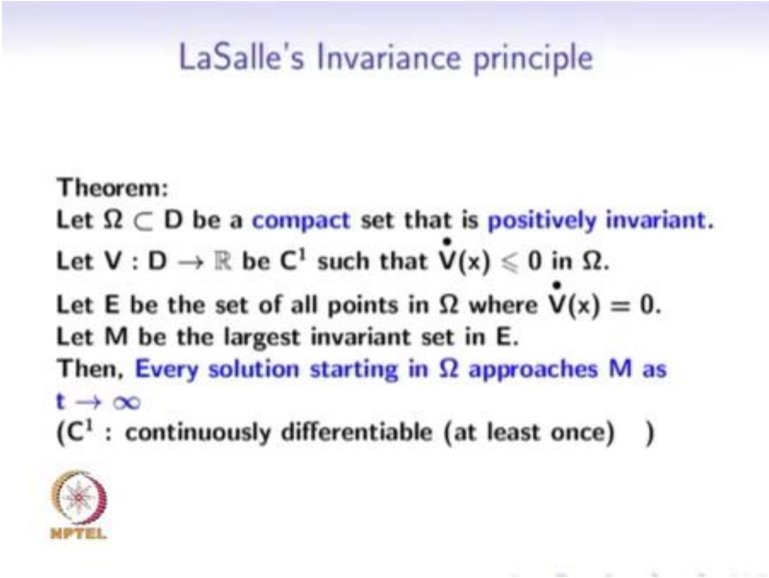

So, in this context there is a Lassalle's invariance principle, so let omega be a compact set that is positively invariant positively invariant means it is positively invariant with respect to the dynamics of the function f. So, we are considering the system x dot equal to f of x here and in that context we have constructed a set omega that is the compact set. So, suppose we could find a function V from this domain D to R and, suppose this function V is C 1, C 1 we only saw means it means continuously differentiable and this function V satisfies the property that its rate of decrease.

So, its rate of change is less than or equal to 0, so it is a strictly decreasing is not what we are assuming non increasing on omega. Now, considering the set E of all those points in this set omega where V dot of x is in fact equal to 0, in fact the set E, we will now look for an invariant set. So, we will now look for the largest invariant set in E, suppose we consider this set M which is the largest invariant set sitting inside E. Then Lassalle's invariant principle says that every solution that starts in omega approaches M as t tends to infinity.

Now, as time tends to infinity it approaches this set M, so C 1 as I said is set of is a function that is continuously differentiable. So, one at least stands for at least once, so what does it mean for a solution to approach a set M this is something we have not seen yet.

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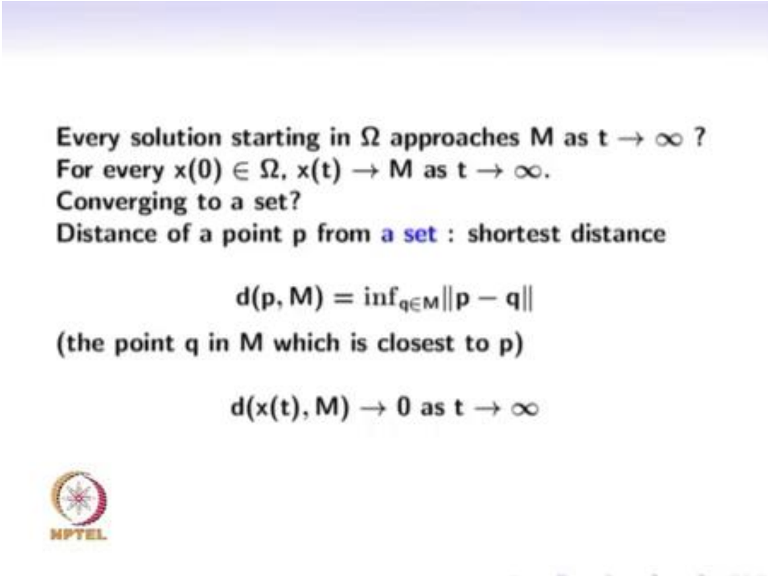

So, every solution starting in omega approaches set M as t tends to infinity, what is the meaning of this statement we will see this quickly before we end today's lecture. So, for every initial condition x 0 in omega x of t tends to M approaches M, as t tends to infinity is nothing but x of t tends to M as t tends to infinity. So, this x of t tends to M means converging to a set, so for that purpose we are yet to define what is the meaning of converging to a set M. So, distance of a point P we can speak of the distance of point P not just from another point, but from a set.

So, it is the shortest distance of P shortest with respect to various points Q, you can take in the set M, so consider the set M and distance. Now, this is nothing but distance of the point P from the set M, this infimum, for infimum you can think of minimum for the time being is the infimum of this norm. Now, you take different points Q and find the distance of P distance between P and Q and you look at the minimum such value as you vary this point Q across the set M.

So, the point Q in M which is closest to P we take, and we take the distance between P and Q and this distance we will call as the distance of P from the set M. So, to converge to a set M means the distance of x of t of the vector x of t at any time t distance of that vector from the set M, this distance is some real number.

Now, it is a positive number, it goes to 0 as t tends to infinity, so you can check that if this point P sitting inside the set M, this distance is 0. Now, it is why because it is at distance 0 for itself which is already inside the set, so it is with respect to this notion of distance of a point P from a set M, and this distance going to 0, this is the notion of approaching a trajectory approaching a set M.

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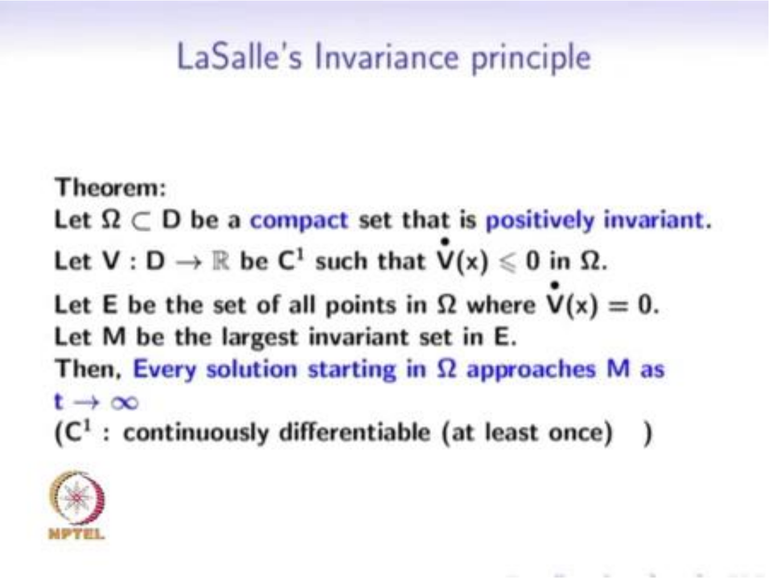

Now, using this we look back at Lassalle's invariance principle, suppose these conditions are satisfied that you could find a compact set that is positively invariant with respect to the dynamics of f. So, if you can find a function V that is continuously differentiable and non increasing its rate of change is less than or equal to 0. Now, on the set omega inside the set omega you construct the set of all points E that are, such that V dot is equal to 0.

So, inside this E you construct the largest invariant set M Lassalle's invariance principle says that every solution starting in omega approaches M as tends t tends to infinity. So, we will see that this invariance principle allows us to use that same Lyapunov function motivated by physical principles to conclude in fact asymptotic stability, this we will see in the following lecture.

Thank you.