Nonlinear Dynamical Systems
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Lecture - 7
Lyapunov's Theorem on Stability

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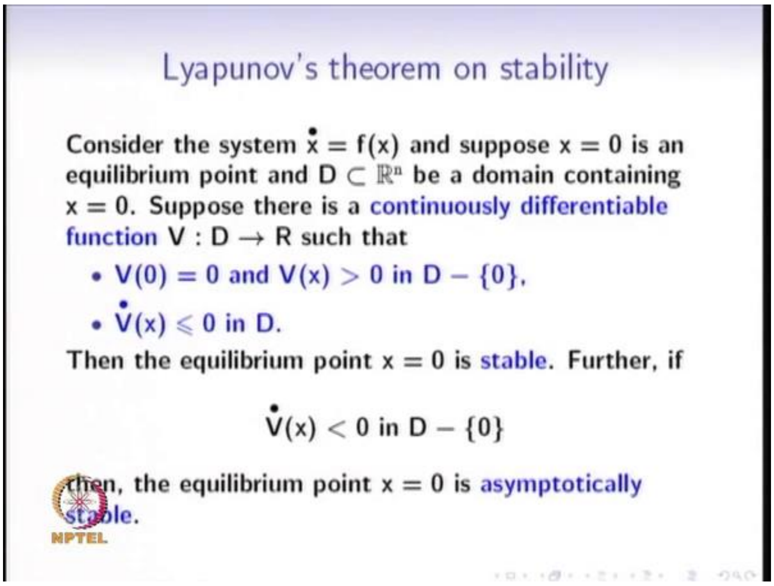

Welcome everyone to the seventh lecture of non linear dynamical systems. So, the last time just saw the definition the statement of the Lyapunov's theorem of stability and asymptotic stability we also saw an outline of the proof. Today we will do a very quick review of the theorem statement, and we will proceed with the proof of both stability and asymptotic stability. So, consider the dynamical system x dot is equal to f of x and suppose x is equal to 0 is an equilibrium point. This is this equilibrium point is a domain D a subset of R n, suppose there is a continuously differentiable function V from this domain to R such that V is equal to 0 at the equilibrium point 0.

It is positive for all other points and its rate of change with respect to time is less than or equal to 0, it is non positive in this domain. If such a V exists which satisfies these properties then the equilibrium point is stable further if the rate of change of V is negative for all points in the domain. So, except the origin of course in that case this equilibrium point is not just stable, but in fact asymptotically stable.

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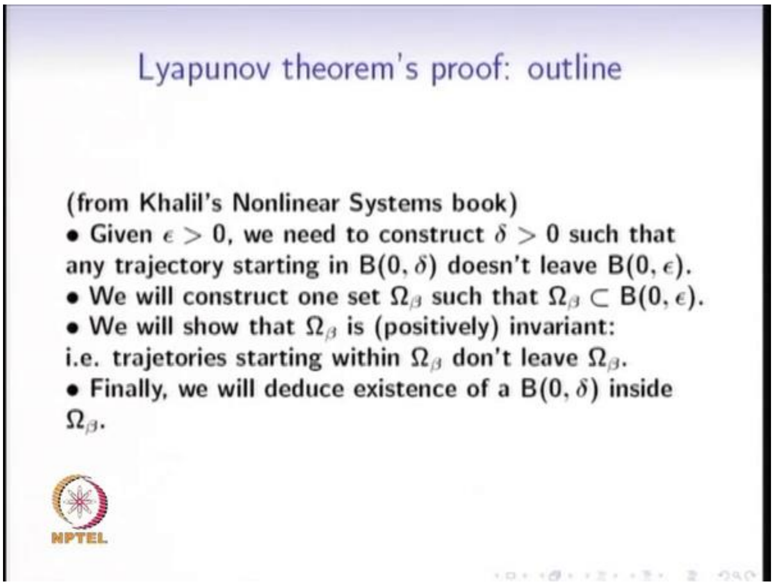

So, the proof that we follow is from the book by Hasan Khalil on non linear systems, so what is what is required to be done in the proof, what is the to prove stability? Whenever somebody gives us an epsilon greater than 0, we need to construct this delta greater than 0, such that any trajectory starting inside this ball 0 comma delta does not leave this other ball B 0 comma epsilon.

So, in this the notation is that the first 0 is the center of the ball and delta is the radius, similarly here so in order to construct this delta. We will construct one set omega beta such that this omega beta is contained inside this ball B epsilon and we will show that this omega beta set is positively invariant. That is trajectories starting inside this set do not leave the set finally, we will also deduce an existence of a ball B delta inside this omega beta.

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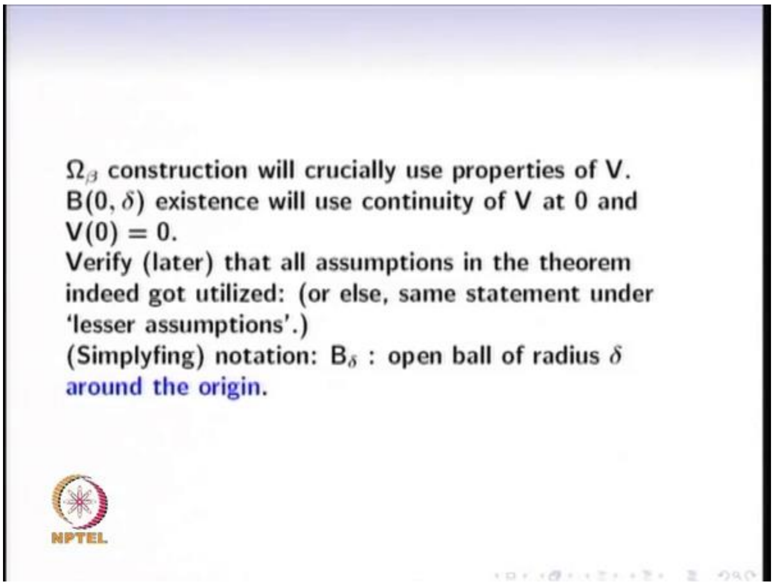

So, to do all this we will crucially use the properties of that Lyapunov function V the existence of this ball B 0, delta will use the continuity of the Lyapunov function at 0. The fact that B and 0 is equal to 0 of course whenever we prove something at the end of the proof it is important to verify that all the assumptions in the theorem statement indeed got utilized in the proof. Else, one could in principle prove a similar statement the same statement and the lesser assumptions, since we did not utilize some of the assumptions we could consider relaxing those assumptions and having the same theorem statement.

So, for the rest of this proof we will, since all the open balls we are considering are centered at the origin we have temporarily we have decided that the origin is the equilibrium point. So, whichever point is the equilibrium point we can always shift the coordinates such that that point is the origin and since 0 is the center we will only denote the radius of the ball. Hence, B delta is an open ball of radius delta the around the origin part we do not require to mention again and again.

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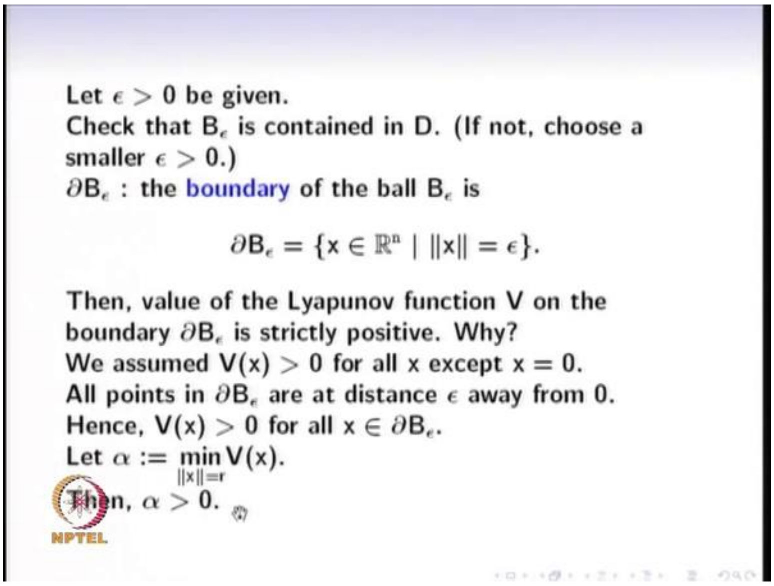

So, let epsilon be given epsilon greater than 0 be given first check that B epsilon is indeed contained in the domain D, if not one can choose a epsilon that is slightly smaller. So, for this ball B epsilon we construct the boundary of the ball, so what is the boundary of the ball B epsilon contains all the points which have radius strictly less than epsilon which have distance strictly less than epsilon. Now, from the origin its boundary is the set of all points whose distance from the origin is equal to epsilon this is the boundary of the ball this is the surface of that sphere on this particular boundary on this ball.

We will now look at the Lyapunov function how that behaves, so notice that this Lyapunov function. The value of Lyapunov function on all the points on this boundary is strictly positive why because it is equal to 0, only at 0 at any other point it is positive. So, all the points on this ball are epsilon away from the origin and hence the Lyapunov function is strictly positive.

So, this is the reason so if it is positive on all the points on this boundary on this del B epsilon, then we will look for the minimum value. But, that B takes on del B epsilon on the minimum over all x, such that mod x the distance of x is equal to R, the norm of x is equal to R. So, norm of x is nothing but the distance of x from the origin for all such points we will look at the minimum value that V takes and let alpha be that minimum value, so we already know that alpha is strictly positive.

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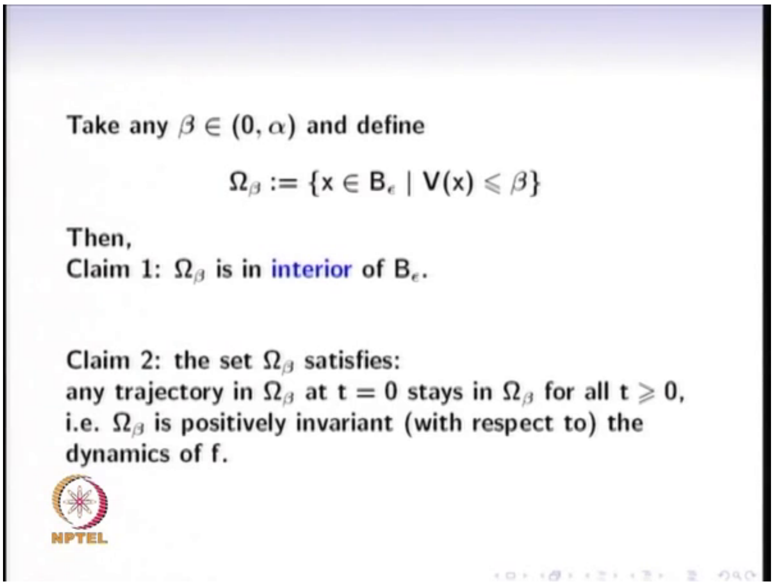

Then we take any beta that is between 0 and alpha in order to take this beta it is required that alpha is positive and alpha is the minimum that V takes on the boundary of D epsilon ball. So, once we have chosen some beta that is strictly less than alpha we will define this set called omega beta, which is the set of all the points in this ball B epsilon such that V of x is at most equal to beta. So, inside this B epsilon ball there are various points and at various points the Lyapunov function takes different values. We will pick all those points where the Lyapunov function's value is at most equal to beta, once we have constructed this omega beta set there are two properties.

So, important properties of the set that we need one is that omega beta is in the interior of this ball B epsilon it does not come to the edge. The set omega beta satisfies this important property that any other trajectory inside this omega beta at t equal to 0, anything that starts inside omega beta stays inside the set for all t greater than or equal to 0. In other words, we use the word that omega beta is positively invariant with respect to the dynamics of f, so these two properties we will first show.

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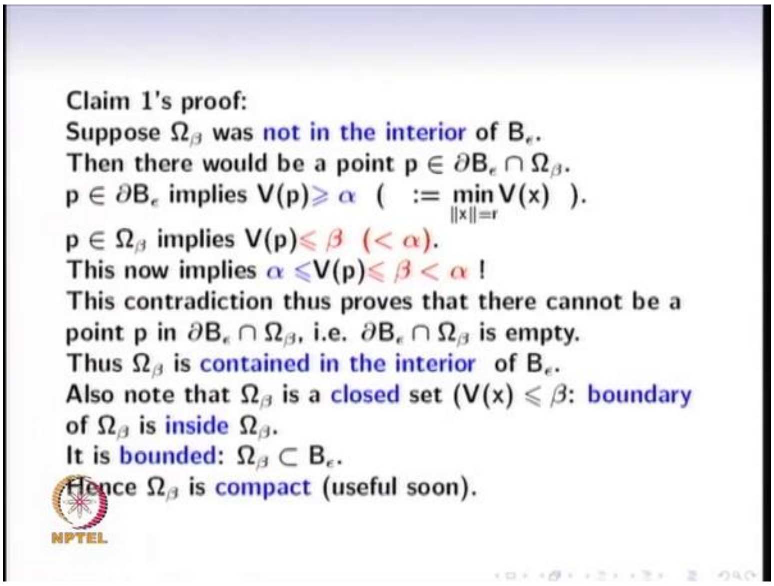

So, the first important property is those omega betas is in the interior of B epsilon, but suppose it was not in the interior. Then there will be a point that is on the boundary of B epsilon and also inside this omega beta set. So, this part of the proof we will replace and hence I am going fast now we come to the second part of the proof.

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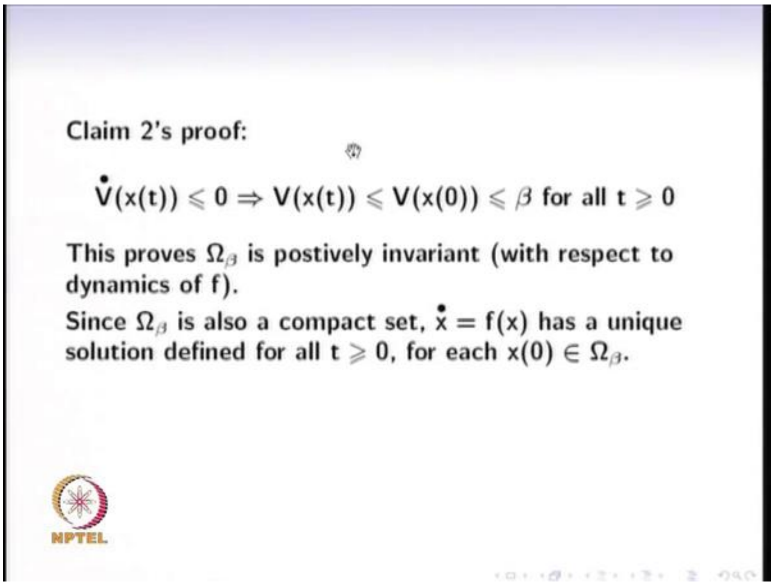

So, what is the other claim what is the other property that omega beta set has we claim that it is also positively invariant, how we prove this. So, we know that V dot x is less than or equal to 0, so what does that mean when we integrate this with respect to time.

But, we see that V at any time is less than or equal to V of x at 0 which was equal to which was at most equal to beta, so this particular inequality is satisfied for all t greater than or equal to 0. This inequality can be obtained simply by integrating this quantity and using the fact that this quantity is less than or equal to 0, yes we can do it a little more slowly on this piece of paper.

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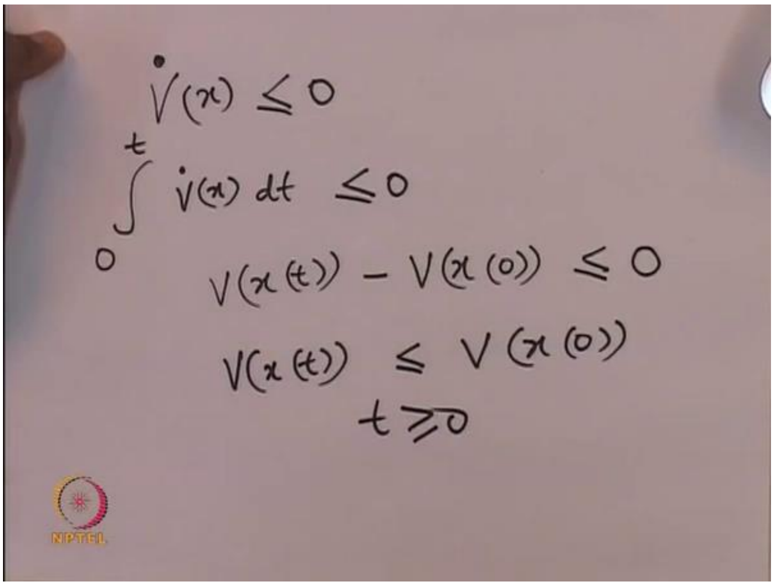

We already have assumed this particular property of the Lyapunov function when we integrate this from 0 to t of V dot of x. This integral is also less than or equal to 0 because they are integrating some quantity that is negative that is non positive. So, this integral of the derivative of a function is nothing but V of x of t minus V of x at 0 final value minus the initial value this itself is less than or equal to 0 which says that is less than or equal to V. Of course, this is expected V is a function that is decreasing with respect to time, hence at any time t that is greater than or equal to 0 this value will be less than or equal to this value, so that is all that is said inside this inequality.

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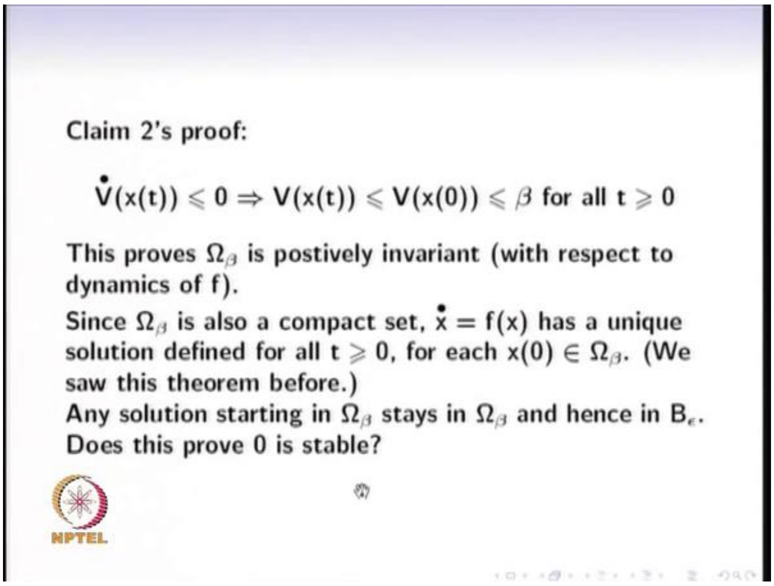

This value itself, this value itself was less than or equal to beta that was method of construction of the omega beta set. Since, we have started inside the omega beta set this value is at most beta, so this proves that omega beta set is positively invariant. If the value is less than or equal to beta at t equal to 0 then for all future time it can only decrease. Hence, that trajectory remains inside the set omega beta, further this omega beta set we also saw just now is a compact set which means it is a closed and bounded set.

Hence, x dot is equal to f of x has a unique solution defined for all t greater than or equal to 0, we saw that if we have a compact set which is positively invariant and the function is just locally lipschitz. Then we are able to in fact able to assure global existence and uniqueness of solution for this differential equation, so any solution starting inside omega beta stays inside the set. But, omega beta set inside it was contained inside the ball B epsilon, hence this omega beta is also there, hence the solution also remains inside B epsilon ball.

So, does this prove that the point 0 is stable, no not yet we want that the result B delta ball such that solution starting with the net remains inside the ball B epsilon. We have only been able to show that any solution starting inside set omega beta stays inside omega beta and hence inside the ball B epsilon.

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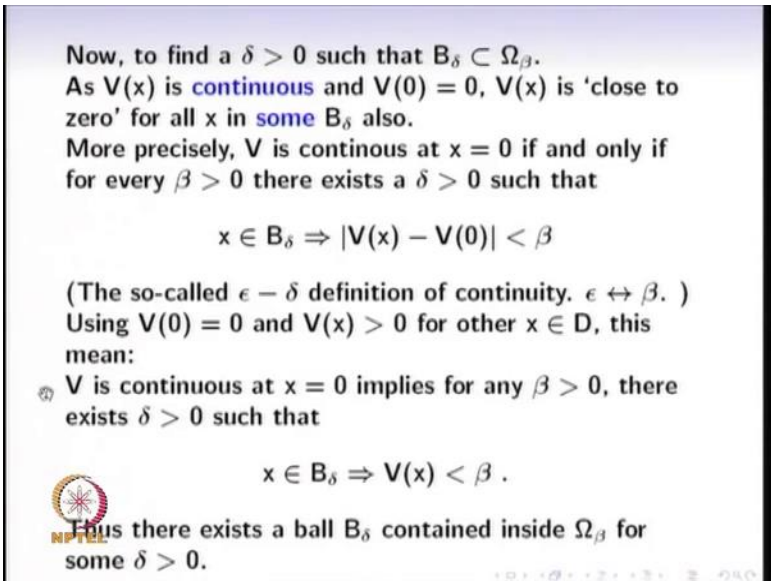

So, how do we find this delta greater than 0 which is contained inside omega beta set, since V is continuous and V of x 0 is equal to 0, V x is close to 0 intuitively speaking. Since, V is continuous at 0 and its value at x equal to 0 is equal to 0, it cannot be very large for points that are close to the origin. So, this particular property is what continuity says more precisely V is continuous at x equal to 0 if and only if for every beta greater than 0. Then there exists a delta greater than 0 such that for all points inside this B delta ball B of x can differ from V of 0 by at most amount beta.

This is the definition of continuity, this is the so called epsilon delta definition of continuity, but since we are requiring epsilon in a different context, we have just replaced epsilon by beta here. Since, V of 0 is equal to 0 and V of x is itself positive we can get rid of this modulus sign here and V of 0 was anyway equal to 0. So, this particular definition of continuity becomes V is continuous at x equal to 0 implies that for any beta greater than 0 there exists a delta such that for all points x in the ball B delta V of x is strictly less than beta. This is nothing but to say there exists a ball B delta that is contained inside the set omega beta for some delta greater than 0, we are able to find some positive delta such that the ball B delta is contained inside the set omega beta.

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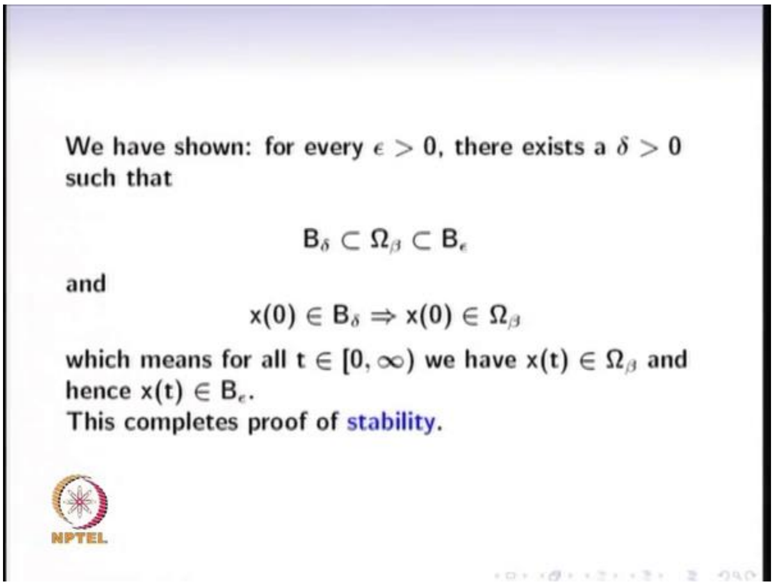

So, what we have shown is no matter what epsilon greater than 0, we start with there exists a delta greater than 0 such that these two inclusions hold B delta is contained inside this omega beta. So, set omega beta is not a ball it is a set only this omega beta set itself is contained inside this larger ball B epsilon further if the initial condition is inside B delta. It means that the initial condition x of 0 is inside the set omega beta also and hence for all future time x of t is inside omega beta. Hence, x of t is contained inside the ball B epsilon also this was precisely what was to be shown to prove that x is equal to 0 is an equilibrium point, so this completes the proof of stability for the Lyapunov theorem.

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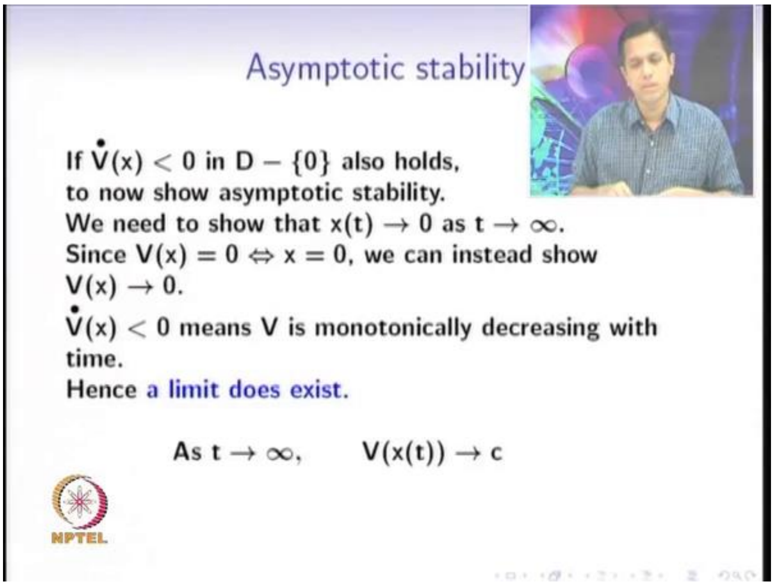

So, what is to be proved for asymptotic stability if V dot of x is in addition assumed to be negative for all points except the origin for all points except the equilibrium point. Then we are yet to show that the equilibrium point is infinite asymptotically stable, so to show that it is asymptotically stable what is to be shown. We need to show x of t tends to 0 as t tends to infinity, but then since V of x is equal to 0 only at the origin we can instead show that V of x tends to 0.

Now, it is because if V of x to 0 that can happen only when x tends to 0 as t tends to infinity, so what does this property that V dot of x less than 0. It means that V is monotonically decreasing with time as a function of time V is decreasing and further it is also bounded from below. But, yes we cannot arbitrarily decrease for this we need to see a small figure of what a monotonically decreasing function can look like.

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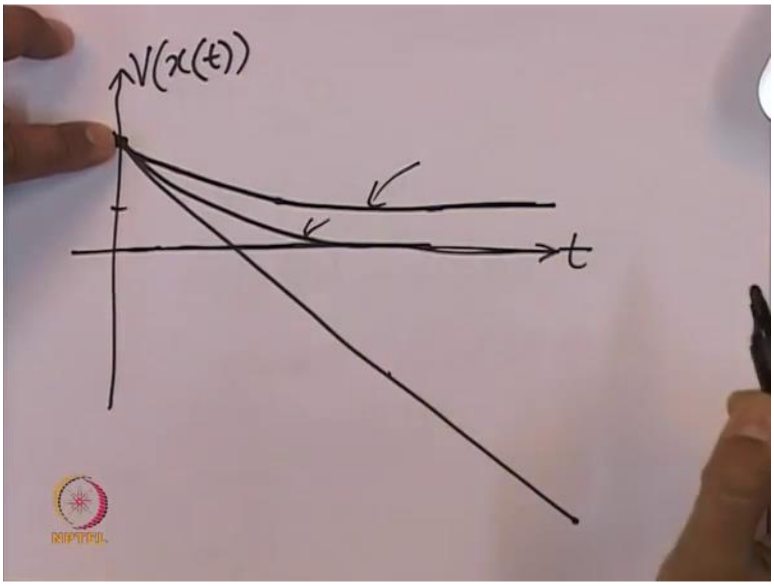

So, this is V which is a function of x, but x itself is a function of time and hence we can draw this, so here is what value of V at x 0 is equal to and the function is only decreasing with time. So, there are three possibilities one it could converge to some non zero positive value other it could converge to 0 or it could converge. It could, it need not converge it could go on decreasing, so at least we know that V is bounded from below. Hence, this is not possible it is bounded from below by a 0, the minimum it is equal to 0 only at x equal to 0 at all other points it is positive. So, it is not possible that any time instant the trajectory goes the function V as a function of time cannot become negative. Hence, it can be only one of these two yes also of course it is not possible that V of x oscillates.

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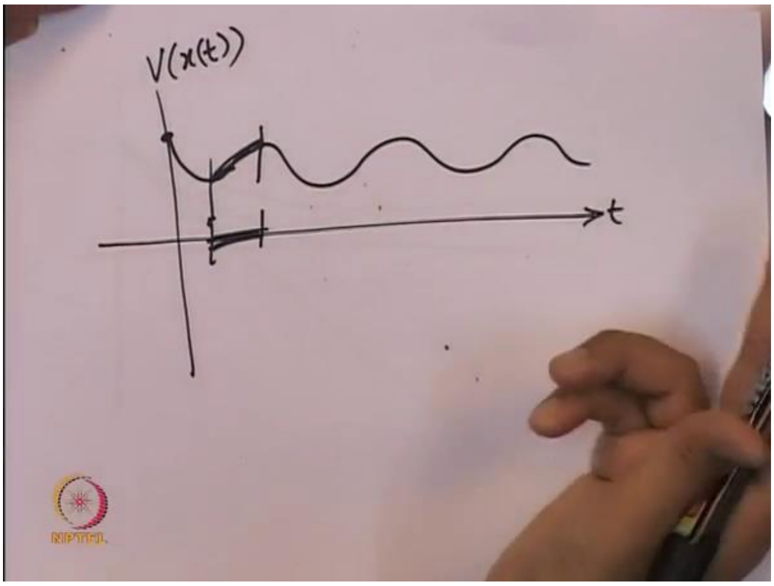

It is not possible that why this is not possible because here is a region where V dot is positive, yes over this region V dot is positive. But, we know that V dot is always less than 0 it is monotonically decreasing if it is monotonically decreasing such oscillations are ruled out.

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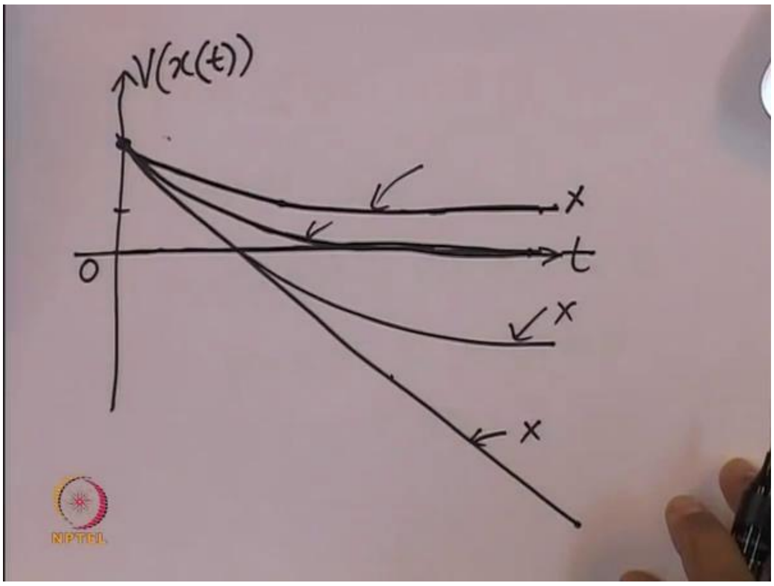

Hence, we are dealing one of these three cases it is going in decreasing arbitrarily too small, too small values or it comes and converges to the 0. It converges to some other value or of course it can converge to some negative value this is ruled out both because we know that V is bounded below by 0, by the 0 function. Hence, it can be only one of these and these cases yes, so we are going to try and use the various properties of V to rule out.

So, this also that we say that the only way we can behave is that for each initial condition it comes and converges to 0 for V of x tending to 0 as t tends to infinity. This can happen only when x tends to 0, so this is what is to be shown to prove that it is asymptotically stable so the first thing is because V is bounded from below.

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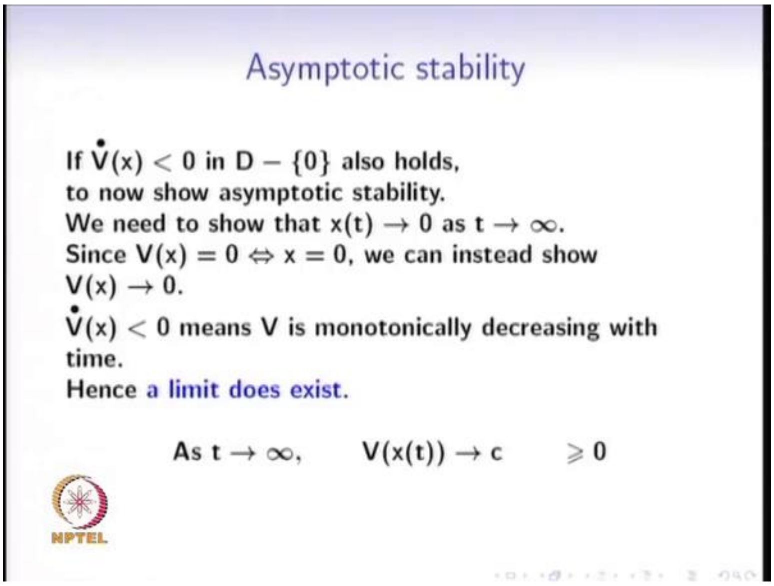

Since, V is monotonically decreasing a limit does exist suppose this limit is c as t tends to infinity, V of x of t converges to c and we also know that the c can be non negative, it cannot be negative.

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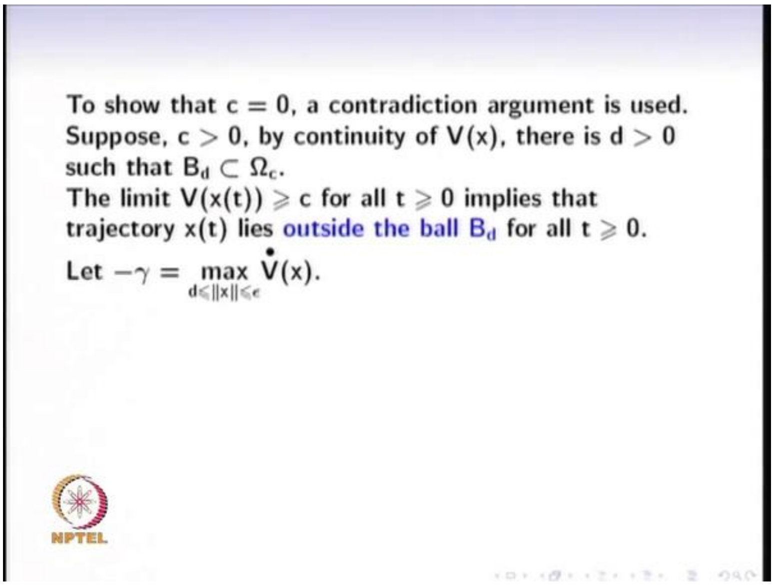

So, to show that c is indeed equal to 0 we will use the contradiction argument what is this contradiction argument, we will assume to the contrary. We will assume that c is positive is greater than 0 and then we will use continuity of V to prove that this cannot happen. So, suppose c is greater than 0 then my continuity of the function V of x like we have proved in the stability case. Here, also if c is greater than 0 then by continuity of the function V there is some d greater than 0 such that the ball d of d is contained inside the set omega c, yes instead of beta greater than 0 we have a c greater than 0.

Hence, we construct this omega c set omega c set is a set of all points where the value V of x is at most equal to c. But, this B d is some ball that is contained inside omega c we are able to construct this B d ball again by using the continuity of the function V at x equal to 0. Now, the limit what does it mean that as t tends to infinity V of x of t is greater than equal to c and the limit is equal to c. It means that for all t greater than or equal to 0 this V of x of t is greater than or equal to c and this just means that the trajectory lays outside the ball B d for all t greater than or equal to 0, so this is a very important argument we have to see a figure.

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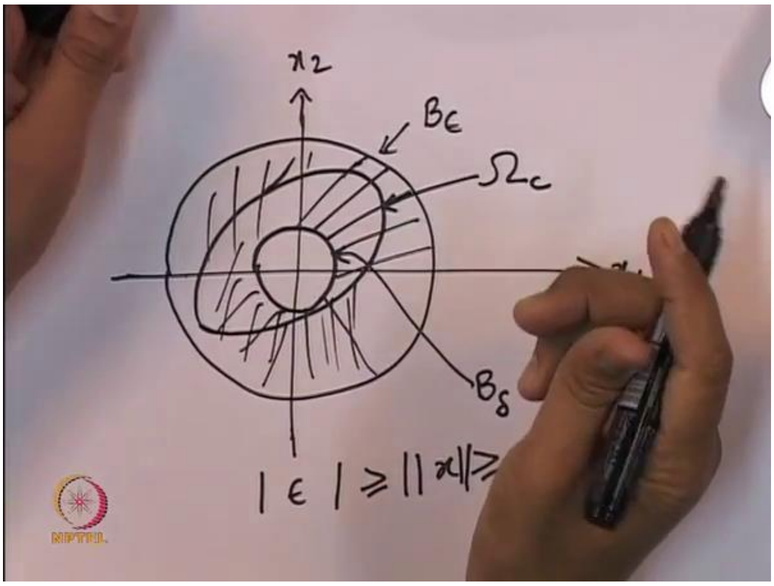

So, this also original ball B epsilon this is our state space x 1 x 2, so we are assuming for the time being that there are only two state components. But, more generally a ball in R n inside this B epsilon ball we have constructed this omega beta set right. Now, this omega beta a, we are going to denote as omega c and inside so the omega set as a, set need not be a ball it can it might be more general set.

But, this omega c set does not come too close to the origin, in other words there is a open ball B delta that is contained inside this omega c set. So, this is this ball which we have called B delta, now look at this region that is outside yes, so x greater than or equal to norm of x greater than or equal to delta. But, the norm of x is less than or equal to epsilon sorry, so epsilon is positive, so this absolute value sign is not required.

So, what does this region signify it signifies this region outside this ball B delta, but inside this ball B epsilon so B epsilon is open, so except for that fact it inside this ball B epsilon. But, it is outside of this ball B delta, so the fact that V of x is always greater than or equal to c means that the trajectory does not enter this set omega set. If it does not enter inside this omega c, it can of course reach the boundary because the boundary is where x of t is equal to c on this boundary it could come. But, does not go inside this set omega c and this B delta ball is contained strictly inside this omega c set and hence we have now concluded that if it starts somewhere here it can at most come up to this omega c.

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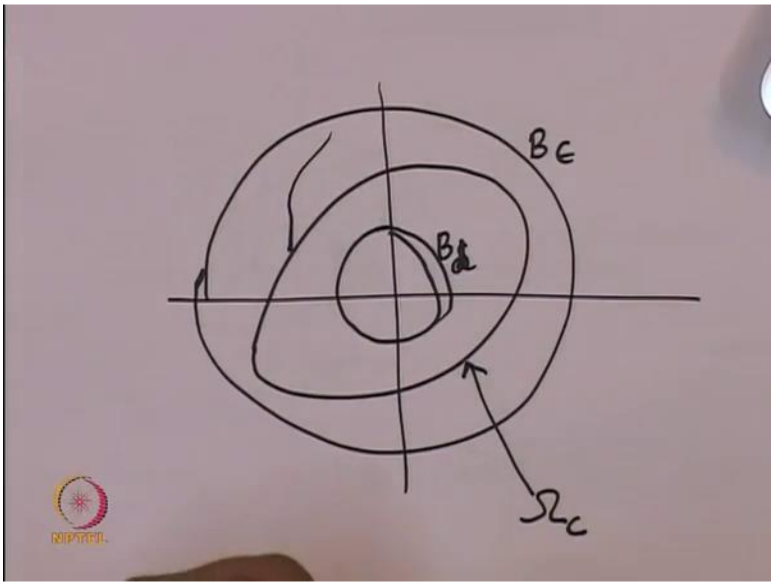

So, this is our ball B epsilon there is the set which is not a circle it is not a ball, this is omega c and inside this there is a ball B delta B d sorry we have now called it B d. Now, we have already shown that if a trajectory starts somewhere here it can only come close to omega c it can come to the boundary. But, cannot go inside in particular it cannot go into this ball B d, so what that is precisely the statement that we are seeing here.

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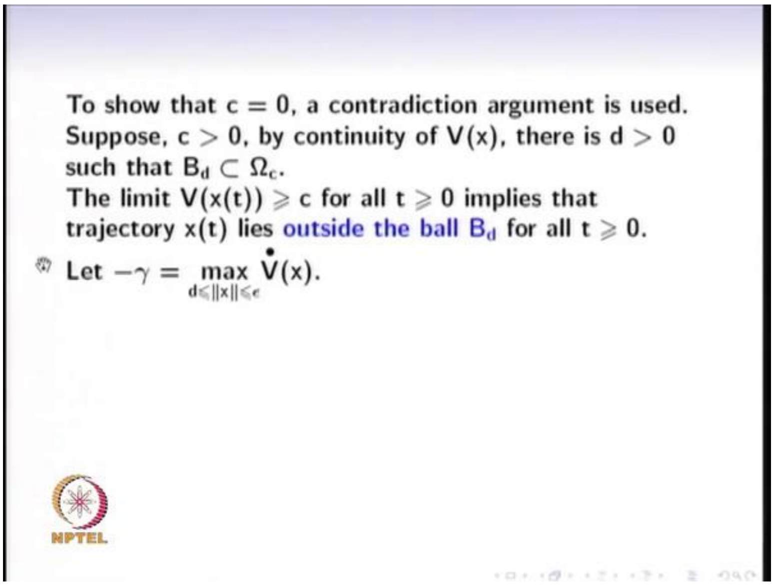

Now, the limit V of x t is greater than or equal to c, it just means that then trajectory x t lies outside the ball B d for all t greater than or equal to 0 and hence we will now look at optimizing this particular V dot function over this set.

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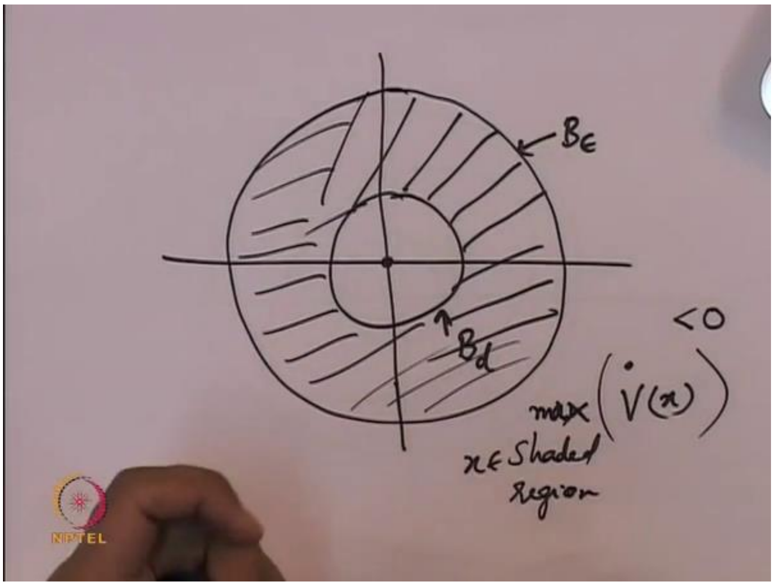

So, over the difference between what is outside B d and inside B epsilon ball, so over this set this is a ball B d this is the ball B epsilon the fact that omega c is something sitting in the middle. So, over this particular shaded region we are going to now look for the minimum value of V dot, yes minimum value of V dot of x. But, at each point V dot is some function it is the rate of change of V with respect to the trajectory at that particular point minimize this particular quantity.

So, over this shaded region for all x inside this shaded region we want to just minimize this quantity, so first important property is sorry, I am sorry for writing the minimum, it is the maximum. Now, we could maximize this particular quantity this particular quantity denotes the rate of change, so it is maximum value is the maximum increase. But, then this V dot is decreasing as a function of time so this V dot at every at every x it is strictly less than 0, why because the point x equal to 0 is ruled out V dot is equal to 0 only at the center.

So, hence inside this region V dot is strictly negative and it is some function, if it is some function that is strictly negative. We can look for the maximum value of V dot over this shaded region even the maximum value is negative why because it is strictly negative for all points here, hence the maximum value is also negative.

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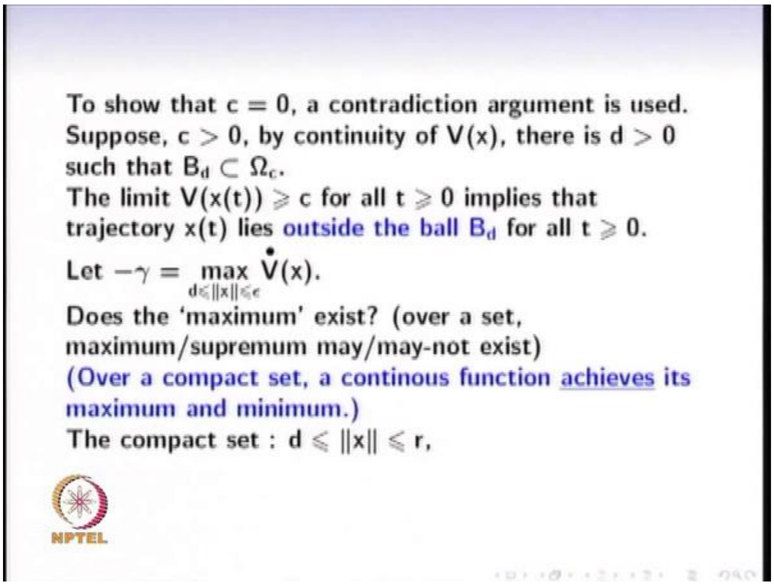

So, let the maximum value be denoted by minus gamma, so here is some quantity gamma that is positive why is that strictly positive because V dot is strictly negative inside this region. Hence, it is maximum value if the maximum exists is also negative, so next question that arises is the does the maximum exist, why because over a set in general the maximum or the supremum may or may not exist. So, we will see some simple graphs where this can happen, but we will use a property that over a compact set a continuous function achieves both its maximum and its minimum. But, yes and this particular set that shaded region is a compact set, so what is the problem that a function may or may not achieve its maximum and minimum values.

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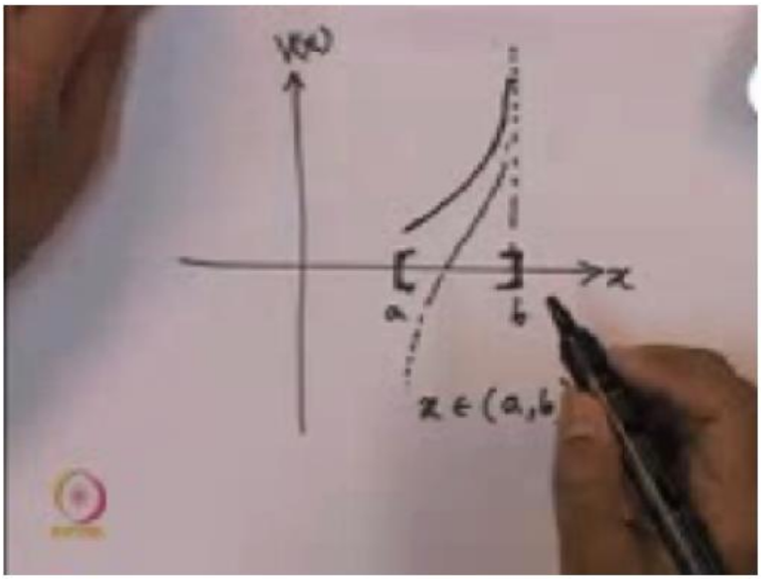

Now, first important property is we can have this particular set for the purpose of this we require to plot V of x versus x, hence x has only one component. So, over this particular set is possible that V of x is defined at every point in this open interval, in the open interval a to b for x in the interval a to b. It is possible that x goes to infinity becomes unbounded as x tends to B it becomes infinity and hence maximum of course, does not exist even the supremum.

But, the value that it can become close to even that value does not exist it is unbounded similarly the minimum value then infimum also may not exist simply because it is unbounded. So, in our case this cannot happen because first of all we are dealing with a closed interval and more over this is a bounded interval. But, another situation where we can have a closed interval a closed set over which the function does not achieve its supremum is if itself the set is unbounded.

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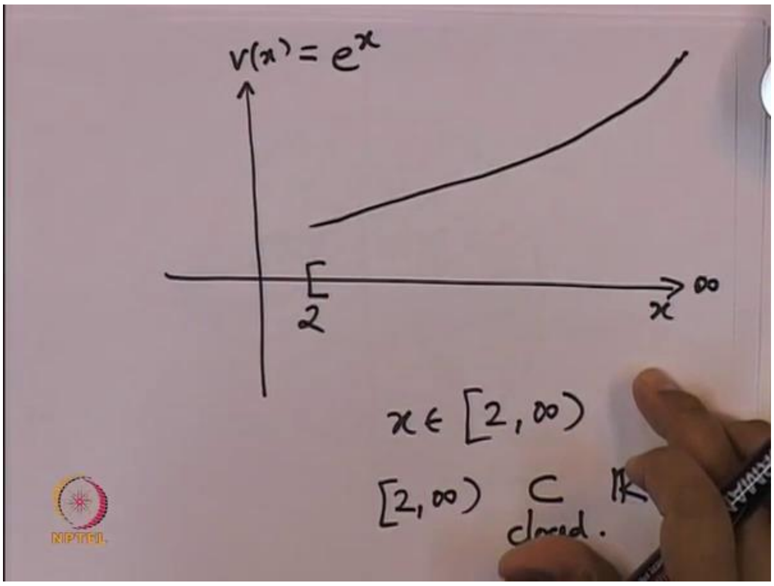

So, consider the interval 2 to infinity, yes this is x this is V of x, so over x in the interval 2 to infinity this is, this set itself is a closed set. It is a closed subset of R n of R 2, infinity is the closed set of R in on this closed set just because a set is closed the supremum may not be achieved, the maximum may not be achieved. The set also required to be bounded, so look at the function now V of x equal to e to the power x, so this particular function over this closed set does not achieve its maximum.

It does not achieve its supremum also there is no number to which it becomes arbitrarily close to and it becomes below that number. But, that number would be called the supremum such a number does not exist for this function simply because even though this set is closed. But, it is unbounded however in our case we are dealing with a closed set and also a bounded set.

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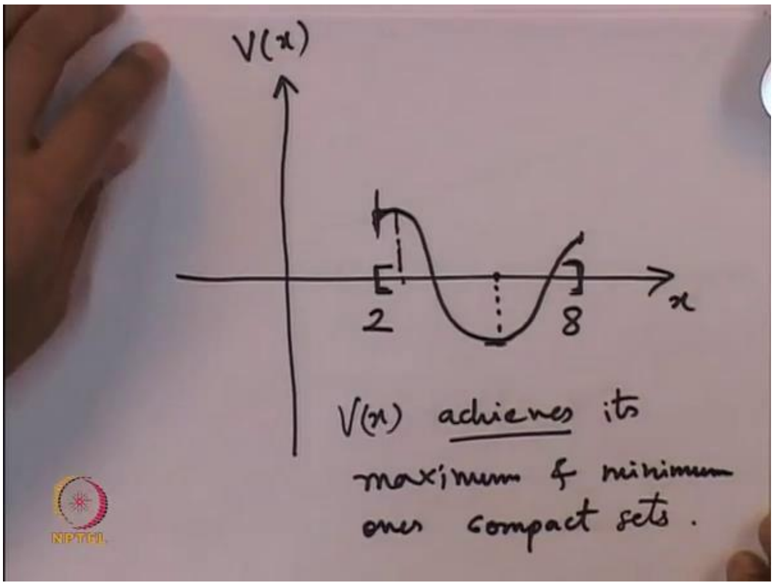

In other words, it is a compact set over compact set a function in fact achieves the maximum and its minimum the interval 2 to 8, this is x this V of x where is this continuous function. So, its continuity is also important suppose this is the interval, so there is indeed a value of x where it achieves its maximum value there is one other value at least where it achieves its minimum value. So, the maximum and minimum are in fact achieved that is why we will say the maximum and minimum of that particular function over that set exists.

So, this is because this interval 2 to 8 is both a closed and bounded set, so V of x achieves meaning of achieve. So, this means there is indeed a point x, where the value of V of x is equal to the maximum value over that set achieves its maximum and minimum over compact set. The compact we saw already was nothing but closed and bounded set of course we have assumed that V is continuous. But, only continuous functions are guaranteed to achieve their maximum and minimum over their compact sets, so we are now back to our case where x has many components.

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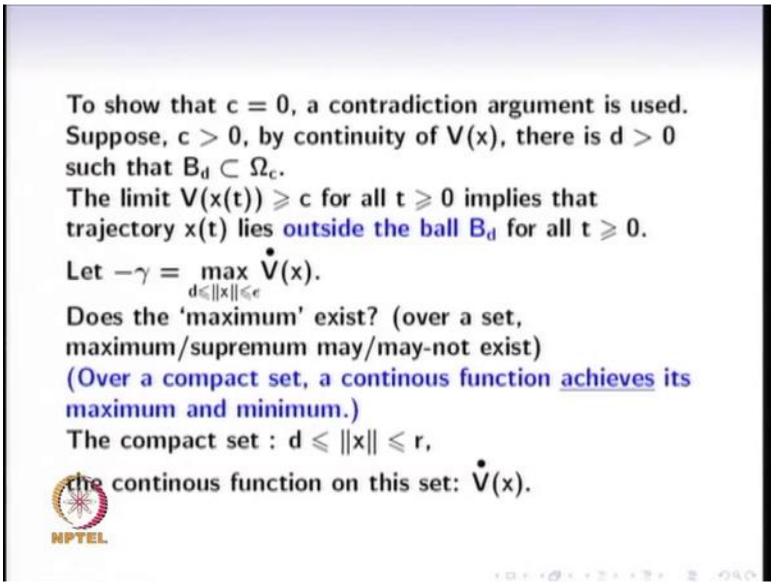

This is a element in R n and we are now looking at what is outside the ball B d and contained inside the ball B epsilon inside the closed ball B epsilon, but it is written r here. So, this is a small type, so the set of all points where the norm of x is greater than or equal to d and less than or equal to r.

So, notice that the boundaries are also included and it is bounded from r epsilon, r is equal to epsilon it is bounded by epsilon. Hence, it is also bounded set on this compact set this maximum value is achieved we really require this property. That is why the emphasis and this maximum quantity is itself is negative and hence gamma is positive. So, the quantity over this compact set the continuous function over the set that continuous function is V dot of x that depends continuously on x.

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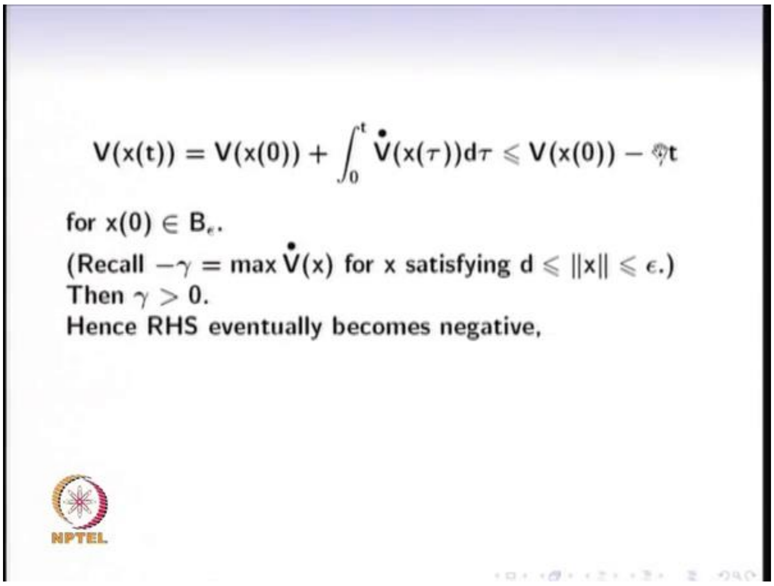

So, since it is we require this property of gamma greater than 0 and that we will use to arrive at the contradiction we are looking for. Now, integrating both sides we see that V of x of t is nothing but V of x of 0 plus this integral from 0 to t of V dot s tau d tau. Now, this particular quantity itself is less than each at each time instant this quantity is less than or equal to gamma. Hence, we will integrate this and this becomes V of x of 0 minus gamma t because gamma minus gamma was equal to the maximum of this and hence we are integrating something instead of this V dot.

We are going to replace V dot by the maximum value that V dot can achieve and hence this quantity to the right hand side will end up becoming larger. But, why would it become larger, because instead of V dot we have done some manipulation by replacing something that can be larger gamma minus gamma is the maximum value that V dot can be over that set. Hence, we have integrated with respect to time of minus gamma and we get minus gamma t here.

So, what we have concluded is at any time V of x of t is less than or equal to V of x of 0 minus gamma t where we call that gamma equal to maximum minus gamma was equal to the maximum of V dot over that set and we also concluded that gamma is positive. Now, notice that this right hand side how does it behave as a function of time this itself is a line with t as the independent variable.

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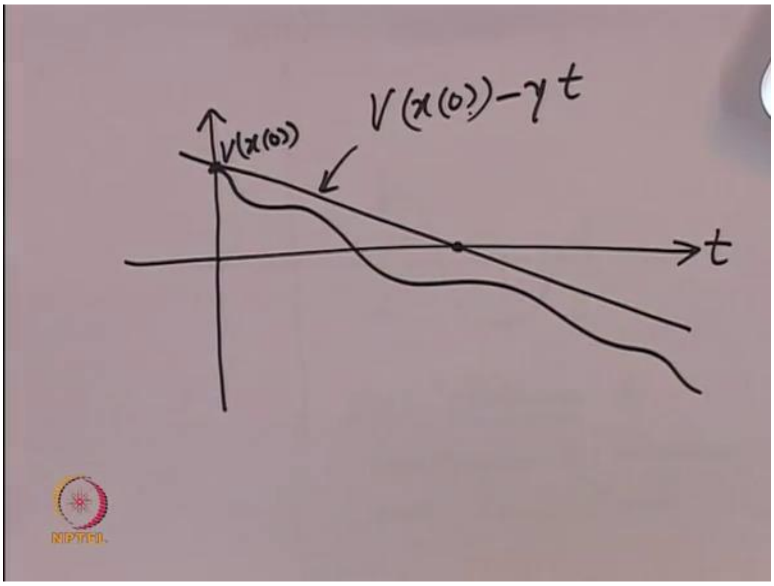

So, this quantity gamma is some positive number and hence this is some line with negative slope with slope minus gamma. Hence, it is decreasing like this since this V of x of 0 at t equal to 0, we get this value and this is where it starts and our is that V of x of t is itself is always below this line. This line itself becomes negative for some time precise value of time depends on the value of gamma, but it is guaranteed to become negative. Hence, this V of x of t, which is what we have concluded is below this line will also end up becoming negative.

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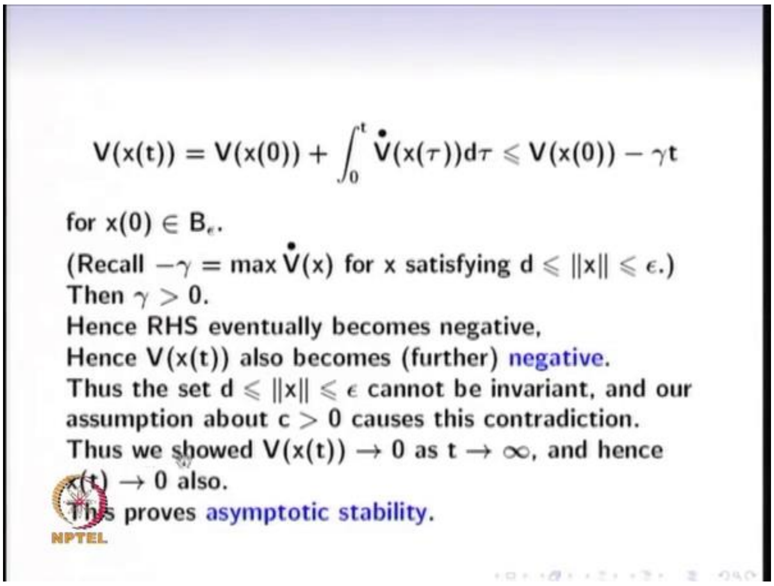

So, coming back to this particular slide V of x of t is less than or equal to this particular quantity this line and this line itself becomes negative. But, no matter V of x of 0 is, no matter gamma value is because gamma is positive this is some line that is sloped downwards. Hence, there is some time instant for which this quantity becomes negative and this quantity which is further less from this will also eventually become negative. Hence, V of x of t also becomes further negative for that time onwards thus this set, so what have we concluded V of x which was guaranteed to be positive has ended up becoming negative.

This is the contradiction that we have finally obtained thus this set d less than or equal to norm of x less than or equal to epsilon the set of all such x cannot be invariant. That is, then property that we used and our assumption that c greater than 0 has ended up causing such a contradiction what contradiction V of x of t eventually becomes negative for some finite t onwards. So, since c greater than 0 has been ruled out we have now concluded that V of x of t converges to 0 as t tends to infinity and hence x of t converges to 0 also this proves asymptotic stability.

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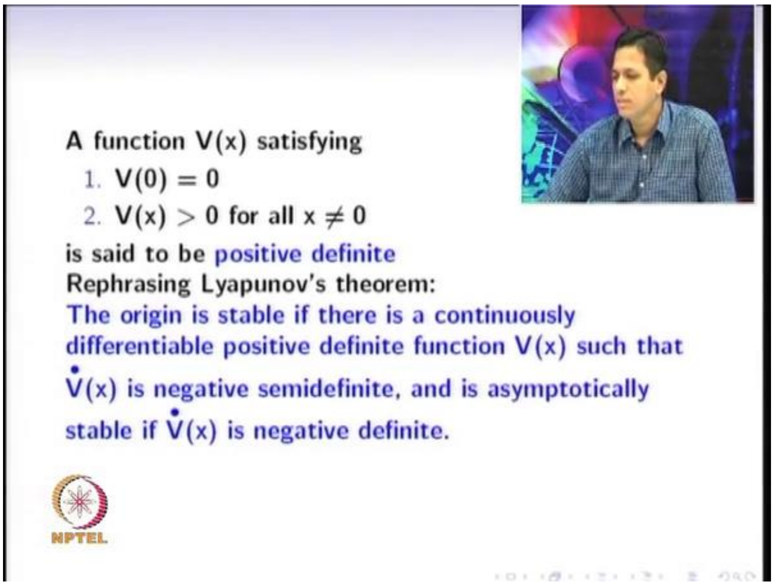

So, this completes the proof of the Lyapunov's theorem on stability and asymptotic stability just some more notation. So, a function V that satisfies V of 0 is equal to 0 and V of x is positive for all non zero points for all points x not equal to 0 such a function V is also called positive definite. So, in these new words we can rephrase Lyapunov's theorem that the origin if there is a continuously differentiable positive. But, definite function V of x such that V dot is negative semi-definite and if V dot is negative definite then that origin is in addition to stable also asymptotically stable.

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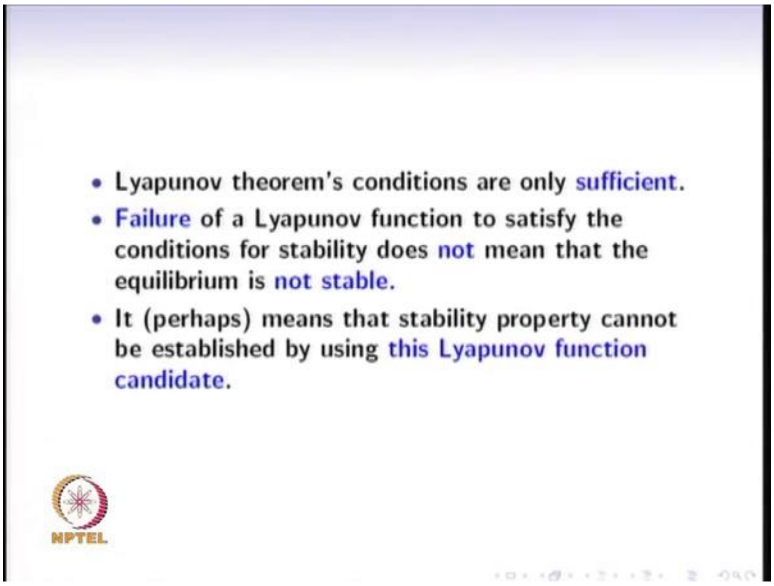

So, it is important to note that the Lyapunov's theorems are only sufficient of there is continuously differentiable function satisfying these properties. Then we can go ahead and conclude that the origin is stable let us see this failure of a Lyapunov function to satisfy the conditions for stability in the Lyapunov theorem. If some Lyapunov function fails to satisfy those conditions it does not mean that the equilibrium is not stable it perhaps. It means that the stability property cannot be established by using that particular Lyapunov function candidate.

So, we will call it a Lyapunov function only if it satisfies that properties if we have a candidate that fails to satisfy those properties we cannot go ahead. Now, conclude the origin is not because it does not mean that it perhaps means that that candidate is to blame that candidate may not be the correct function. But, we might, we should perhaps be looking for other candidates which will satisfy the conditions that are stated in the Lyapunov's theorem of stability. So, before we go to some more results about Lyapunov's theorem not just for locally stable for globally stable we should draw figure of how this Lyapunov function is helping.

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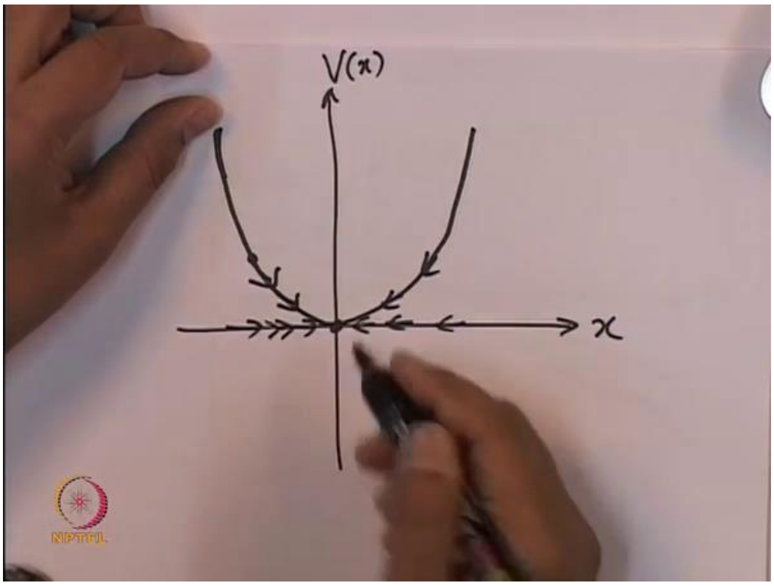

So, this is our x this V of x, so we want to now conclude that all points are on all the arrows are directed towards the origin we are able to do this by the existence of some function V. So, this function V is positive that is why it has to always lie above the graph above the x axis and here because the directions are always directed towards the origin. It turns out that this function V itself when differentiated with respect to time is always decreasing, in other words if we are able to find a function V. But, that is decreasing as a function of time if it is decreasing strictly and function itself is positive then the only way it can happen is that that particular equilibrium point is asymptotically stable.

So, it is not possible that there is such a function which is decreasing which is decreasing, strictly decreasing along the trajectories. Now, it is itself positive is equal to 0 only at that point that function V if such a function exist, it is not possible that this particular equilibrium point is unstable. It is guaranteed to be asymptotically stable this is what the Lyapunov function says we can take a very simple example for this purpose.

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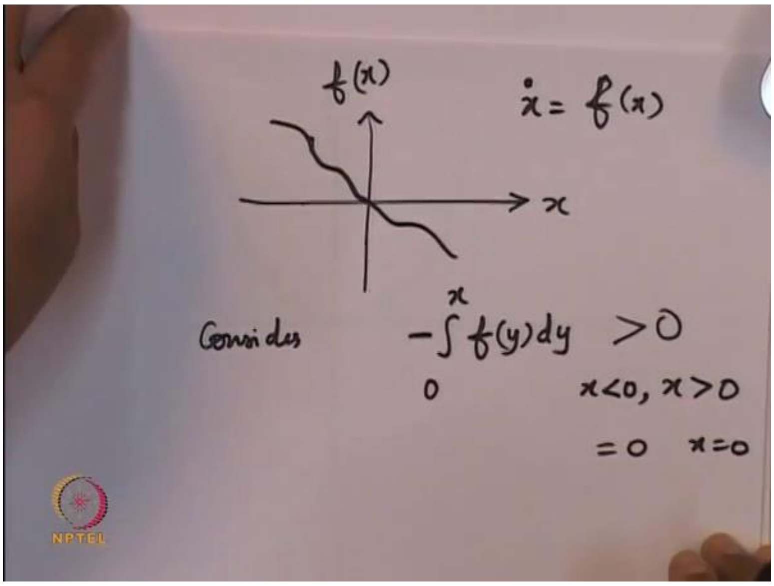

So, consider suppose this was our dynamics x dot is equal to f of x in which graph of f was equal to like this, so here is some function, so we know that if we if f satisfies this. So, it is important to note that in this example we are plotting f versus x while we are actually going to construct a Lyapunov function which is also positive except at the point x equal to 0. Now, rate of change of V with respect to time V dot is required to be negative, so this dot all our by our convention all the dot signifies rate of change with respect to time f is itself a function of x.

So, consider integral from 0 to x of f of f of y d y it is customary to use different variable here and here, so this particular function can we say that this is positive. But, for this particular graph what does this particular function signify it signifies the area from here to here, so it seems like this graph for x positive. This graph is lying below the x axis and hence this greater than 0 is not satisfied the negative of this function seems to satisfy this property that is for x greater than 0. This area has the sign that the area is negative area under this function f for x greater than 0 because this function lies below the x axis the area's negative.

After putting this sign this quantity is positive at least for x greater than 0 for x equal to 0 it is the integral of some function for the width 0. Hence, this is equal to 0 for x equal to 0 what about for x negative for x negative we will instead consider the integral from x to 0 from x to 0 this particular quantity has positive area. Hence, from 0 to x it has negative area, so after this negative sign has been put this again this one is satisfies, so in fact this greater than 0 satisfies for both x less than 0.

Now, for x greater than 0 only for x equal to 0 it is equal to 0, so perhaps this one will serve as our Lyapunov function candidate it is some function that is positive for all non zero x and is equal to 0 for x equal to 0. So, we will now show that this particular Lyapunov candidate indeed serves the, satisfies the properties of Lyapunov function.

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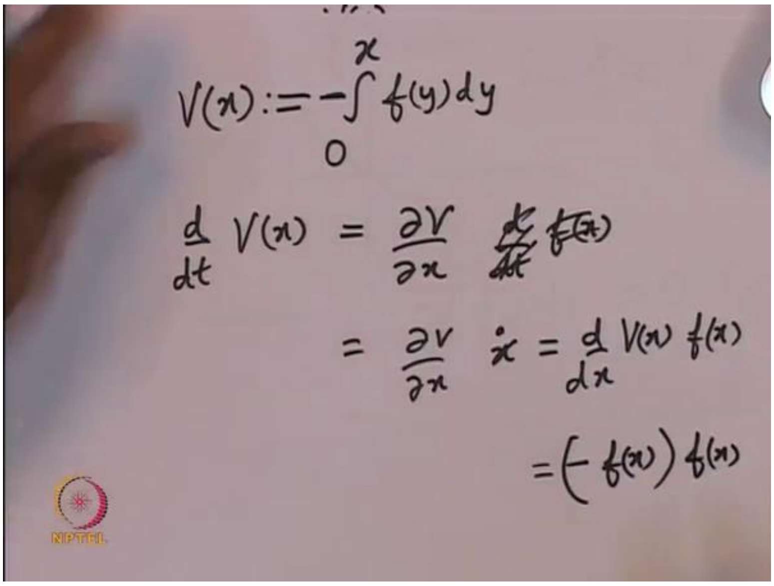

So, V of x is now defined as integral from 0 to x f of y d y, now what is the next thing d by d t of V of x is now equal to derivative of this with respect to x. So, this is nothing but del V by del x times d by d t of f of x del V by del x times, sorry this is equal to del V by del x times x dot n because this is a composite function. But, V is a function of x which is itself a function of t to differentiate this quantity with respect to time we will first differentiate V with respect to x. Then differentiate x with respect to time and of course, V is a function of only one component x in this case.

So, this partial derivative could also be replaced by this ordinary derivative, so this is d by d x of V of x times x dot is nothing but f of x. So, what does it mean to differentiate this particular function with respect to x, here is the function which is growing with respect to x growing at what rate precisely this f y. Now, when you differentiate such a function with respect to the end points then we get precisely the quantity that is inside, sorry we had a negative sign in our definition of Lyapunov function. But, the negative sign is really crucial, so this is nothing but minus f of x this is the rate of change of V of x times f of x.

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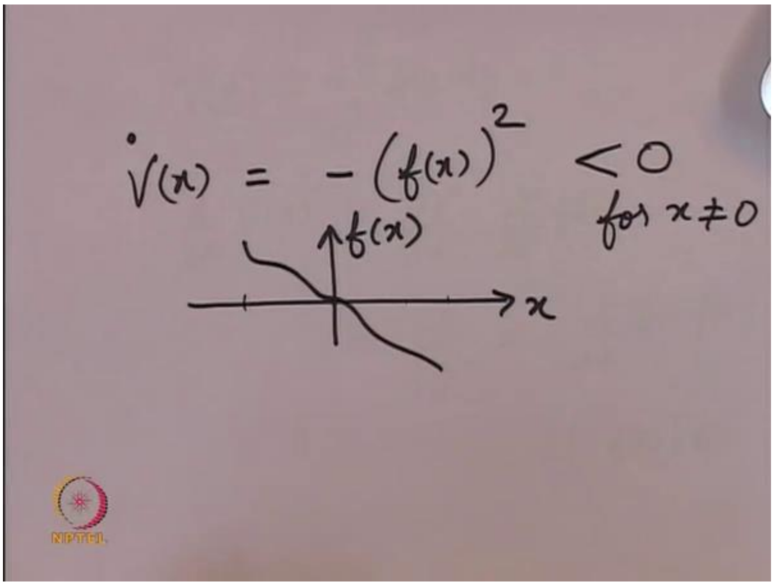

So, we have finally concluded that V dot of x is equal to minus of f of x whole square and recall that f of x graph that was drawn was like this. The graph indicated that f was equal to 0 only at the equilibrium point x equal to 0 what are all the equilibrium points x equal to 0 only. So, only at that point f is equal to 0 which means that that is the only that is then only equilibrium point for this interval that we have drawn the graph it is an isolated equilibrium point.

Hence, this particular quantity is less than 0 for x non zero, since f of x is non zero, for all non zero x this particular quantity is less than 0 is strictly negative. So, here is a function V which is obtained as the integral of other function hence it is automatically continuously differentiable also and it is positive. We already saw its rate of change is negative and this proves that this particular point is asymptotically stable.

So, this is one way in which without knowing the precise formula for f by just using the property that f was positive for x less than 0 f was negative for f x greater than 0. By using just the continuity property of f itself, we have been able to show that this equilibrium point is in fact asymptotically stable equilibrium point, we could in fact take for example these two dimensions.

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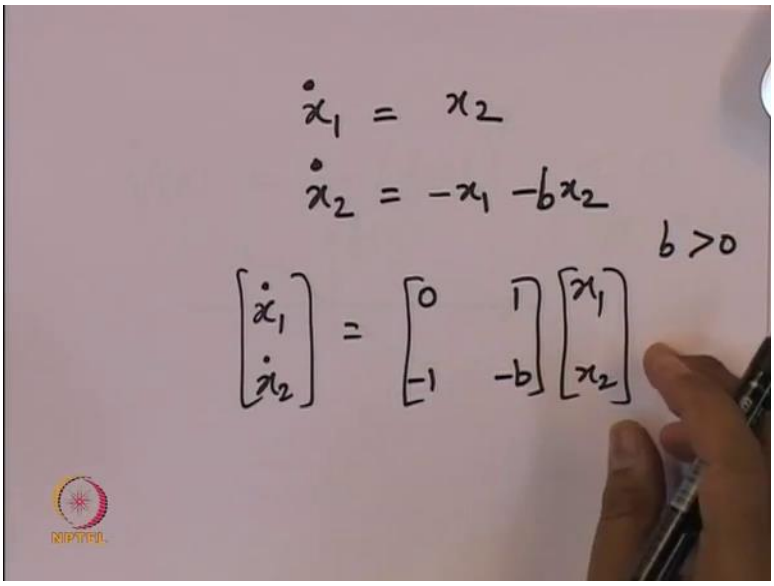

Consider x 1 dot is equal to x 2 and x 2 dot is equal to minus x 1 minus x 2, so rate of change of x 1 is just equal to x 2 rate of change of x 2 is equal to minus x 1 and also b times x 2 where b we will assume as positive. So, this is an example of a pendulum with friction, so we will see this example in more detail when we consider relaxing the conditions in the Lyapunov's theorem. But, at least this particular example we will use and check whether this is whether the equilibrium point is stable and asymptotically stable.

But, of course this is the linear example we can write x dot is equal to a times x and one could in principle fund the Eigen values of this matrix. Now, already concluded that all the Eigen values are in the left half plane this is just one way of doing it we will now use the Lyapunov function argument for proving that the equilibrium point is stable.

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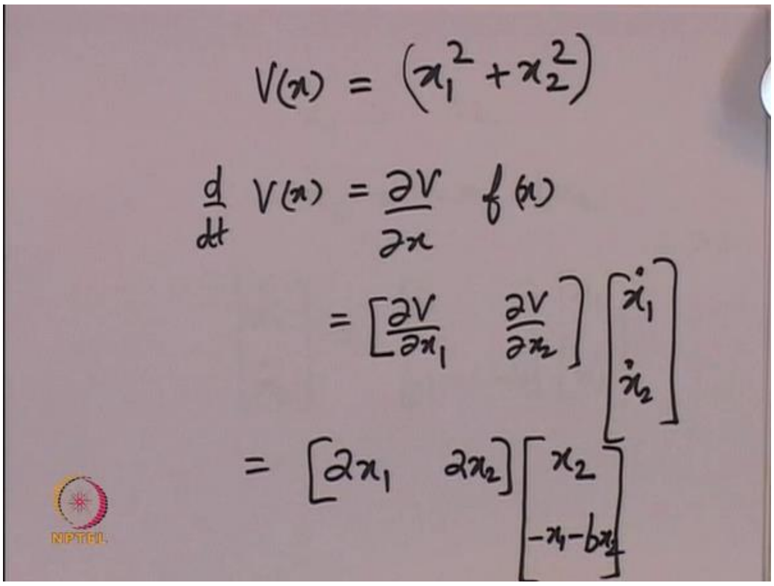

So, take the Lyapunov function it requires some effort to guess to guess a candidate from physical for physical system, sorry it is possible to take the actual energies and sum them up. But, for more general system it requires an effort with can come with experience of how to guess a Lyapunov function candidate and then try to show that it is stable. Now, this is why because if one candidate does not serve the purpose it might require some effort to guess another candidate and still succeed in showing that the equilibrium point is stable or asymptotically stable.

So, here we take this particular candidate so d by d t of V of x is nothing but del V by del of x times f of x f of x is nothing but x dot. So, partial derivative of V with respect to the vector x is nothing but del V by del x 1 del V by del x 2 times x 1 dot x 2 dot. So, the partial derivative of this function with respect to x 1 is nothing but 2 x 1 partial derivative of this function with respect to x 2 is nothing but 2 x 2 x 1 dot was equal to x 2.

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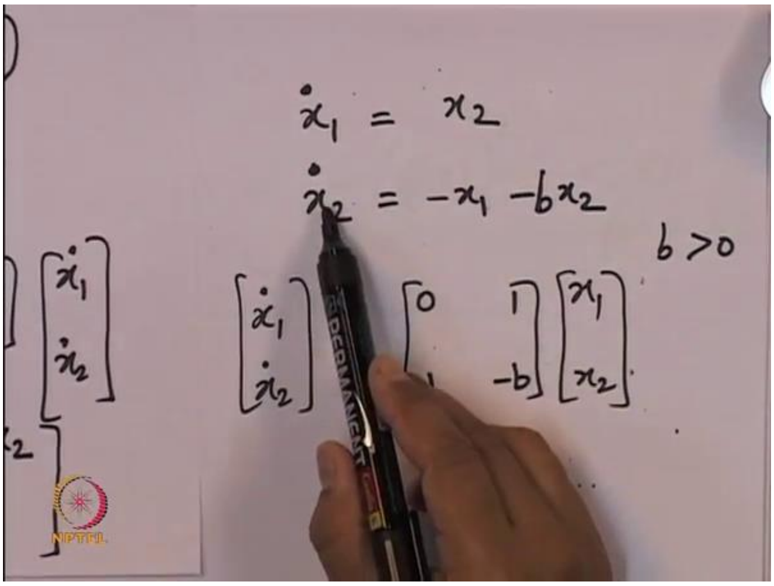

Let us see this pervious, here x 1 dot was equal to x 2 and x 2 dot we will write as minus x 1 minus B x 2 minus x 1 minus B x 2.

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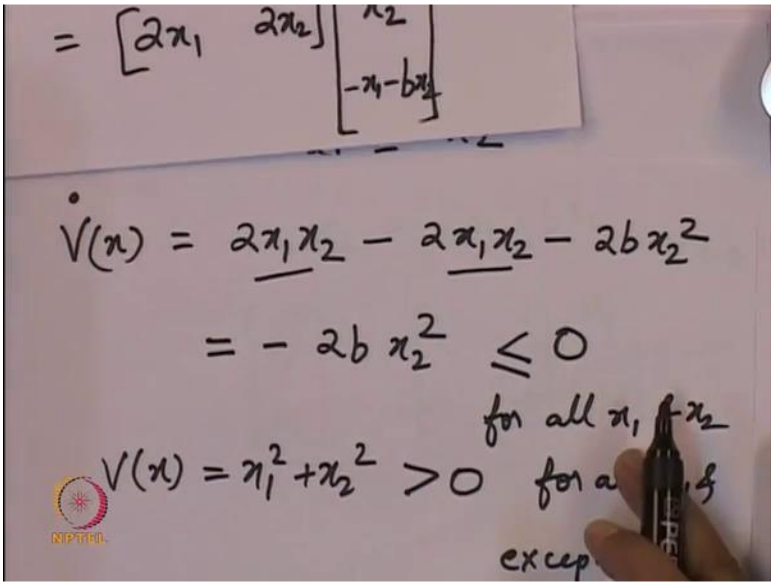

So, upon evaluating this particular we get 2 x 1 x 2 minus 2 x 1 x 2 minus 2 B x 2 square this is what we get as equal to V dot of x. So, we can cancel this and this and this is nothing but minus 2 b x 2 square by since we have assumed that b is positive. This particular quantity is less than or equal to 0 for all x 1 and x 2 it turns out that x 1 does not appear at all in the definition of this in V dot of x. But, in any case small x 1 x 2 is less than or equal to 0 and also we forgot to verify that V of x itself which is equal to x 1 square plus x 2 square is greater than 0.

But, for all x for all x 1 and x 2 except of course except 0 comma 0 except when both components are equal to 0 except for that that case this Lyapunov this particular function is positive. So, in other words it is a positive definite function that is why it is a candidate and we have in fact checked that its rate of change is also non positive. It is not equal to 0 for all x 1 x 2 and hence the equilibrium point for that particular dynamical system is a stable equilibrium point that is all we have been able to show.

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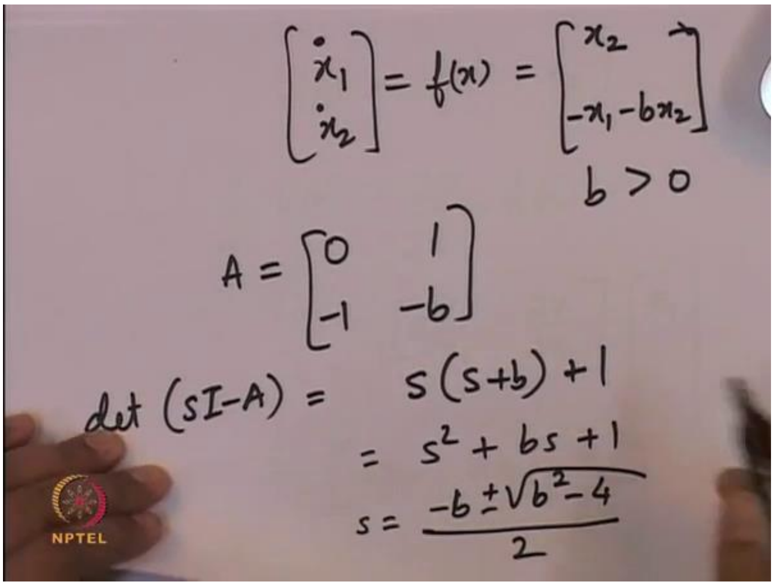

So, what have we shown that this particular function x 1 dot x 2 dot is equal to f of x given by x 2 minus x 1 minus b x 2 for some b greater than 0. This particular system's equilibrium point is the point s x 1 and x 2 equal to 0 both equal to 0 the origin that equilibrium point. We have shown is stable, because we have demonstrated that there is one Lyapunov function whose rate of change is always non positive. So, does that mean that this is not asymptotically stable, let us check that matrix a we had got was 0, 1 minus 1 minus b. Now, when we do s I minus A determinant we get s times s plus b plus 1 equal to s square plus b s plus 1.

So, here what are the roots of this particular equation these are some two points whose product is equal to plus 1, in other words they both have the same sign. Moreover, their sum is equal to minus b, so the roots are nothing but s is equal to minus b plus minus b square minus 4 A c b is nothing but d b square minus 4 over 2. So, the sum of the two roots is equal to minus b and the product is equal to 1, so these are some two points on the unit circle.

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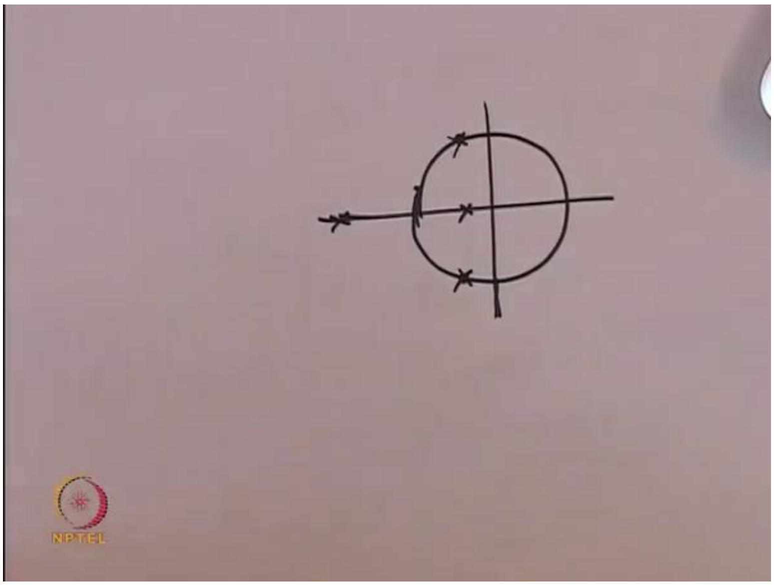

So, how do the two roots look these are two points on the unit circle, so they are to be conjugate pairs their product is equal to 1 and their sum is equal to minus B because the sum is negative. They have to be on the left half of the complex they cannot be on the imaginary axis also because then the sum would be equal to 0. So, we know that the matrix has Eigen values in the left half complex plane as a result they are either here or here.

They are here and here such that the product is still equal to one depending on the value of d, so here two points which are both in the left half complex plane and hence we know that the origin is asymptotically stable. But, our Lyapunov function candidate only helped us to prove that the origin is stable we were not able to use that particular candidate. So, to show that the origin is asymptotically stable we will resolve these issues in the next few lectures, but it is important to note that the Lyapunov theorem is only a sufficient condition.

If we were able to find out a V that satisfies those conditions then we can go ahead and compute something about stability or asymptotic stability, if V does not satisfy those. Perhaps, we should spend those efforts on finding another V or there is a possibility that the equilibrium point is indeed unstable for that also. We will see some conditions on V; these are the things that we will cover in the following lecture.

Thank you.