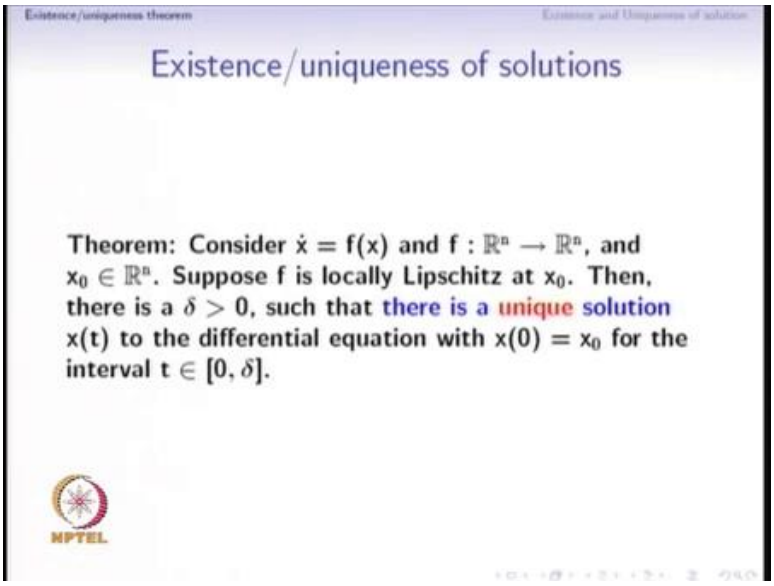
Nonlinear Dynamical Systems
Prof. Madhu. N. Belur
Prof. Harish. K. Pillai
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture - 05
Existence / Uniqueness Theorems

Welcome everyone to the fifth lecture on non linear dynamical systems. In the previous lecture we had seen existence and uniqueness theorems for solution to a ordinary differential equation with a given initial condition. So, let us just quickly see that theorem again.

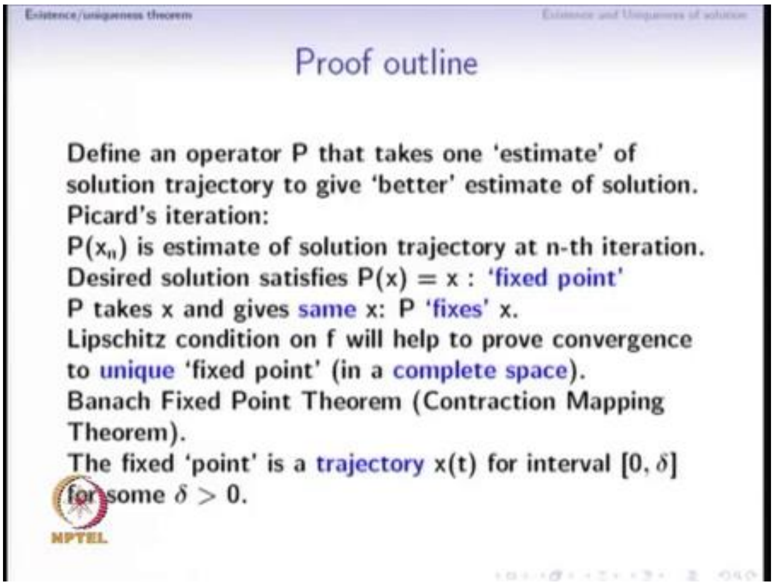
(Refer Slide Time: 00:34)



So, consider the differential equation x dot is equal to f of x and f is a map from R n to R n and x naught is a vector in R n which is specified as the initial condition. So, suppose f is locally Lipchitz at the point x naught then the statement of the theorem is then there is a delta greater than 0 such that there is a unique solution x of t to the differential equation x 0 is equal to x naught for the interval t 0 to delta.

So, there are two important statements in this theorem. First is there is a solution x of t to the differential equation this is the existence part. Second, there is a unique solution that solution which is guaranteed to exist is also guaranteed to be unique for an interval 0 to delta. So, beyond time delta there is no guarantee either of existence or uniqueness that requires preyed condition on f.

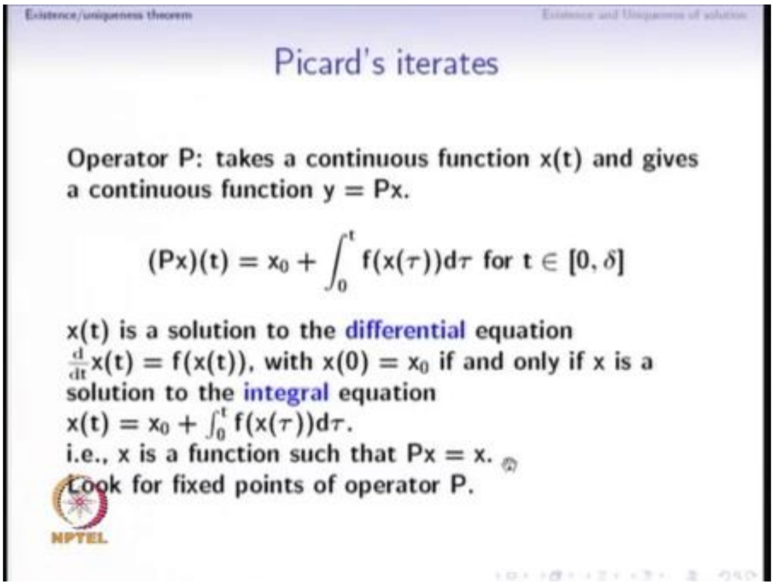
(Refer Slide Time: 01:37)



So, we had just begun seeing the proof, so what is the outline e the proof we define an operator P that takes one estimate of solution trajectory to give better estimate of the solution. So, this is what we call the Picard's iteration, so P x n is estimate of the solution trajectory at the n-th iteration and desired solution. We, will consider the operator P such that the desired solution is satisfied P of x equal to x, so this particular trajectory x which satisfies P of x equal to x. So, we call this trajectory the fixed point why? Why do we call it the fixed point? Because P takes x and gives the same x the operator P will be constructed.

So, that the fixed point is precisely the solution to the differential equation the Lipchitz's condition on f will help to prove convergence of this operator P that takes one estimate and gives a better estimate upon each iteration. So, the Lipchitz's condition will help to prove convergence to a unique fixed point also in a suitably complete space. So, for that purpose we just saw the statement of the Banach fixed point theorem which is also called the contraction mapping theorem. So, a fixed point in our situation is a trajectory x of t for the interval 0 to delta for some delta greater than 0.

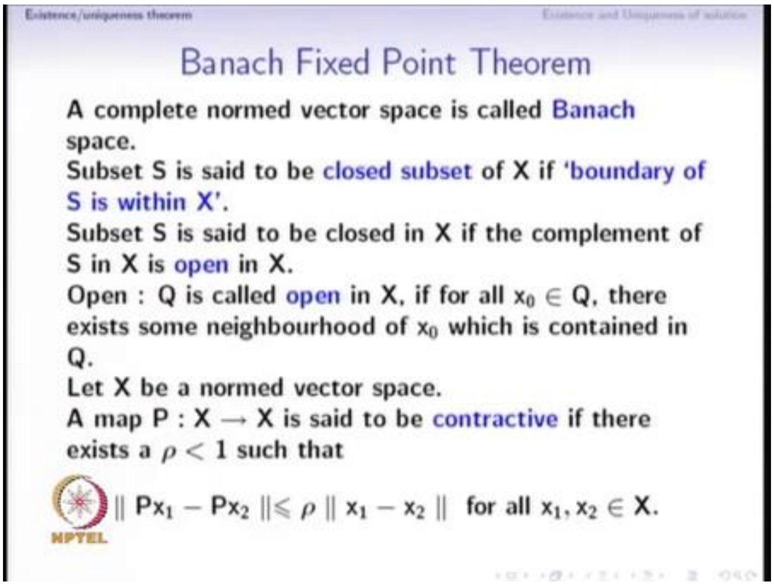
(Refer Slide Time: 03:04)



So, what are the Picard's iterates so the operator takes a continuous function x of t and gives another continuous function y is equal to P of x. So, we define this operator P as like this P of x is a function of time a value of P of x at time t is equal to x naught plus the integral from 0 to t of f of x tau d tau. This is the definition of P of x at time t where t varies over the interval 0 to delta, so we already saw that x of t is a solution to the differential equation x dot is equal to f of x. With this initial condition, x 0 is equal to x naught if and only if x is a solution to the integral equation x of t equal to x naught plus integral 0 to t f of x tau d tau.

Notice that x occurs on both sides of this equation this is an integral equation, similarly the differential equation also x occurs here and here. So, this solution to the differential equations and solutions to the integral equations are the same, moreover this integral equation we have seen the equation also allows us to say that x is a function such that it is a fixed point of the operator P.

(Refer Slide Time: 04:27)

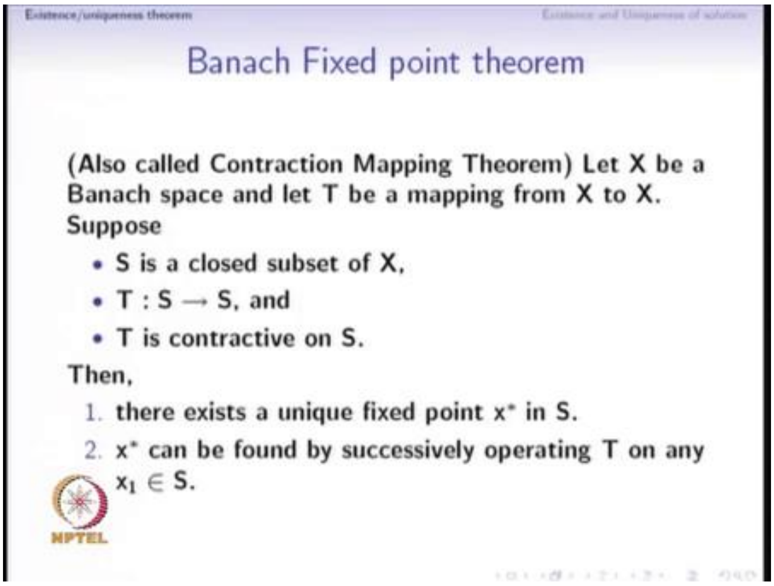


So, in this condition we began seeing the Banach fixed theorem, so we for that purpose we saw the definition for normed vector space and the notion of complete. So, a notion of a complete normed vector space we called the Banach space, a subset as of a set x is said to be a closed subset of what we called the boundary of the set S is also, sorry there is mistake here.

The boundary of S is within S, so it is required to have S in place of x here, but a more precise and correct definition is a subset S is said to be closed in x if the complement of the set S in the set x is open in x. So, this brings us to the definition of S, so when do we call a set open q which is the subset of x is called an open set. If for every x naught for every point x naught in q, there exist some neighborhood of x naught which is also contained in q. So, let x be a norm vector space a map P from x to x is said to be contractive if there exists some real number row strictly less than 1.

So, some positive number row which is strictly less than one such that this particular inequality satisfied for all x 1 x 2 and x we required the notion of contractive in the definition of the, in the statement of the Banach fixed point theorem. So, for that purpose, we are reviewing this definition even though an operator P may be defined from x to x it may turn out to be contractive over only a subset S. Now, that is the situation where Banach fixed point theorem is able to conclude about a fixed point.

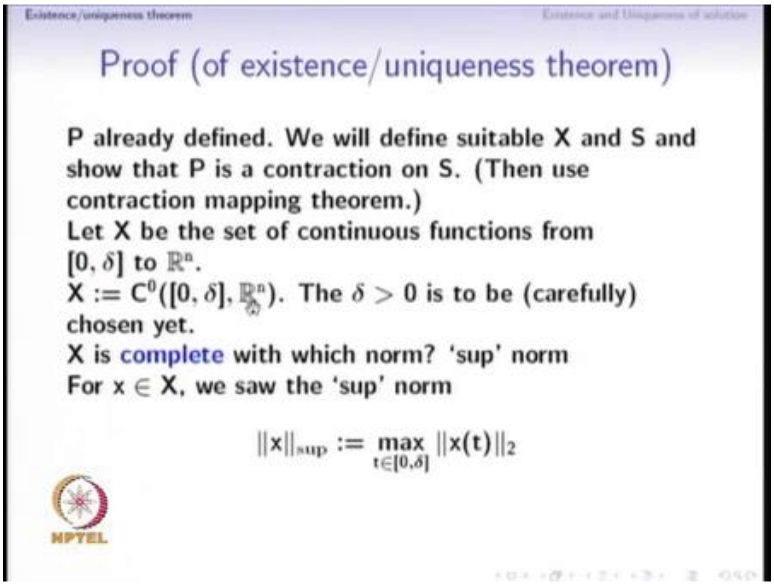
(Refer Slide Time: 06:20)



So, what is the contraction mapping theorem what is the Banach fixed point theorem, let x be a Banach space and let t be a mapping from x to x. Suppose x suppose S is a closed subset of x and suppose t is a map which also takes x into S, S is a subset of x this t which takes x into x need not take subset S into subset S. But, suppose that also satisfies this property that it takes S into S and further on S, T is contractive which means that there exists a number row such that inequality.

We saw on the previous slide holds for all x 1 x 2 in S if these three properties are satisfied then the contraction mapping theorem says there exists a unique fixed point x star in S. So, this statement has two important claims first is it claims that there exists a fixed point, second it also claims that this fixed point is unique in S. The next important statement of the theorem is this x star can be found by successively operating t on any initial x 1 in S we take any initial point x 1 in S and make t act on x 1. Then we make t act on t of x 1 and when we do this successively then it will converge to x star the unique fixed point in S.

(Refer Slide Time: 07:53)



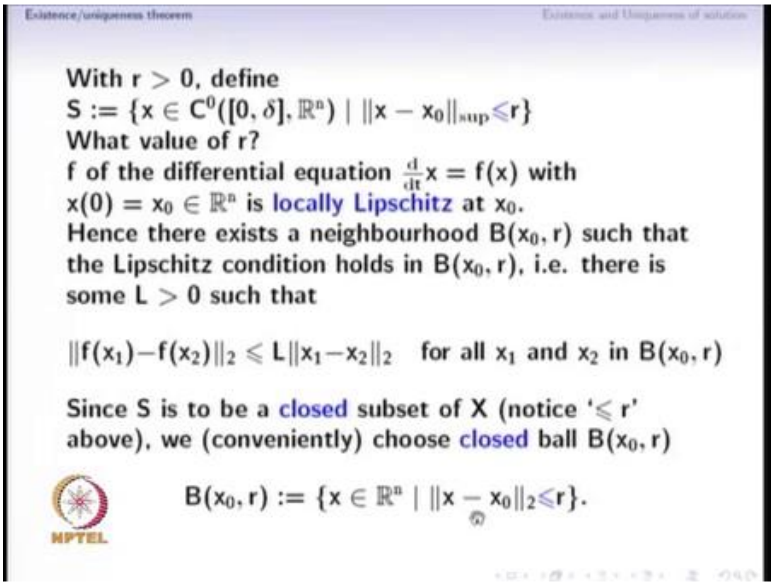
So, how do we use the contraction mapping theorem for the proof of existence and uniqueness we already defined the operator P. We will now define a suitable x and S and the subset S, and we will show that this operator P we already defined is a contraction on S. Then we will use the contraction mapping theorem, so what is this x, so x we will define is a set of all continuous functions from this interval 0 to delta to R n.

So the notation for x is C 0 from this domain to this co domain R n this 0 means that it is required to be just continuous it could be differentiable twice differentiable that is an extra property. But, we are asking for all functions that are at least continuous and, hence this 0 appears here, so for what interval it is defined from 0 to delta the time duration that delta is to be carefully chosen yet. Now, we can ask the question is x complete with respect to some norm after all for the contraction mapping theorem we require a Banach space x which norm is x complete.

So, we already saw that for a point x in X, we saw the sup norm for this space of functions for this space of continuous function. We define the sup norm as the maximum as t varies in the interval 0 to delta of the Euclidean norm of x of T at any time T x of t is a vector in R n.

We can take the conventional two norm the Euclidean norm and this Euclidean norm itself is a function of time and we will see what is the maximum of that norm function as t varies from 0 to delta. That is called the sup norm, it is also called the max norm, so we already saw that with respect to this sup norm this space of continuous functions on this Interval to R n is a complete normed space.

(Refer Slide Time: 10:10)



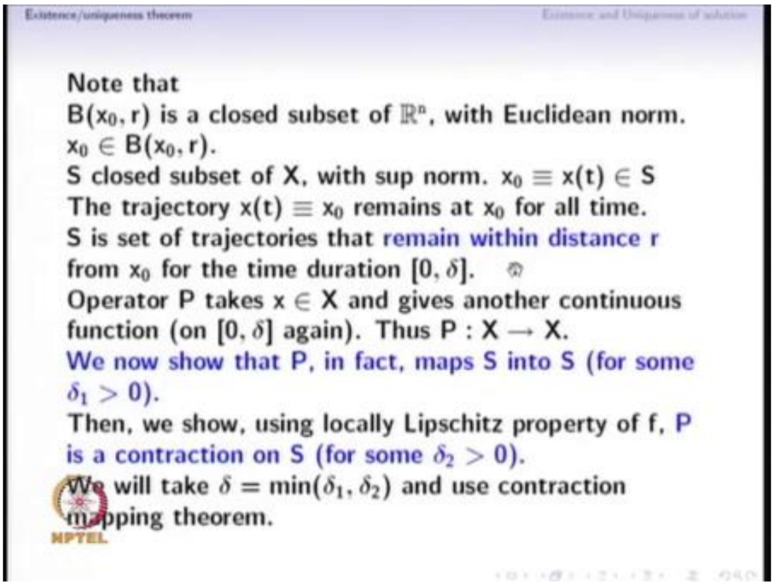
The next important property was the next important requirement was to define the closed subset S. So, we take some r greater than 0 and we define the set S of all this contingency functions in this particular set which satisfy the property that x minus x naught. So, x naught here is actually just a vector, but we also think of it as a function we will this in more detail. But, the distance from this x naught is the supremum of the distance from this as t varies from 0 to delta is at most R.

So, we take all those continuous functions which satisfy this sup norm condition and we pick these functions and put them into the set S how do we chose the value of R for this definition of S. So, we have differential equation f in the differential equation d by d t x equal to f of x we already are given that f if locally Lipchitz at the point x naught.

So, what is the significance of x naught x at time t equal to 0 is equal to x naught, so because it is locally Lipchitz, we know there exists a neighborhood B the ball B centered at x naught and of distance and of radius equal to R. So, this closed ball we will very soon define it to be a closed ball, we know that because x is locally Lipchitz at point x naught. There exists such a ball such that the Lipchitz condition holds inside this ball to say that it that the Lipchitz condition holds means that for all x one and x two inside this ball this inequality satisfied.

So, we pick this R from the locally Lipchitz property of the function f also this way we have defined is a closed subset of S. It is a closed subset because of this inequality being a non strict inequality, hence notice that less than or equal to R about. Now, for the same reason we will conveniently choose the ball b x naught, R as the closed ball, so b x naught comma R is defined to be a set of all x such that the distance from x naught is at most equal to r.

(Refer Slide Time: 12:44)



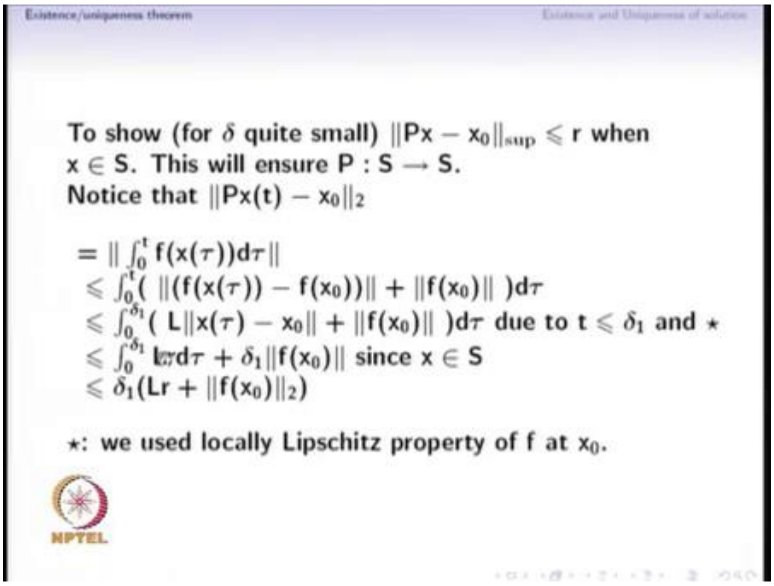
So, before we go further in the proof let us quickly note that b x naught comma R is the closed subset of R n with Euclidean norm and the point x naught is a element of the closed ball it is inside this center of this ball. On other hand, S the subset S is the closed subset of x the space of continuous functions over the interval 0 to delta and this space x has the sup norm because we are dealing with two types of norms. Here, one the norm over R n the Euclidean norm and the other a norm over x the sup norm because we are dealing with these two norms. It is very important to be careful about which norm at each place we user the norm function, so for this subset S we have this particular function x of t which is always equal to x naught.

So, always equal meaning as time t varies from 0 to delta x of t is the constant function it is equal to x naught, so this constant function is also an element of the set S. So, what is the meaning of that the trajectory x of t is always is equal to x naught, it remains at x naught for all time t. So, what is the set S is, S the set of trajectories that remain within distance R from the point x naught for the time duration 0 to delta. So, what does the operator do operator P do x and X and gives another function again on the interval 0 to delta.

So, as we see a map of P from x to X, we now show that in fact t maps S into S for some delta. So, take this small notice that we had delta as some number that was to be chosen yet so for delta suitably small it will show that P maps not just x into x. But, in fact S into S then we will use the local Lipchitz property of the function f to show that P is in fact a contraction on S. Again, for a sufficiently small delta this we will call delta 2 greater than 0 and once we have these two conditions delta 1 and delta 2, one which ensures that P maps S into S.

But, another which ensures that P is a contraction on S we have to define the delta Equal to minimum of the 2 minimum of delta 1 and delta 2 and since delta 1 and delta 2 are both positive the minimum of the two will be a delta. So, that meets the conditions in the theorem for this delta we will use the contraction mapping theorem.

(Refer Slide Time: 15:40)



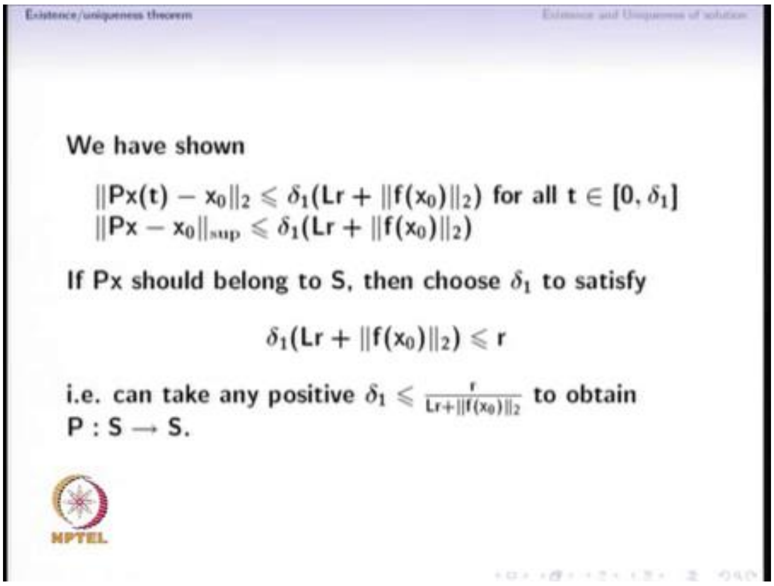
So, the first part of the statement wad to show that for delta quite small P x minus x naught sup norm is less than or equal to R where S is in when x is in S. Now, to show this particular inequality we will imply that P takes an element x in S and gives you a function which is also in S. So, why does it give a function again in capital S because the distance of P of x from x naught from the constant function x naught in the sup norm is at most R. Now, if we show this will ensure that P is a map from s to S, so in order to show this what is that P of x of t minus x naught the 2 norm is equal to this.

So, once we take the norm function inside the integral sign it turns out that this right hand side will become larger. So, this inequality this norm of integral 0 t, t of x of tau d tau is less than or equal to integral 0 to t of this whole thing inside the brackets notice that f of x naught. While doing this particular quantity can increase because of the triangular inequality, now what we will do we will integrate not just from 0 to t. But, from 0 to delta 1 after all t is some number at most delta 1, so if we integrate this positive quantity up to delta 1 it is only going to become larger.

Once we do this, we will also use the Lipchitz property of the function f and replace the first term in the norm with x tau minus x naught times l and because f is locally Lipchitz at the point x naught. This is satisfied for all x and x naught inside the ball this other quantity we just leave as it is. So, we have used the Lipchitz the locally Lipchitz property of f, since x of tau minus x naught is at most equal to R why because the function f is inside S.

Hence, this particular quantity is at most R, so we have replaced x of tau minus x naught in the two norm by R that is the maximum distance. It can be away from x naught and the second quantity because it is integral of a constant we have removed f of x naught and replaced integral of d tau by delta 1. Finally, we see that this particular quantity which we are integrating this is also constant it is not varying as a function of tau. Hence, we called we equated that to delta 1R and after taking delta 1 common we obtained this expression.

(Refer Slide Time: 18:47)

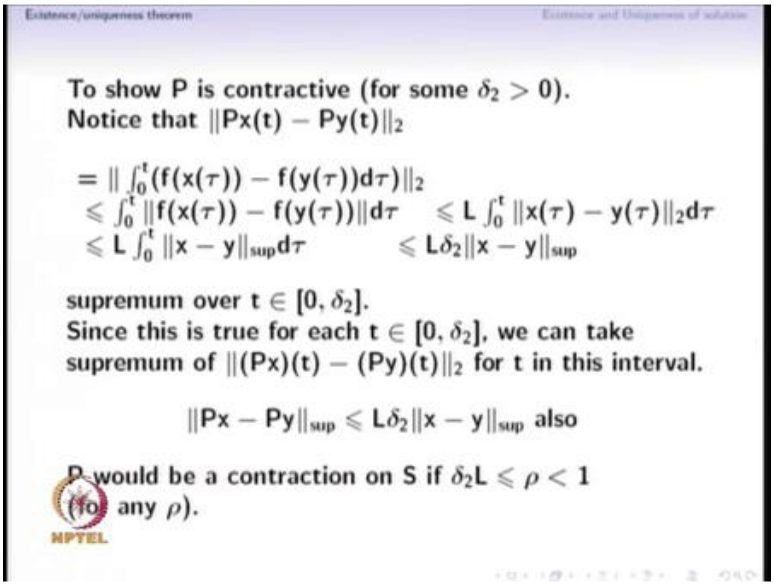


So, what have we shown we have shown that the two norms of this particular function at any time t is bounded from above by this quantity and what is that on the right hand side there is no t. So, for all time t the left hand side which depends on time t is bounded from above by this particular number which does not depend on time t.

So, in fact if we take the supremum on the left hand side even the supremum will be bounded from above by the same quantity. So, what does this show that this particular P of x minus x naught in the sup norm is at most equal to this, now we will choose delta 1 such that P of x belongs to S. So, if P of x should belong to S then choose delta 1 to satisfy delta 1 times l R plus f of x naught to norm is at most equal to R.

If we choose this delta 1 such that this is satisfied then we see that P of x minus x naught in the sup norm is bounded from above by R and hence P of x goes into S. So, we can take any positive delta 1 that is less than or equal to R times R divided by this quantity and we will then get that P max S into S we can take delta 1 equal to this.

(Refer Slide Time: 20:18)



The next important step was to show that P is a contraction on S, so for some delta 2 greater than 0 which we will carefully choose. Now, we will show that P is contractive for this purpose what is P of x t minus P of y t into norm is equal to the norm of this. From the definition of the operator P, we see that we obtain this and when we take the norm inside the integral sign. Then we get that this is at most equal to this by using the locally Lipchitz property of the function f inside the ball b of x naught, R. We see that this quantity is bounded from above by this after taking the l outside this integral sign.

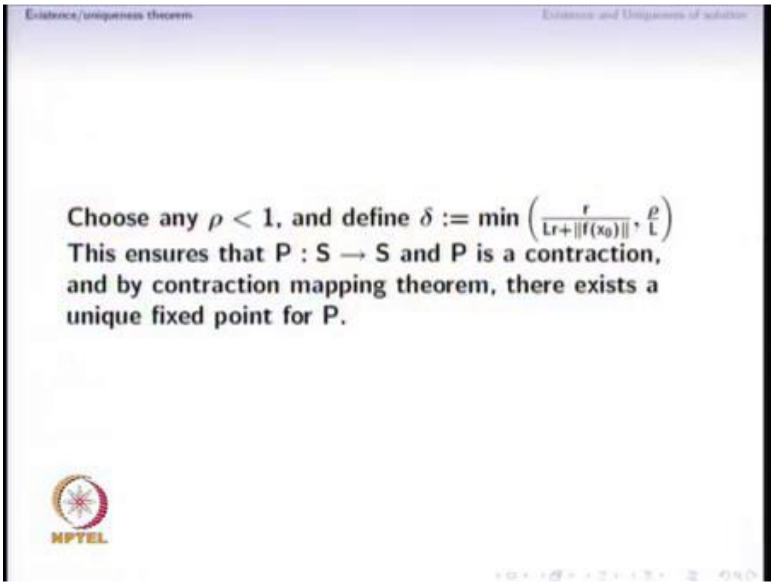
Moreover, this particular quantity we have written here is at most equal to this why because x and y both at any time t we can take the difference between them in the two norm and integrate them. But, instead of taking at any time tau we could also look at the maximum difference between them and this maximum difference is only going to be larger. Hence, we have obtained at this particular inequality is less than or equal to this particular quantity by replacing a sup norm.

Here, by replacing the two norms there with the sup norm, here this quantity can only become larger and hence this inequality not equal. Finally, this quantity which we are integrating it is over the interval 0 to t, but we could go ahead and integrate up to delta 2, this quantity because it is a norm it cannot be negative. But, when we integrate further instead of time t only up to delta 2 then we see that we get l times delta 2 times sup norm of x minus y. So, here the supremum is being taken as t varies from 0 to delta 2 and, here also the sup norm was being taken as t varies from 0 to delta 2.

So, what have we obtained they have obtained that P of x of t minus P of y at time t the difference norm of that the two norm of that is bounded from above by some number, by some quantity that is independent of time t. So, this is true for each time t I n the interval 0 to delta 2 and hence we take the supremum of this quantity, even the supremum will be bounded from above by the same number L times delta 2 times sup norm of x minus y. So, finally we have obtained this inequality sup norm of P x minus P y is at most equal to L times delta 2 times sup norm if x minus y.

So, this should give us a hint as to how to choose delta 2, so that P is a contraction on S that was the objective of doing this inequality. So, that would just be a contraction it would be a contraction if this particular quantity l times delta two is strictly less than 1. So, if you set delta 2 times L equal to some number rho and that number rho is strictly less than 1 then we will obtain this contraction on S.

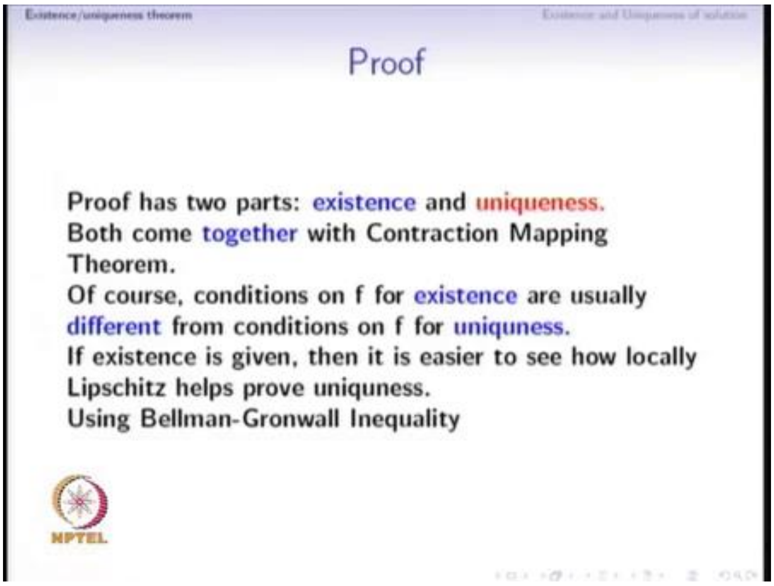
(Refer Slide Time: 23:57)



So, finally we do as follows choose any role strictly less than 1 and define delta to be the minimum of these two quantities. So, notice that this we had called as delta 1, the second one we had called as delta 2 and the minimum of these quantities when we take that as delta it will ensure both, it will ensure that P is a map from S to S. It will ensure that P is a contraction and once these two are guaranteed by the contraction mapping theorem.

We know that there exists a fixed point in S for the operator P and, moreover there exists a unique fixed point at P unique fixed point inside the subset S.

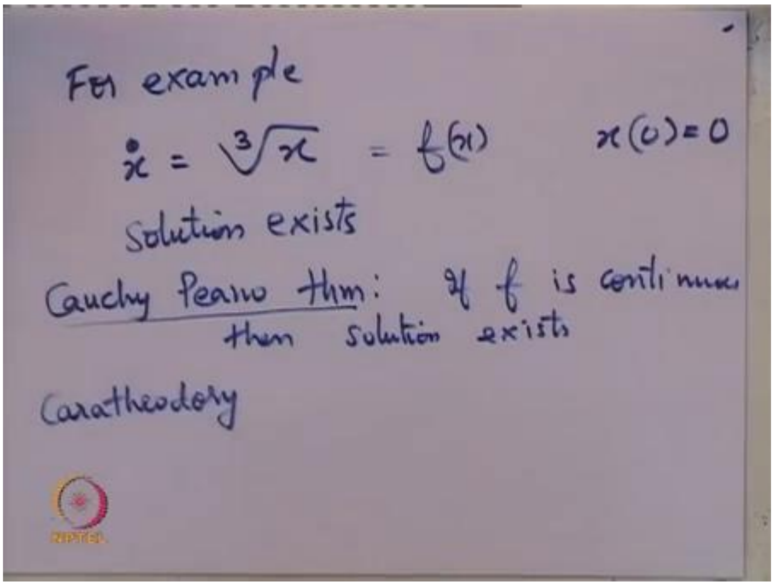
(Refer Slide Time: 24:41)



So, this proof has 2 parts, so this completed the proof just a small discussion about the proof it has two parts, one about the existence and one about uniqueness. So, notice that both come together with the contraction mapping theorem the contraction mapping principle assures us both existence and uniqueness.

But, of course in general the conditions on f for existence of a solution to the differential equations are different from conditions on f. So, for uniqueness of the solution to the differential equation these conditions are usually different and suppose the existence is given. Suppose due to some particular property on f it turns out that we have a particular solution to the differential equation.

(Refer Slide Time: 25:32)



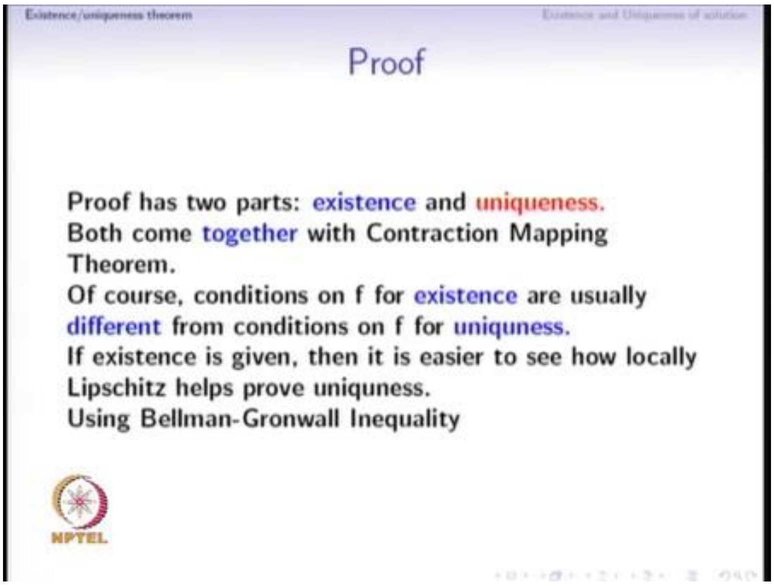
For example, for this particular differential equation solution exists for the claim that a solution exists we are not able to use the theorem of existence and uniqueness. But, why we are not able to use because the theorem for existence and uniqueness requires this particular function f of x to be locally Lipchitz at the point of initial condition. Suppose the initial condition is equal to 0 then we already saw that this particular function is not nor locally Lipchitz at x equal to 0.

Hence, we are not able to utilize the theorem however we know that the solution exists why we are able to conclude that the solution exists because of certain other properties of the function f. Now, for example there is a Cauchy Peano theorem that says that if f is continuous then solution exists for certain situations it is possible that we are not interested in uniqueness. But, we are interested in just existence of a solution because of which locally Lipchitz property on of the function f might be too severe might be too harsh. So, function f may not be locally Lipchitz because of that hat particular theorem we are not able to utilize to claim existence and uniqueness.

But, just existence might also come under milder conditions on f, so Cauchy Peano theorem is one of the various statements that relaxes the conditions on the function f and at least gives us existence. So, it says that if a function is continuous at the point x naught then a solution exist over a small interval 0 to delta there is also another result.

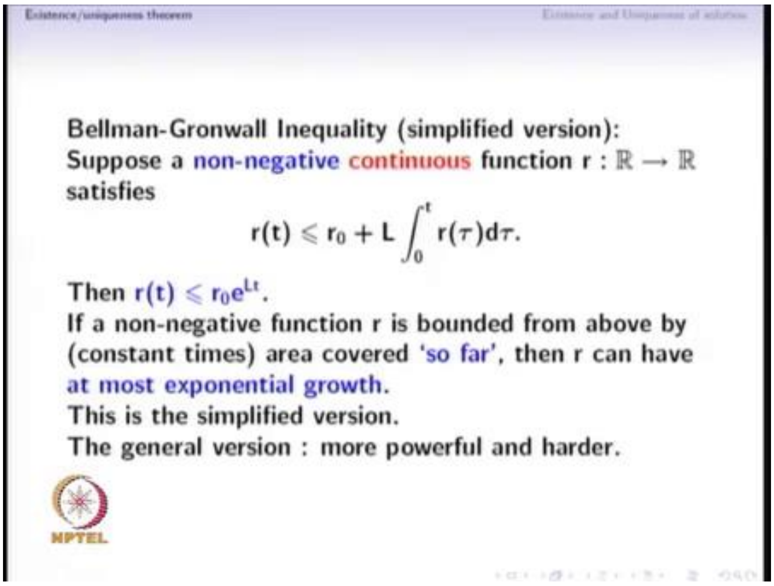
There is a result by Caratheodory which says that which is also under milder conditions on the function f not even continuity it turns out that f is not even continuous it is still possible that a solution exist. But, then we do not go into that that is also said to be a solution in the sense of Caratheodory. So, in general it is important to keep in mind that the conditions for existence and the condition for uniqueness are not usually the same.

(Refer Slide Time: 28:16)



Hence, if existence is given by some other particular by some other property then it might be easier to show that locally Lipchitz ensures uniqueness of the solution. So, one of the ways to prove uniqueness of the solution under assumptions of existence of solution to the differential equation is by using the Bellman Gronwall inequality, this is the result we will see now.

(Refer Slide Time: 28:44)



So, what is the Bellman Gronwall inequality we will see only the simplified version suppose a non negative continuous function r that takes R to R that takes real values, and gives real values. Suppose this non negative continuous function R satisfies that r of t is at most R naught plus L times the integral 0 to t r of tau d tau. So, what is on the left hand side this is like an integral inequality, R appears on both sides R appears here and also here.

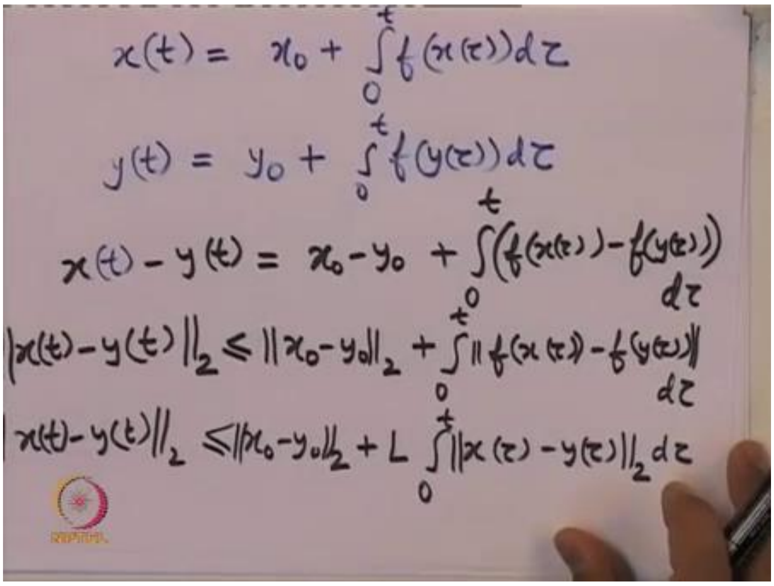
So, R at any time t is at most some constant plus another constant L times the integral from 0 to t of the same function r if r is continuous non negative function that satisfies this property. Then R of t is at most equal to R naught times e to the power L t, so notice that if we have a function R that satisfies this integral inequality in which R appears in both sides.

Then we want to make a claim about R being bounded from above by this other function that is now not depending on the right hand side. So, r now appears on the left and side, so what is this Bellman Gronwall inequality say it says that if a function R is bounded from above by some constant times the area covered so far this. But, so far it has been included to indicate that the integral from 0 to t of that same function r of tau that is the area covered so far.

Now, if r is bounded from above by such a constant time S the area covered so far then r can have at most exponential growth this is the simplified version of the Bellman

Gronwall inequality. The general version is much more powerful and harder to both understand and prove, so we will use the Bellman Gronwall inequality to prove that if we assume the existence of a solution to the differential equation. Then locally Lipchitz property of the function f in fact proves uniqueness of the solution to the differential equation. So, this also helps us to look into sensitivity of the solution to the differential equation to initial condition.

(Refer Slide Time: 31:17)



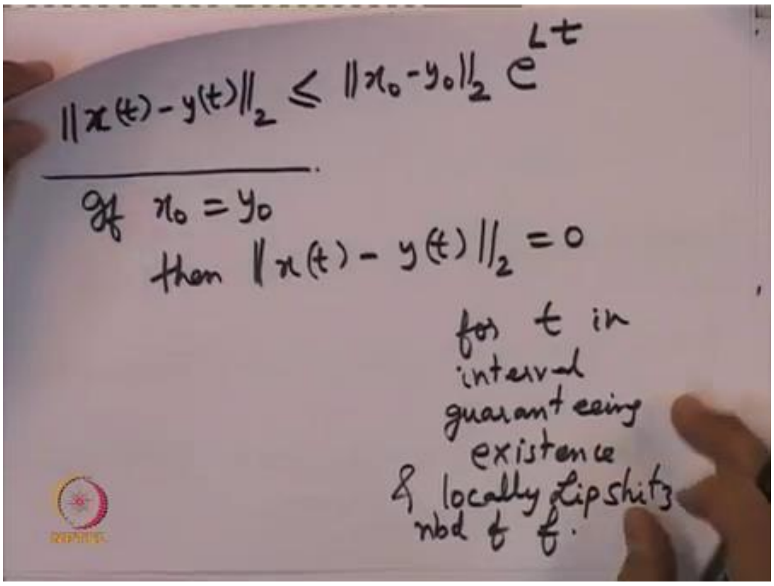
So, consider this solution the first solution to the differential equation is guaranteed to exist then we know that this equation, this is the integral equation. Suppose another solution y of t is solution to the integral equation with this initial condition. Now, we can ask of x naught and y naught are close by does it mean that x t and y t are also close by, so let us just take the difference.

Now, let us see what happens to the distance x of t minus y of t is less than or equal to by the triangular equality x naught minus y naught to norm plus integral from 0 to t of f of x tau minus f of y tau d tau. Now, using the locally Lipchitz property of the function f we can simplify this particular term in this integral.

So, this is less than or equal to x naught minus y naught plus l times integral from 0 to t x tau minus y tau all the norms appearing in this page are the two norms. So, we see that x t minus y t the norm of that satisfies this particular inequality that kind of appears in the Bellman Gronwall inequality.

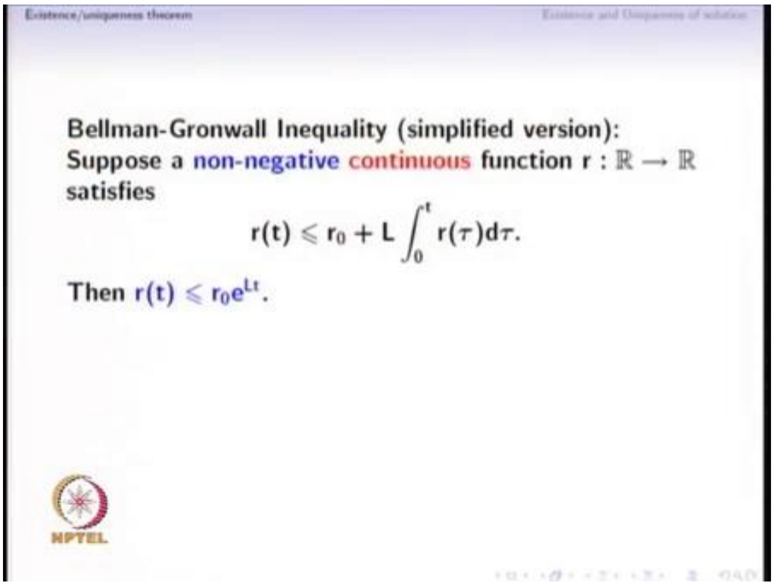
So, this non negative function norm is a non negative function it is also continuous both x and y are continuous because their integral of some function which is also continuous. So, f is locally Lipchitz it is also continuous and integral of this continuous function j is in fact differentiable and of course x and y are continuous. So, their difference is also continuous, so this non negative continuous function satisfy is bounded from above by a constant plus l times its own integral so far.

(Refer Slide Time: 34:27)



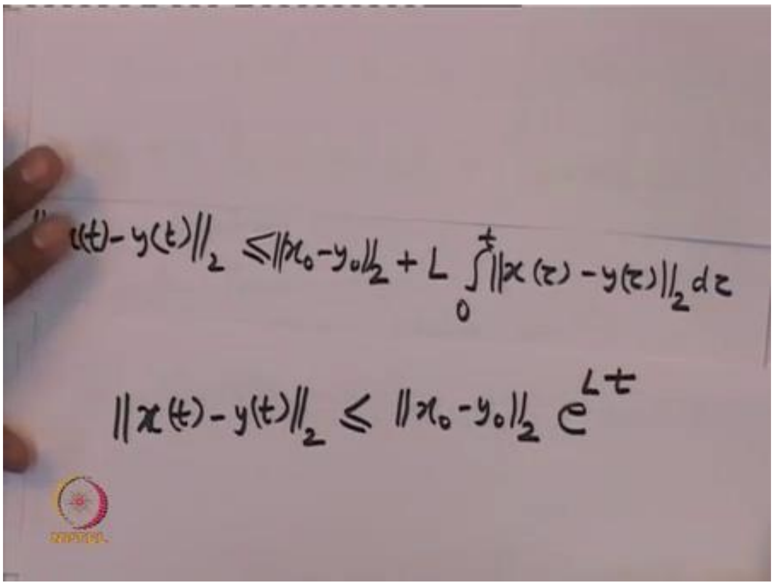
Now, we use the Bellman Gronwall inequality Bellman Gronwall result to say that because of the statement in the inequality in the Bellman Gronwall principal, these two norms is bounded from above by 2 times e to the power L times t, so let us just compare these two inequalities.

(Refer Slide Time: 35:05)



So, after having a look at statement of the Bellman Gronwall inequality we see that on the left hand side r of t non negative continuous function is bounded from above by this particular integral on the right hand side. So, R naught plus L time integral from 0 to t r of tau, if this is what r satisfies the non negative continuous function then r of t is bounded from above by this r naught, this constant times e to the power L times t.

(Refer Slide Time: 35:38)

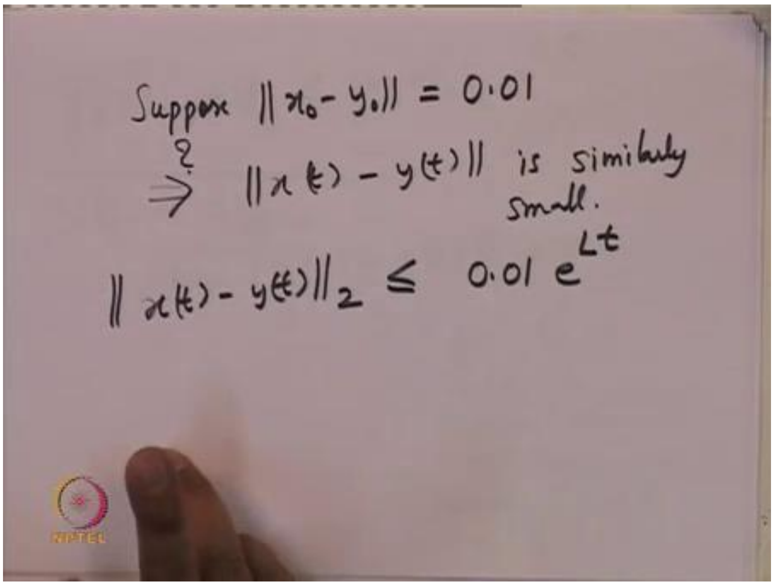


So, when we apply to this particular difference of two solution to the differential equation we see that x of t e to the power r of t is bounded from above by some constant x naught minus y naught 2 norm plus L times the integral of r of tau d tau. Now, this quantity which we are integrating here is same as the situation here of that is the case. Then by using the Bellman Gronwall inequality we are able to conclude that the x of t minus y of t two norm cannot be larger than x naught minus y naught 2 norm times e to the power L time t.

So, this helps us to conclude uniqueness how do we show uniqueness if x naught is equal to y naught then x of t minus y of t 2 norm equal to 0 for t in for t in what interval for t in interval guaranteeing existence. Now, t also should be restricted to an interval, so that the solution remains inside the domain of x where it is locally Lipchitz. But, locally Lipchitz neighborhood of f t should be restricted to a small enough interval where both these are satisfied. So, if t is sufficiently small then we see that this 2 norm is equal to 0 if the initial condition is same if the initial condition if the initial condition is same.

Then the difference in the trajectories is forced to be equal to be 0 why because this difference in trajectories is less than or equal to 0 times this number at the same time this inequality also tell us tells us to what extent the solutions are sensitive to the initial conditions. So, suppose we know that these two initial conditions are not same, but they were close by in other words the distance between them was equal to 0.01.

(Refer Slide Time: 38:07)



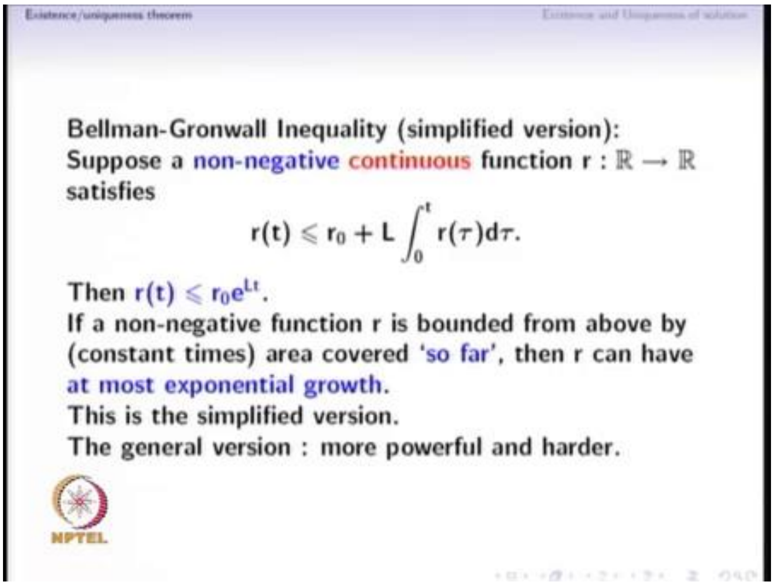
Suppose, x naught minus y naught 2 norm is equal to 0 point 0 one does that imply this is the question we ask in the context of sensitivity x t minus y of is similarly small. So, what is similarly small as small as this particular amount or at least of the order of this, so we know now that if the initial conditions are this much away. Then x of t minus y of t 2 norm is less than or equal to 0.01 times e to the power L times t, so for that duration we can compute e to the power L t.

Now, this is something some number that of course grows as t increase because L is a positive number it is a locally Lipchitz condition L that appears in a locally Lipchitz condition f. So, it is a positive number, so it is indeed a function that is growing, but we are now asking about sensitivity to initial conditions. So, if the initial conditions are close then the solutions at any time t are apart from each other, but at most this distance from the initial condition times this number.

So, the fact that this number becomes large is not the topic of discussion, now the topic of discussion, now is how sensitive is the solution to the initial condition. So, if the initial condition is order 0.01 apart then at any time t, x t minus y t is also order 0.01 apart, so this explains sensitivity to initial condition if L is small, in fact these both are not growing too fast are apart.

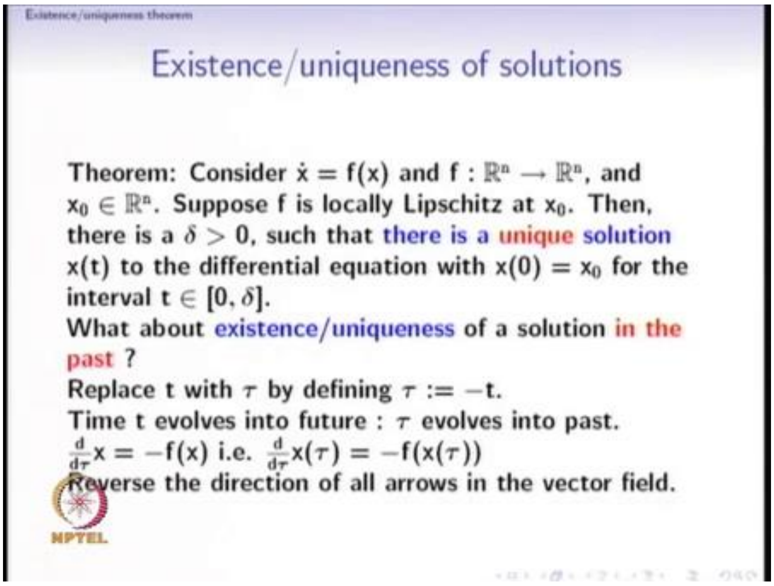
So, using this particular Bellman Gronwall inequality, if under some particular theorem we already have existence then we can see that locally Lipchitz property of f guarantees uniqueness also. So, why it guarantees uniqueness because if the initial conditions are close to each other then the solutions are also close in fact if the initial conditions are equal to each other then the solutions are equal.

(Refer Slide Time: 40:36)



So, this completes the proof of existence and uniqueness to the solutions of a differential equation, and also completes sensitivity of the solution to the differential equation sensitivity with respect to the initial condition after having finished the proof, this is a good moment to see a closely related topic.

(Refer Slide Time: 40:56)

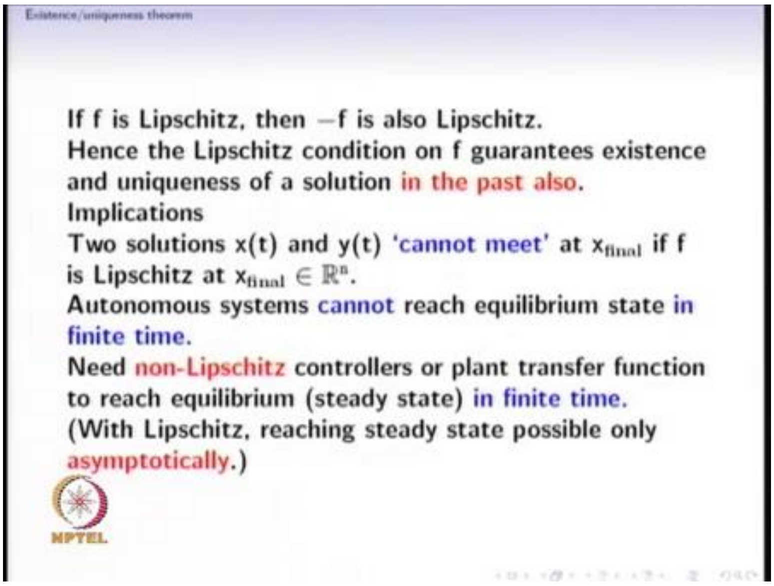


So, we saw the existence and uniqueness of solutions, so let us have a quick relook, so consider the differential equation x dot is equal to f of x where f is a map from R n to R n and at the initial condition x naught. Suppose x if locally Lipchitz then there is a delta greater than 0 such that there is a solution and there is a unique solution in fact x of t to the differential equation x 0 is equal to x naught for the interval 0 to delta. So, please note that we are starting from t equal to 0 to some delta greater than 0 see this is an interval in positive time for the future there is a solution.

So, a unique solution for some time in the future an important question is there a unique trajectory in the past, so what about existence and uniqueness of a solution in the past. So, for this particular issue we can easily modify our theorem to replace t with tau by defining tau equal to minus t, so as t evolves into the future tau evolves into the past.

So, the differential equation x d t is equal to f of x becomes d tau x d by d tau of x equal to minus f of x, in other words d by d tau of x of tau where x is a function of tau. Now, is equal to minus f at x of tau, so how does one obtain a vector field for this dynamical system we just reverse the direction of all the arrows. So, in the vector field of the differential equation x dot is equal to f of x why because each arrow is not f of x, but minus f of x.

(Refer Slide Time: 42:55)

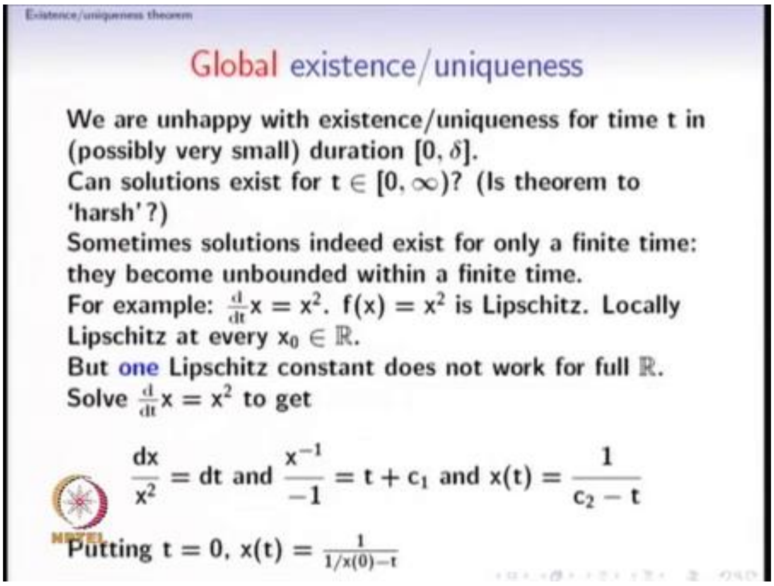


Now, see f is Lipchitz, notice that minus f is also Lipchitz, hence the Lipchitz condition on f guarantees existence and uniqueness of a solution in the past also. So, what are the implications of these particular observations with two solutions x of t and y of t cannot meet at x final, if at a point x final if f is Lipchitz at that point x final. If f is locally Lipchitz at x final then it is not possible that there are two past trajectories x of t and y of t which have the same final condition x final. Similarly, autonomous system everything that we have been doing so far is for autonomous system.

So, one of the properties that we can claim about autonomous systems is that the autonomous systems cannot reach the equilibrium point. But, the equilibrium stage in finite time why because whenever it reaches an equilibrium state that equilibrium state already had one past which was same point for all time. But, there cannot be another trajectory that comes and meets this equilibrium state is locally Lipchitz at this equilibrium state. So, if you want to have a particular system if you want to design a controller in steady state in finite time and remain there.

Then it would require non Lipchitz controller or plant transfer function to reach the equilibrium in this case we interpret the equilibrium as the steady state. If you reach the steady state in finite time then one would need either non Lipchitz controllers or non Lipchitz plant transfer function. But, why is that because with Lipchitz we can reach the steady state only asymptotically it is not possible to reach in finite time.

(Refer Slide Time: 44:51)



So, another important topic is we have been seeing only local existence and uniqueness conditions what is local about it we saw that there exists a solution. But, it is unique only for an interval 0 to delta even existence could not be guaranteed for large enough time, but it could be guaranteed only for a time interval 0 to delta. Now, all that was guaranteed was that delta was greater than 0, but it is possible that this delta is a very small value and we are unhappy with this result about the existence.

So, uniqueness for so small an interval possibly, so it is possible that can solutions exist over the interval 0 to infinity is it that the solutions indeed exist and they are unique. But, our theorem is not able to guarantee it is the theorem too harsh is it that it is assuming coming locally Lipchitz property on f, because of which we are able to guarantee existence. So, uniqueness only for a small interval 0 to delta, but there might be some other result some other way of proving that the solutions exist from 0 to infinity.

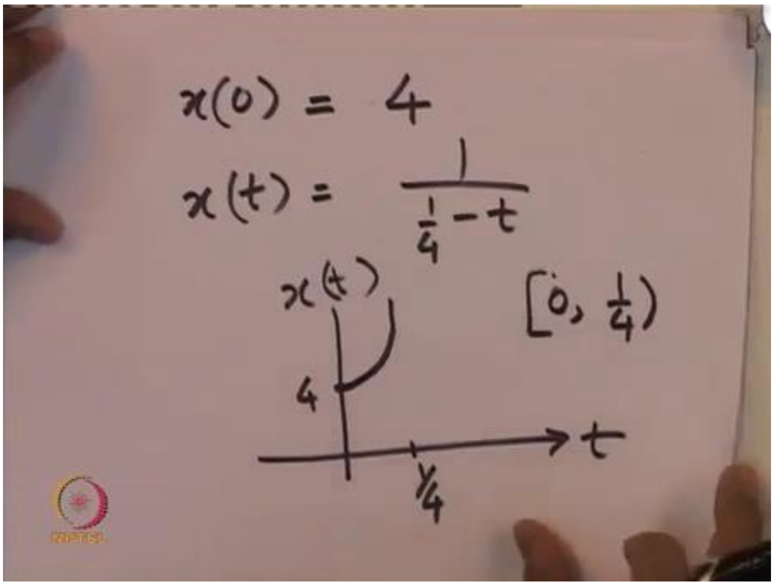
So, the conditions assumed in our theorem too harsh because of which they are able to prove only local existence and uniqueness for this we will see one small example. So, it is indeed true that sometimes solutions indeed exist for only a finite time, so our theorem can also accordingly claim existence and uniqueness only for a short interval. But, why would they exist only for a finite time, because it is possible that the solution becomes unbounded in finite time.

So, consider the differential equation x dot is equal to x square, so x dot is equal to x square means that f of x is equal to x square. So, notice that this is Lipchitz in fact it is locally Lipchitz at every x naught in r. So, please note that this dot here does not mean that is multiplication of x dot and f of x it is the end of a sentence x dot is equal to x square is a differential equation. Now, for this differential equation f of x equal to x square and this particular function f is locally Lipchitz at every point x naught.

But, notice that one Lipchitz constant does not work for the full r solve, so we can explicitly solve this differential equation x dot is equal to x square to get d x by x square equal to d t. Now, on integrating both sides we get x to the power minus 1 r divide by minus 1 equal to t plus some constant c 1 and upon rearranging this minus 1 and x of t.

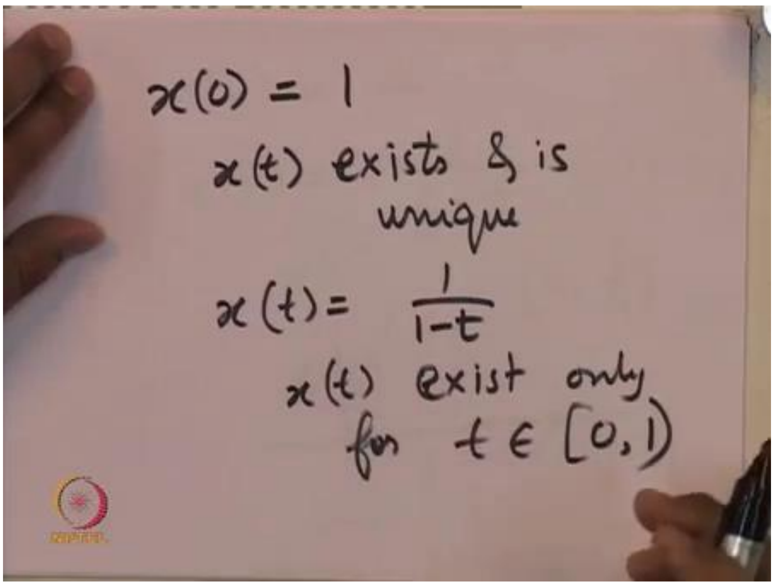
We will call minus c 1 equal to c 2 and we get x of t equal to 1 over c 2 minus t, so when we put the initial condition at t equal to 0. Suppose it was at x naught x 0 then when we substitute we get x of t equal to one over one over x naught minus t, so let us just make this a. So, our differential equation this is not how to the solution to our differential equation looks, so let us see what this means.

(Refer Slide Time: 48:21)



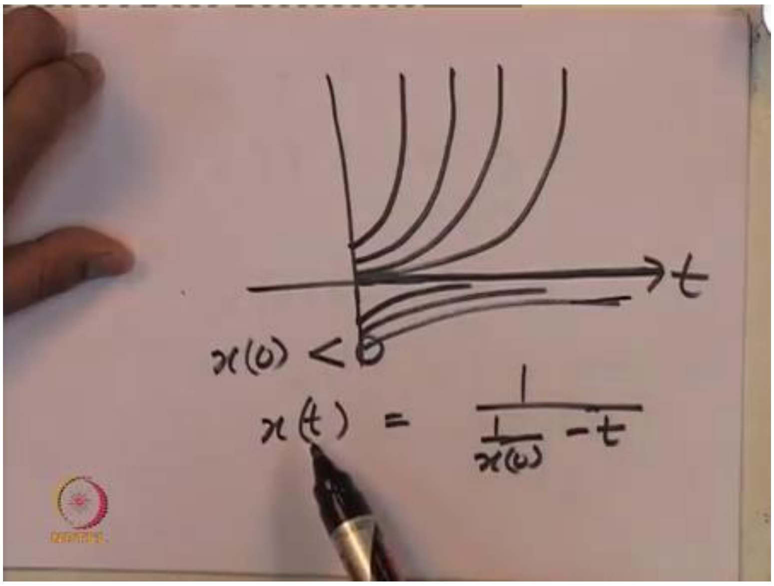
If x of 0 is equal to some number, let us say 4 then we see that x of t equal to 1 over quarter minus t. So, we see that for t equal to 0 of course it is equal to 4 and as t tends to 1 by 4 this quantity becomes unbounded, so a graph of x versus t x starts from 4 and it becomes unbounded. So, within a small interval up to 1 by 4 it is already so large that it is unbounded, so we have solutions defined only over for this particular initial condition. So, we are able to define existence of a solution only from 0 to 1 by 4, while it is a closed interval on this side it is an open interval for t equal to 1 by 4, we do not have a solution this solution does not exist.

(Refer Slide Time: 49:25)



So, what we have seen is when the initial condition is equal to 4 we had solution only up to 1 by 4, suppose the initial condition was equal to 1 then we have a unique solution for some delta. But, when we try to increase this delta we see that x of t is exists and is unique how long can we extend this. So, we see this that by explicitly solving the differential equation we get x of t equal to 1 over 1 minus t x t is defined exists only for t in the interval 0 to 1, 0 to 1 for this particular initial condition. So, for each initial condition it become unbounded in a finite time in how much time it becomes unbounded that time depends on the initial condition.

(Refer Slide Time: 50:29)



So, a solution, so a set of solutions to this differential equation this starts below the solution for some more time if it is at 0 of course it remains at 0 for all future time because it is locally Lipchitz at 0. So, the solution cannot emanate out of the equilibrium point there is a unique trajectory and hence it remains always at 0. But, if x of 0 is negative then what happens then x of t is equal to some number, some number 1 over x naught which is negative minus t. So, the solution always exists when it is negative then we see that the solutions are coming close by, so we see that if x of 0 is negative then the solutions exist for all future times.

But, they are not becoming unbounded in finite time and they are all approaching 0, but if x of 0 is positive then the solution grows and becomes unbounded in a very short amount of time in finite time. Hence, we cannot have global existence of solution when initial condition is positive but, it is it appears that we can have global existence of a solution and x of 0 is negative. So, it appears like for certain situations there exist solution from t equal to 0 to infinity while there are other situations for the same differential equation.

But, there are certain other initial conditions for which the solutions exist only for a finite amount of time in which case we cannot have global existence of the solution let alone global uniqueness. So, for this particular differential equation we might have some additional assumptions under which we might have unique solution from 0 to plus infinity. But, it is possible that for certain initial condition those conditions of the theorem do not hold in which case we do not have global existence. So, those additional conditions, how to formulate this? The topic we will see in the following lecture.

Thank you.