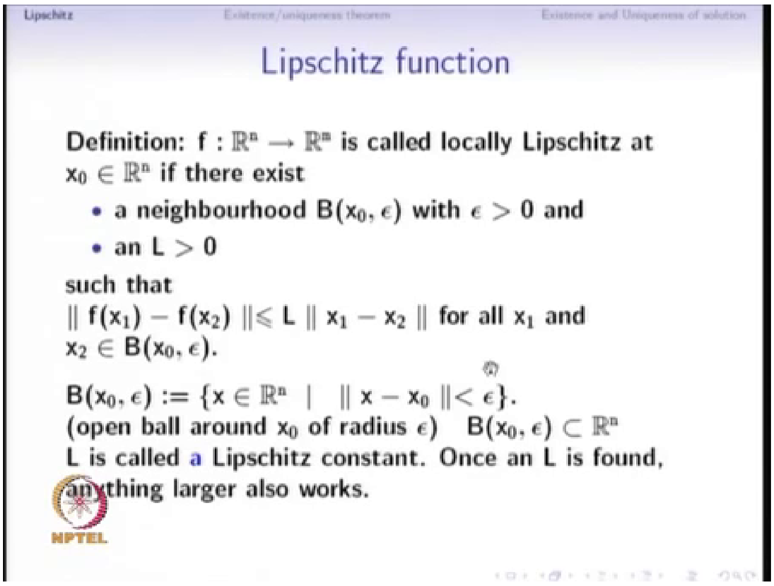
Nonlinear Dynamical Systems
Prof. Madhu N. Belur and Prof. Harsih K. Pillai
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture - 4
Lipchitz Functions

Welcome to lecture number four of Nonlinear Dynamical systems, we have just began seeing what a Lipchitz function is.

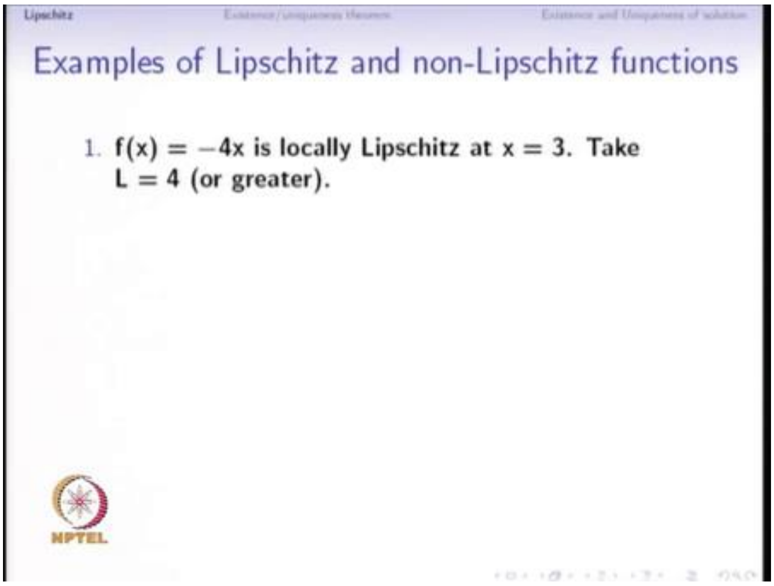
(Refer Slide Time: 00:26)



So, let us just recapitulate the definition, so function f from R n to R m is called locally Lipchitz at a point x naught in R n, if there exists a neighborhood B. A neighborhood here in this case is defined to be a ball centered around x naught with radius epsilon and the radius is greater than 0. There should exist a neighborhood and some constant l, such that this inequality is satisfied for all points x 1 and x 2 in that neighborhood of the point x naught.

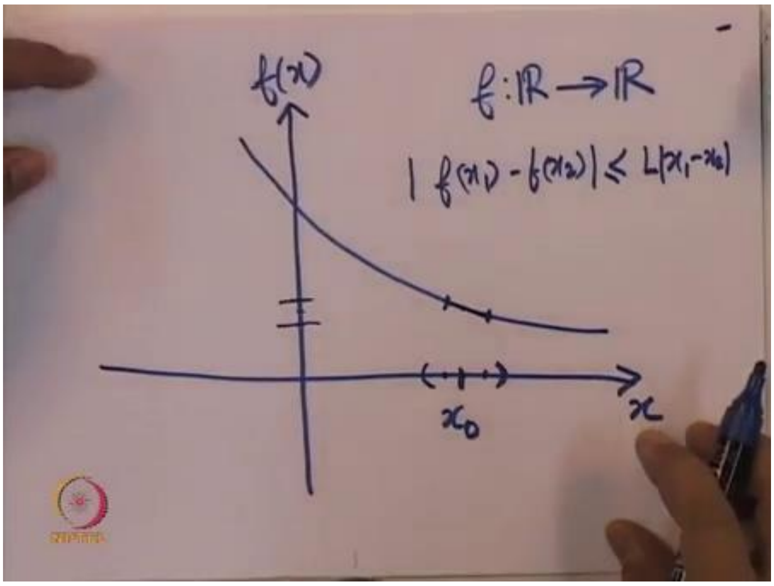
So, we are using a ball which is an open ball which means that the distance of every point in that ball is strictly less than epsilon from the center x naught that is why it is called a open ball. So, L is called a Lipchitz constant once an L is found we can see that anything larger also can be put in here and that inequality and this inequality will be satisfied with a larger L also, in general L depends on both x naught and on epsilon.

(Refer Slide Time: 01:38)



So, some examples of Lipchitz functions f x is equal to minus 4 x is locally Lipchitz at x equal to 3. We will let us see that it is locally Lipchitz everywhere in fact it is globally Lipchitz this is what we will see very soon, but we had just began drawing a graph, what is the meaning of Lipchitz?

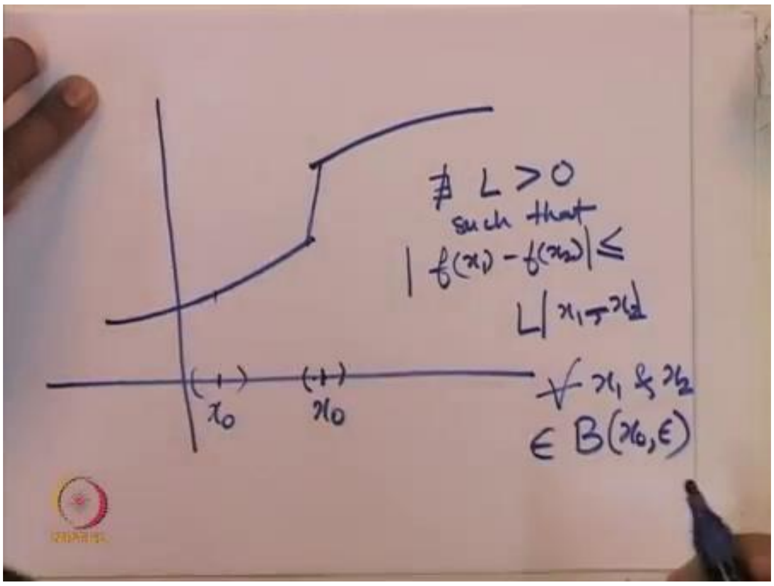
(Refer Slide Time: 01:59)



For the special case, so here is a function, so suppose we are interested in checking if the function is Lipchitz at this point locally to say it is locally Lipchitz at this point. It means we should be able to find a ball in this case; it is an open interval such that we take any two points inside this ball and we look at the corresponding values and those values we take here so on. One side of the inequality f x 1 minus f x 2 is less than or equal to L times x 1 minus x 2.

This inequality just means that if we connect those two points by a line, then the slope of this line should have absolute value at most l. So, can we find a number L such that L puts a upper bound on the absolute value of the slope. Even though this is decreasing the slope is not positive here, but we look at the absolute value of the slope and that should be bound from above by a number l. So, we will quickly see that a discontinuous function will not satisfy Lipchitz property at the point of discontinuity.

(Refer Slide Time: 03:26)

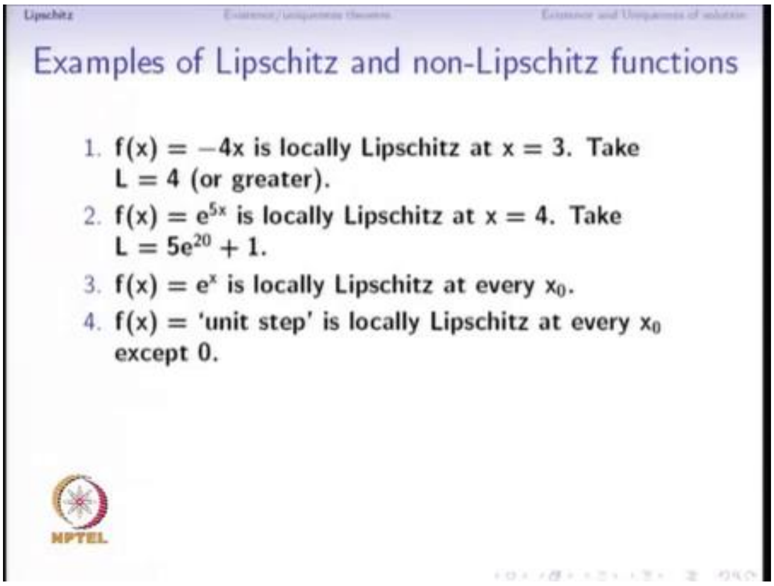


At other points, it could satisfy the property of Lipchitz, but at the at the point of discontinuity because this is the function and at x naught, we have this discontinuity. So, if at x naught how much ever small neighborhood, we take, we are forced to take points both from the left and the right. When we connect these two points in the left and the right, we see that this line could have a slope, which becomes more and more vertical. The line connecting these two points becomes more and more vertical, it will become vertical if and only if you take the point x naught and just before it, but you cannot find.

There does not exist a number L greater than 0 such that such that f x 1 minus f x 2 is less than or equal to L times x 1 comma x 2 in which these inequalities satisfied for all x 1 and x 2 for all x 1 and x 2 in the call. So, for how much ever small epsilon we take because we are forced to take the point both form the left and the right it turns out that this line connecting the point from the left and connecting the point from the right. This particular line has a slope that is not bounded in absolute value, so because of that we see that there does not exist an L such that this inequalities satisfy for all x 1, x 2 in the ball for a particular x 1 and x 2.

We might be able to find the l, but that L will work only for that x 1 and x 2 in that ball, but we want this inequality to be satisfied for all points x 1, x 2 inside the ball of radius epsilon strictly greater than 0. So, this is how we see that if it is discontinues, it cannot be locally Lipchitz at the point of discontinuity at another point x naught. Suppose, this is another point x naught at this point, it can very well b locally Lipchitz as long as in a in this case as long as we take a ball that does not contain this discontinuity, we can see that the function is locally Lipchitz.

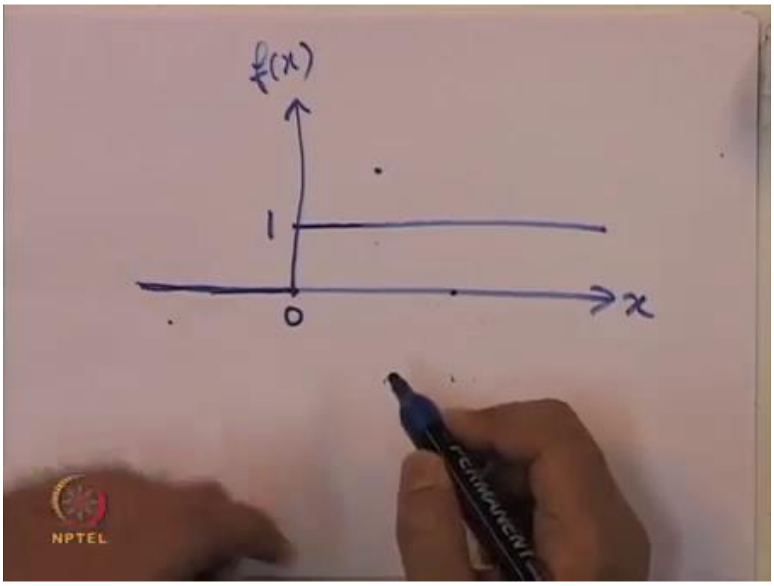
(Refer Slide Time: 06:01)



So, let us see some examples, so f x equal to minus 4 x is locally Lipchitz at x equal to 3, so take L equal to 4 take the Lipchitz constant L equal to 4 or anything larger consider the function f x equal to e to the power 5 x. This is locally Lipchitz at point say x equal to 4 for this particular point, we can take the Lipchitz constant L equal to 5 times e to the power 20 plus 1. So, notice that we are taking the slope of the function evaluated at x equal to 4 and we take something that is slightly greater how much greater we take decided on how big the open ball around the point x equal to 4.

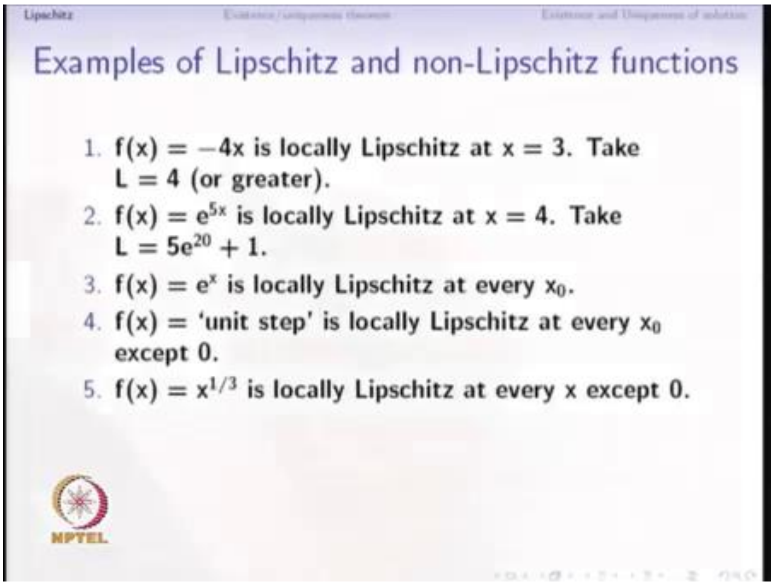
Since we are interested in just an open ball of radius greater than 0, we can take a Lipchitz constant slightly more than the slope at the point x equal to 4. Consider the function f x is equal to the power x, this is locally Lipchitz at every point x naught which ever point x naught. We are able to find a Lipchitz constant L that will work for all points inside a suitable ball the unit step is locally Lipchitz at every point x naught except the point x equal to 0, so what do we mean by locally, what do we mean by unit step.

(Refer Slide Time: 07:27)



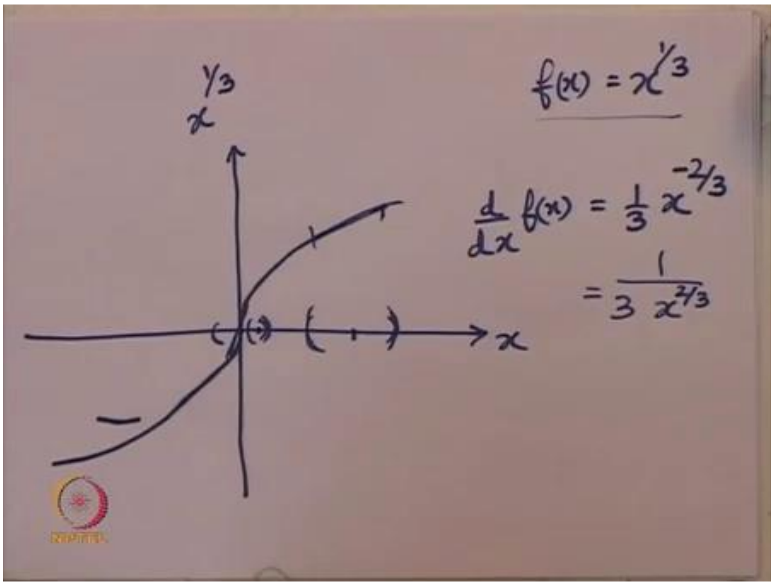
This is like our step function this is equal to 0 up to here, then suddenly it jumps to the point 1, so this particular function up to x equal t 0 it is equal to 0, for x greater than 0, it has jumped to 1. This is locally Lipchitz at every point except x equal to 0, at x equal to 0 we saw that there is this discontinuity. Hence, there will not be a constant L that will work for that inequality for all points in a ball how much ever small the ball may be.

(Refer Slide Time: 08:19)



So, in other words the unit step is locally Lipchitz at every point x naught except the point x naught this particular function f x is equal to x to the power 1 by 3 is locally Lipchitz at every x. Again, except x equal to 0, this is a very important example, we will see this again and again, so let us just draw the graph of f x is equal to x to the power 1 by 3.

(Refer Slide Time: 08:46)

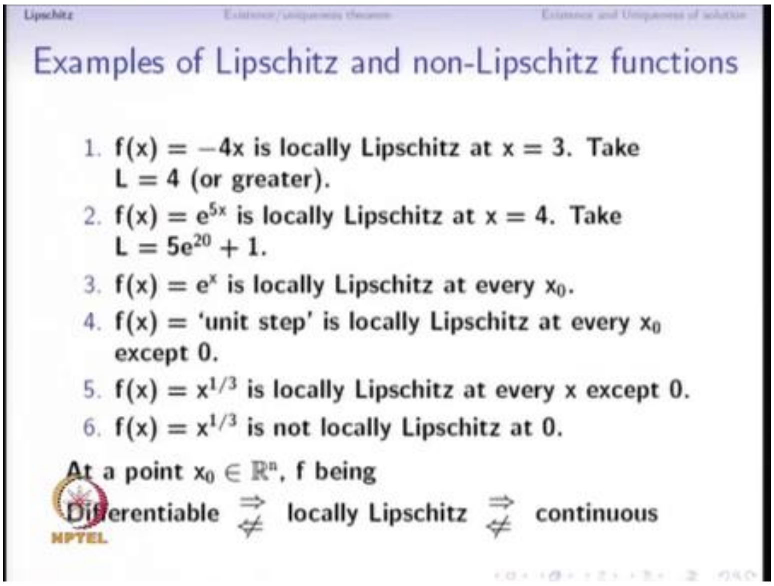


This particular graph we see that this is how the graph looks, so we see that at x equal to 0, this particular curve becomes vertical almost vertical. If we try to evaluate d by d x of f of x, we get 1 by 3 times x to the power minus 2 by 3 which is nothing but 1 over 3 times x to the power 2 by 3. So, we see that as x tends to 0, this quantity becomes unbounded and hence and hence we see that the slope is not bounded about the point 0. So, how much ever small neighborhood we take about the point 0 when we connect two points, we see that it could have a slope that is unbounded.

We are not able to find a number L such that the Lipchitz the inequality in the Lipchitz condition definition that inequality will be satisfied for all points inside that ball. Such an number L we will not be able to find, that is why we will say this particular function is not locally Lipchitz at the point x equal to 0. At another point, say here we are able to find a ball such that there will be a number l. In other words, whatever ball we take as long as we do not include the point 0 when we connect this, we can take the farthest and the nearest. We can see where this slope is maximum and we can choose the number L accordingly that number L will work for everything.

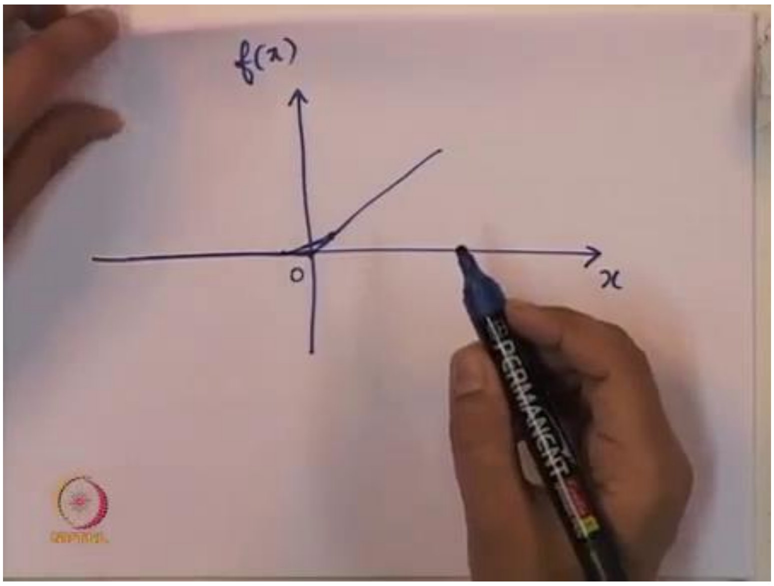
We have to see where the absolute value of the slope is maximum and we take a Lipchitz constant that is equal to that or more and that will work for all points in that ball. Hence, we see that as long as we do not take the point its equal to 0, how much ever close we take, if it is away from x equal to 0, we can find a ball such that there is a number L satisfying the Lipchitz inequality. Hence, except the point x equal to 0, there is a number L that will satisfy the Lipchitz inequality, hence it is locally Lipchitz at every point except the point x equal to 0.

(Refer Slide Time: 11:34)



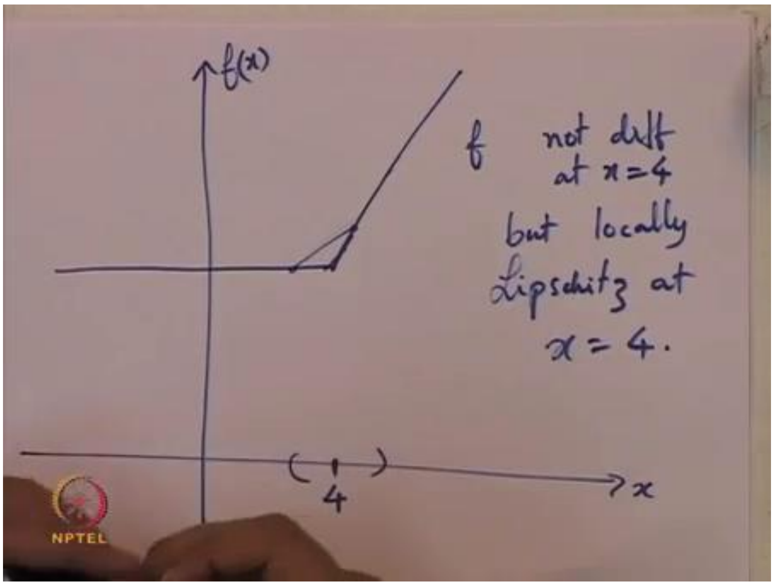
Some more examples, so the same function f x is equal to x 2 the power 1 by 3 is not locally Lipchitz at x equal to 0. So, we could just we could conclude that at a particular point x naught in R n if f is differentiable, then it is indeed locally Lipchitz at that point x naught. If it is locally Lipchitz at a point x naught, it implies that it is continuous locally Lipchitz at a point x naught means it is continuous at that point x naught conversely. If the function f is continuous at a point x naught, it does not imply that it is locally Lipchitz at that point x naught. If the function f is locally Lipchitz at the point x naught, it does not mean that it is differential at the point x naught. So, we can see some examples about this just to see why differentiability is not assured by locally Lipchitz property, we will see that locally Lipchitz only requires that the slope is bounded.

(Refer Slide Time: 12:38)



Slope between any two points, if we take the unit ramp say it is continuous. Now, so we see that the derivative does not exist at this point because the left hand limit of the derivative and the right hand limit of the derivative are not equal to each other. Hence, at the point 0, the function f is not differentiable, but we see that since the slope is bounded, we take any two points, we connect them by this line the slope is bounded, so I will just draw a bigger figure.

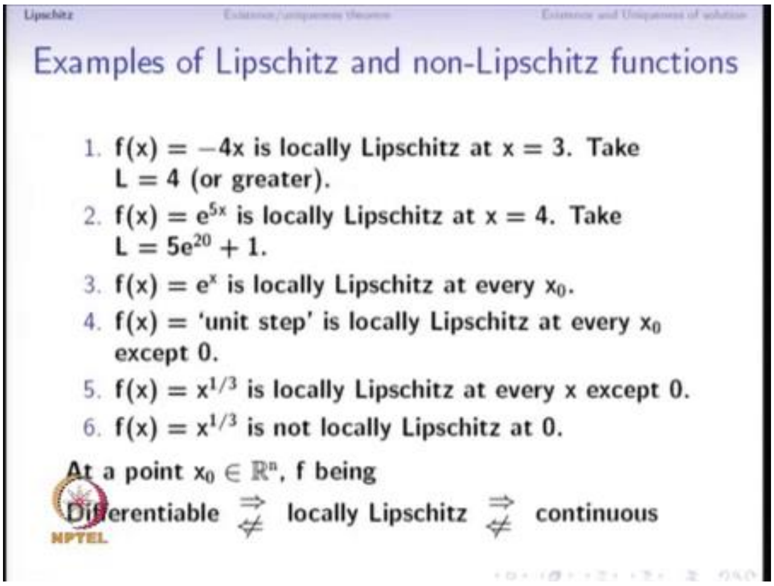
(Refer Slide Time: 13:22)



Take this particular example, so at the point let us say 4, this we have drawn a figure such that the graph of f verses x is continuous, but it is not differentiable at this point. The left hand limit of the derivative and the right limit of the derivative are not equal to each other. However, we take any tow points and we connected we see that this line is guaranteed to have a slope that is less in absolute value than the slope of this line. So, we could take any two points here may be on the same side and the line slope in absolute value is an upper bound for the absolute values of the slopes of any two points close to the number 4.

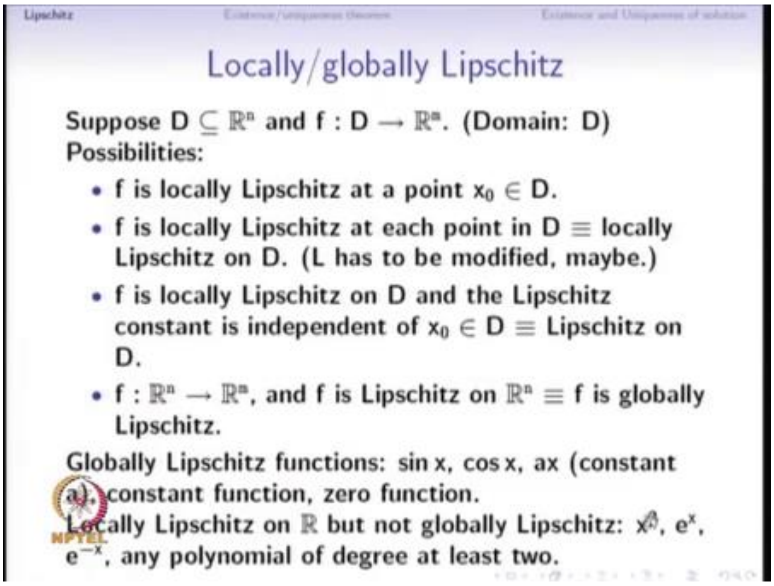
In other words, we can find a ball here such that we can find the number L that will satisfy the Lipchitz inequality for all the points in that ball. In other words, here is an example of f that is not differentiable at x equal to 4, but locally Lipchitz at x equal to 4. In other words, if somebody tells us that this particular function f is locally Lipchitz at the point x equal to 4 and one asks us does it imply that f is differential able at the point x equal to 4, answer is no, it is what the statement here says.

(Refer Slide Time: 15:15)



So, if the function f is differentiable it does imply that it is locally Lipchitz at the point x naught. If it is locally Lipchitz at the point x naught it is also continuous at the point x naught, but if it is continuous at the point x naught, it does not imply that it is locally Lipchitz at x naught. We saw x to the power 1 by 3 as a important example and if it is locally Lipchitz at a point x naught, it does not imply that it is differentiable at the point x naught, for that we saw an example now.

(Refer Slide Time: 15:43)



So, we will just spend a few we will spend one slide on the difference between locally Lipchitz and globally Lipchitz. So, for this purpose we need to see to what extent the number L depends on x naught and epsilon, so consider a domain d a subset of R n and f a map from d to R n. So, f need not be defined on whole or R n it is defined on a domain d for domain is a open connected subset of R n in this case. So, to say that d is a domain in R n, it means d is a open and connected subset of d open and connected subset of R n.

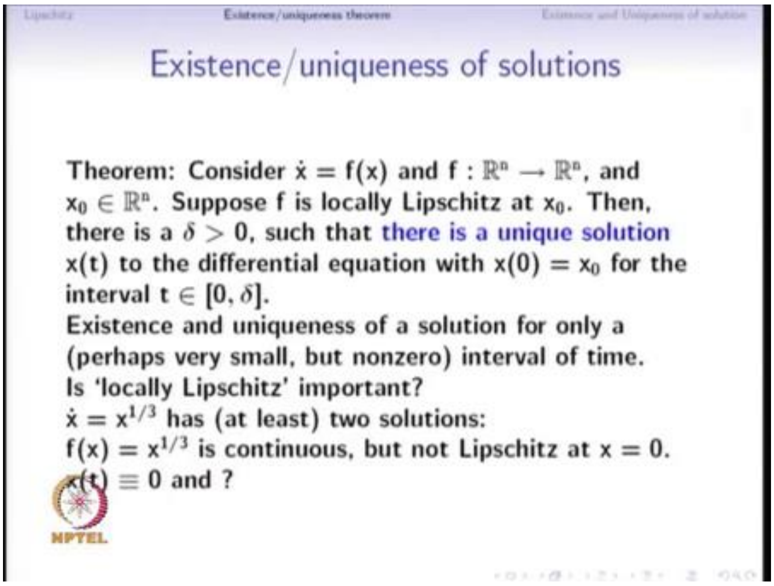
So, for a function f that is a map from d to R n, there are various possibilities f could be locally Lipchitz that a point x naught in d, we saw the definition for locally Lipchitz at a point. We could also have a situation where at every point in d f is locally Lipchitz at that point this we will say is, we will use a word f is locally Lipchitz on d. So, what is the significance here L might have to be modified depending on the point x naught, if x naught is another point the slope absolute value of the slope might be larger because of which L might have to be made larger.

So, at each point in d f is locally Lipchitz, but the Lipchitz constant L has to be modified depending on the point x naught perhaps L has to be modified. We will say f is locally Lipchitz on d if it is the case that f is locally Lipchitz on d and the Lipchitz constant is independent of the point x naught in d. Then, we will say f is Lipchitz on d the word locally is no longer relevant, since we can find one Lipchitz constant L that works for the entire domain d. Finally, when the domain d is a whole of R n if s is Lipchitz on R n that is there is a constant L that works for every point x naught in R n.

Then, we will say f is globally Lipchitz, so examples of globally Lipchitz functions are sin x cos x a constant a times x the constant function itself and the zero function. It is possible to decide for each of this function a Lipchitz constant L that works for the entire domain of these functions. Entire domain in this case is whole of r examples of functions which are locally Lipchitz on R, but not globally Lipchitz r x square e to the power x, e to the power minus x in fact any polynomial of degree 2 or more.

So, these all examples are locally Lipchitz on R, but not globally Lipchitz, why because this x square, it slope could become very large depending on the point. Hence, there is no one number L that works on the whole of R similarly, e to the power x and e to the power minus x can have slope which are very large in absolute value.

(Refer Slide Time: 19:12)

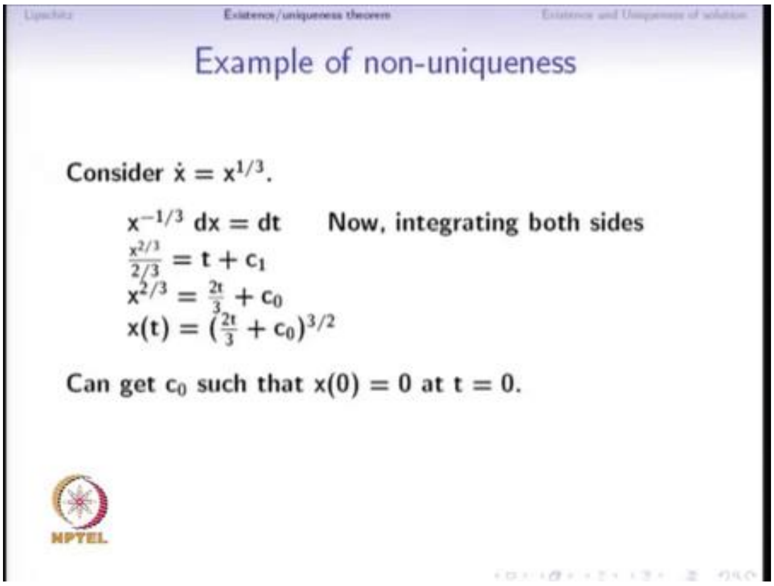


We finally come to the existence and uniqueness theorem for solution to a differential equation. So, consider x dot is equal to f of x consider this differential equation where f is a map from R n to t n and consider the point x naught in R n assume f is locally Lipchitz at z naught. Then, there is a delta greater than 0 such that there is a unique solution x of t to the differential equation to the differential equation x dot is equal to f of x with the initial condition x 0 equal to x naught. So, such a solution such a unique solution exists for the time interval t belonging to 0 to delta.

So, there is some number delta that is strictly positive because of it, there is this interval of time 0 to delta for which we have a solution. Moreover, the solution is unique that is what it says with the initial condition x 0 is equal to the point x naught where the function f was assumed to be locally Lipchitz. So, it turns out that existence and uniqueness of a solution is being guaranteed for only an interval of time it is possible that this interval of time is very small, but it is guaranteed to be a non zero interval of time it is not just one point, but it is an interval 0 to delta.

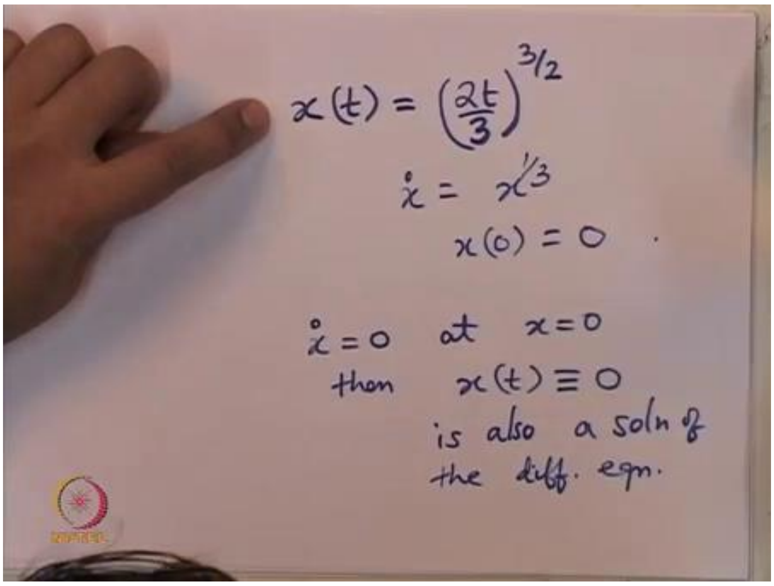
We can ask the question is locally Lipchitz important after all we have spent analyzing the significance of Lipchitz locally Lipchitz its relation to differentiability and continuity. We could ask is this locally Lipchitz property really crucial for the existence and uniqueness of the solution to the differential equation. For this purpose, we will see that the differential equation x dot is equal to x to the power 1 by 3 has at least two solutions, why because x to the Power 1 by 3 turned out to not be locally Lipchitz. Hence, uniqueness was not guaranteed by this theorem and we will see that there are two solutions what are these solutions this is what we will see now.

(Refer Slide Time: 21:32)



So, consider this differential equation x dot is equal to x to the power 1 by 3, so we can rewrite this as x to the power minus 1 by 3 d x equal to d t. Upon integrating both sides, we see that we get this equality where c 1 is some constant and re writing these terms x to the power 2 by 3 is now equal to 2 t by 3 plus c 0. So, z 0 and c 1 are related by the constant 2 by 3, for convenience we have renamed c 1 times 2 by 3 as c naught. Then, upon taking suitable powers we see that x of t x of t is equal to 2 t by 3 plus c naught this to the power 3 by 2. So, we see that this is also a solution to the differential equation and how do we calculate c naught, we can ask the question suppose at t equal to 0. The solution, the differential equation was at 0 x satisfied x 0 is equal to 0 at t equal to 0 by substituting this we can get c 0.

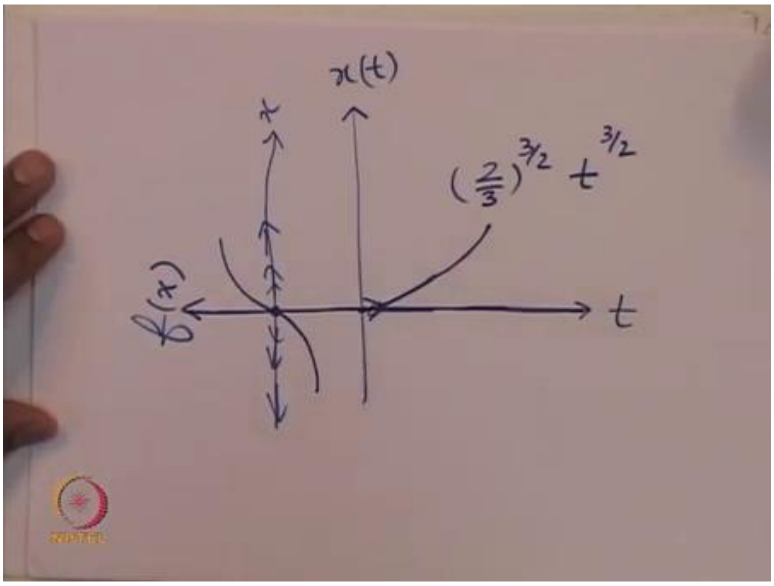
(Refer Slide Time: 22:47)



So, we see that x of t equal to 2 t by 3 to the power 3 by 2 is a solution to the differential equation which differential equation x dot is equal to x to the power 1 by 3 with the initial condition x 0 is equal to 0. Is this the only solution? no because if x naught equal to 0 at x equal to 0, then we also know that x of t is also a solution is also a solution of the differential equation. So, we see that if x is equal to 0 at t equal to 0, then x dot is equal to 0, why is x dot equal to 0, because we put x equal to 0 here and cube root of 0 is nothing but 0 and hence x dot is equal to 0.

Then, we conclude that x is equal to 0 for all time t this is one solution to the differential equation, but we see that x t is equal to 2 t by 3 whole to the power 3 by 2 is also a solution to the differential equation with the same initial condition. So, for the same initial condition, we see that we have two solutions to the differential equation let us just draw a graph of x verses time.

(Refer Slide Time: 24:22)

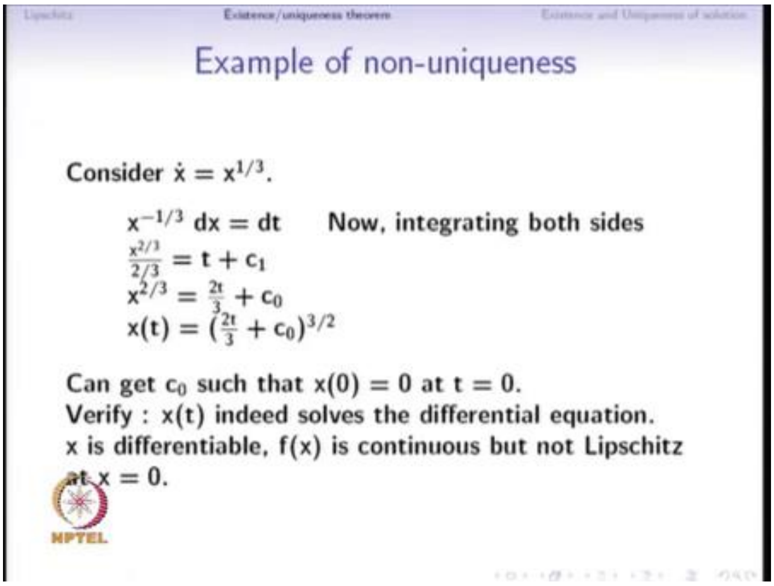


So, here is this x that was 0 until t equal to 0 and form here it is growing as 2 by 3 to the power 3 by 2 times t to the power 3 by 2, this is a graph of x verses time t and in addition the differential equation also has equivalently equal to 0. As a solution, the differential equation, in other words at this point t equal to 0, there is this solution that comes out of x equal to 0 and it also continuous at x equal to 0 has another solution at this point. The vector field we see that it is pointed, sorry this is something we have to plot like this. So, the graph of x to the power of 1 by 3 is like this, we see that of course there is this instability, but at z equal to 0 itself the arrow had length 0 because the graph f crossed the x axis at x equal to 0.

Hence, if it is at the point x equal to 0, then it continuous to be at the point x equal to 0 this is what our vector field diagram told us, but here we see that in addition to continuing to be at 0. There is also this possibility that it comes out it emanates out of the equilibrium point without requiring a perturbation while this figure says that this equilibrium point is a unstable equilibrium point. We see that upon perturbation, there are points that are trajectories that are going away from the equilibrium point, but here is an example because of this non.

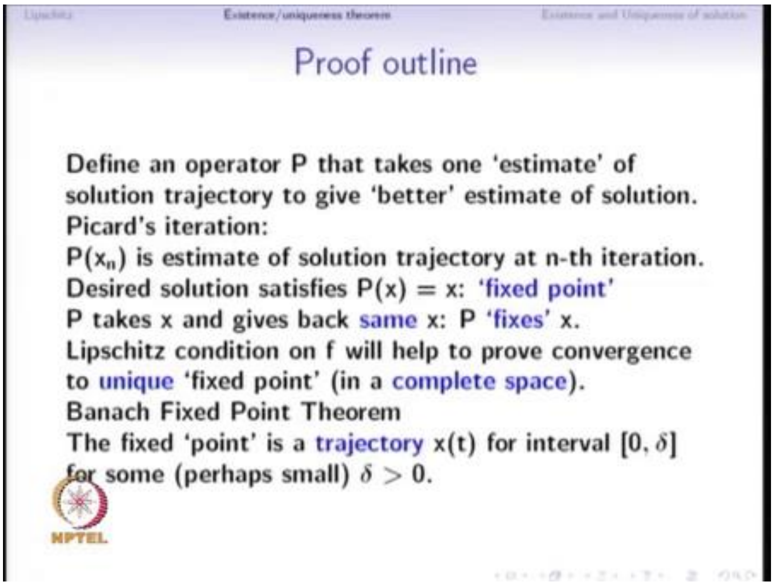
Locally, Lipchitz property at the point x equal to 0 without a perturbation also we see that there is a solution emanates out of the equilibrium point thus making us ask what is what exactly is the definition of an equilibrium point. When there are solutions that can emanate out of the equilibrium point even without requiring a perturbation for these purposes. We will form now on eventually assume that a function f when studying a differential equation satisfies locally Lipchitz at every at every point x naught. This is an example where without locally Lipchitz property, it turns out that can be non uniqueness of solutions to the differential equation.

(Refer Slide Time: 27:04)



So, we see that the solution x is differentiable, we are we were able to get explicitly the solution x as a function of time we can also differentiate this as a function of time. We see that it solves this differential equation f of x is continuous, but it is not locally Lipchitz at x equal to 0. That is why the previous theorem would not guarantee existence of solutions existence and uniqueness of solutions and here is an example where uniqueness fails.

(Refer Slide Time: 27:39)

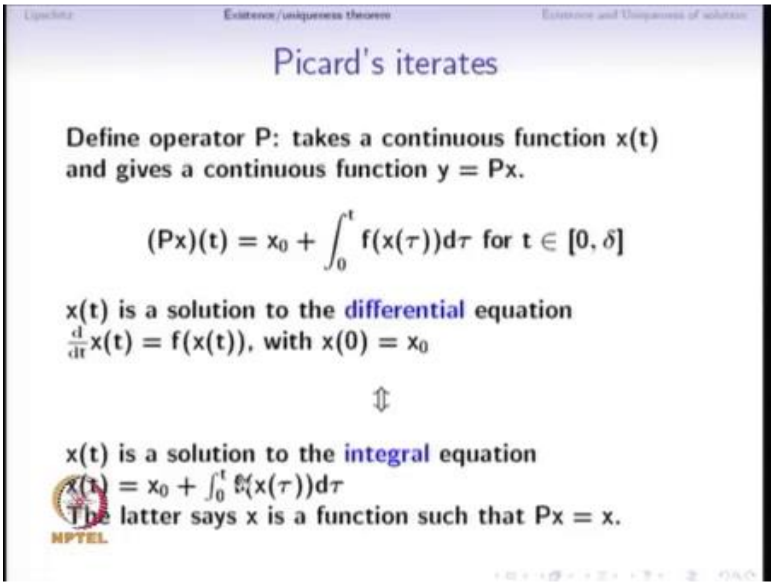


So, because of the importance of this result of existence and uniqueness of solutions to differential equation under locally Lipchitz Property, we will because of its importance, we will see the proof. So, the outline of the proof will be as follows, we will define an operator p that takes one estimate of the solution trajectory and gives the better estimate of the solution. So, this operator p takes a trajectory a trajectory which is a solution to the differential equation on the interval 0 to delta and it gives a better estimate of the solution.

So, this is going to be call Picard's iteration because of Picard's work in this area. So, p x n is an estimate of the solution trajectory at the n th iteration, we will define p such that the desired solution will satisfy p x equal to x. In other words, x will be a so called fixed point, why do we call it fixed because this desired solution to the differential equation takes x and gives back the same x in other words. While it takes different x n and gives back x n plus 1 possibly x n plus 1 is different from x n while this is possible.

In general, we will construct p such that the desired solution x will satisfy p x equal to x, in other words p fixes x, so the Lipchitz condition on f will help to prove convergence of this iteration to a unique fixed point. This fixed point would be unique provided we are looking for a so called complete space this is these are something that we will define precisely. So, for this purpose we will use a so called Banach fixed point theorem, so please note that the point in this context is a trajectory x of t for the interval 0 to delta possibly the interval delta the interval 0 to delta is a very small interval. In other words, delta is only slightly more than 0, but it is positive which means that interval is not just a point, but it is the interval of time of length delta.

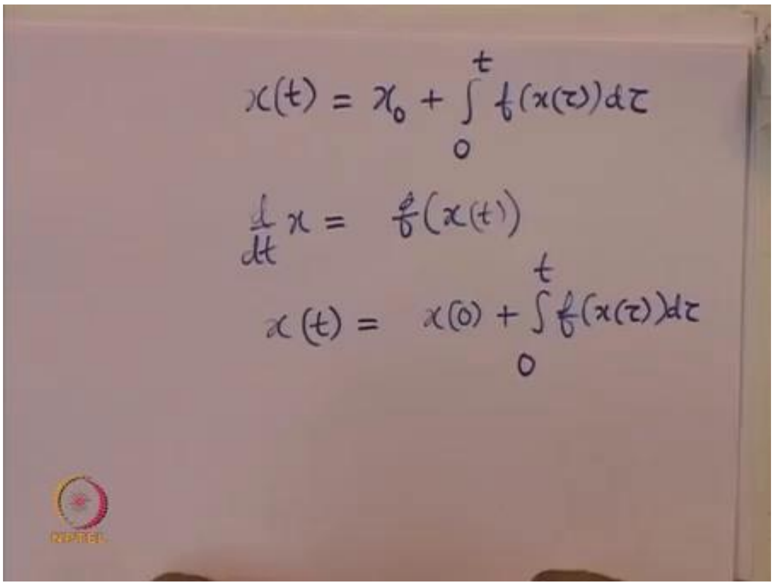
(Refer Slide Time: 30:11)



So, how will we define this Picard's iterates, so define the operator p that takes a continuous function x of t and gives another continuous function y is equal to p of x. How will we define it p x is again a function of time p x at any time, t is defined as x naught plus the integral from 0 to t of f of x tau d tau where t belongs to the interval 0 to delta. So, t comes in here and this integral is being added to 0, so we see that 4, this t equal to 4 this particular trajectory is also equal to x naught so these are different functions of time that start form x naught. So, what is the significance of this operator p with our solution to this with our differential equation?

We see that x t is a solution to the differential equation d by d t of x is equal to f of x. With this initial condition x 0 is equal to z naught if and only if x t is a solution to the integral equation to this integral equation x of t equal to x naught plus integral from 0 to t F of x tau d tau. So, by differentiating this right hand side, this is something you can quickly check, y is solution to this equation also solution to the differential equation.

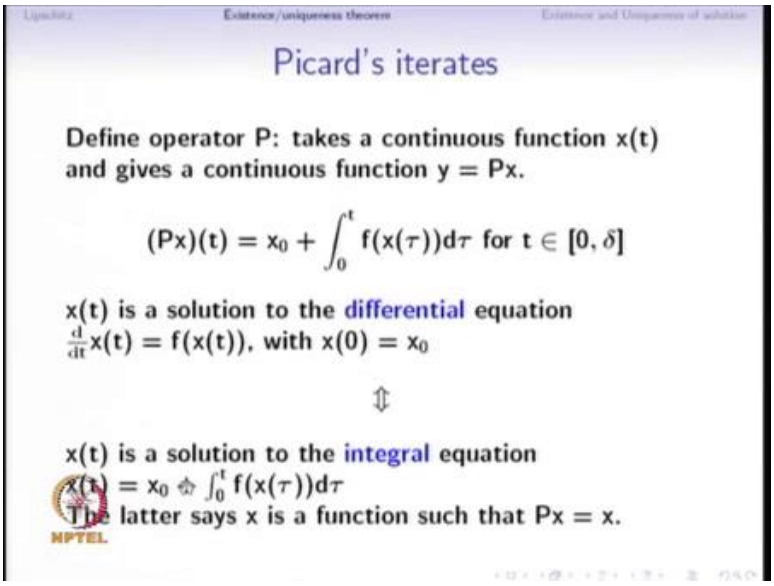
(Refer Slide Time: 00:26)



We see that x appears on both sides, here is x and x is also inside this integral, so d by d t of x is equal to del t of this is a is equal to 358 and since t appears only here this is nothing but f evaluated at the n point. So, this is what our differential equation wants, in other words this solution to the integral equation is exactly a solution to this differential equation. Also, when we integrate this on both sides, we see that x of t also satisfies this initial condition plus integral from 0 to t of the right hand side.

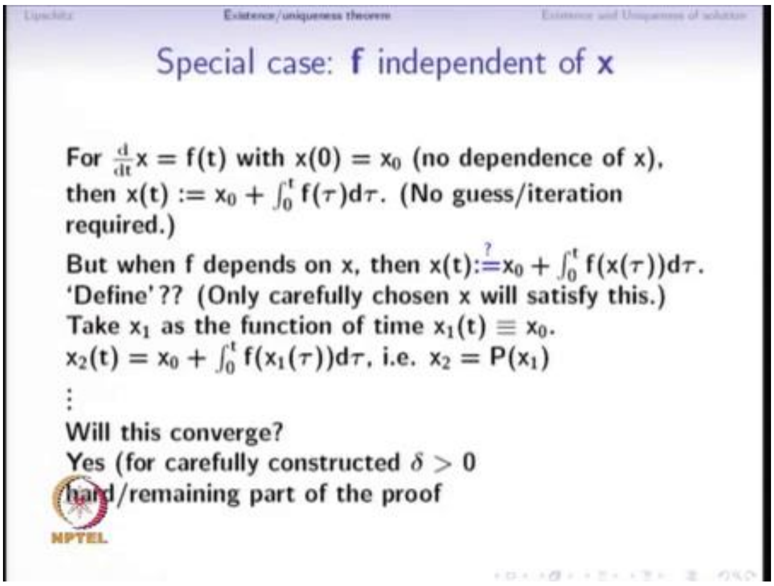
In other words, solution to the differential equation when integrating both sides is differential equation; we obtain exactly the integral equation. Thus, solution to the integral equation is a solution to the differential equation and solution to the differential equation is also a solution to the integral equation. However, we see that here also x appears on both sides and here also x appears on both sides and it is not clear that going from a differential equation to the integral equation is genuinely a improvement. We will see that obtaining an integral equation allows us to use Picard’s iteration.

(Refer Slide Time: 33:27)



So, latter what is important is that when X satisfies this integral equation that x is also a fixed point of this operator p why because the right hand side is nothing but p of x. What x we have put in here is exactly, what is also here, that is why this particular integral equation says that p of x equal to x.

(Refer Slide Time: 33:56)



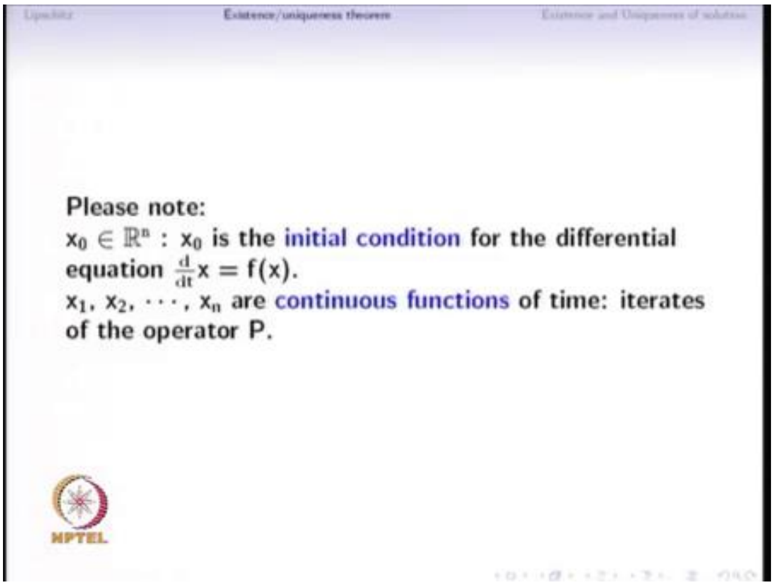
So, we will first see the special case and f is independent of x may be f is allowed to depend on time, only here we will assume that f is a function of time explicitly and there is no dependence on x. So, consider this differential equation d, d t of x is equal to f of t with this initial condition x naught. So, there is no dependence of f on x then we are able to integrate this x of t, we can then define as x naught plus this integral from 0 to t of f of tau d tau. So, there is no guess or iteration required to define this such a definition x will always satisfy this differential equation, but when f depends on x, then we are not able to define this why because x appears both on the right hand side and left hand side of this integral equation.

So, the word define can no longer b used here, so only a carefully chosen x will satisfy this equation and that is indeed the solution to our integral equation. So, what we will do now is we will take x 1 as some function of time, it turns out that within a small neighborhood of the actual solution, which solution we take will not matter. So, we will take x 1 of t equivalently equal to x naught, so we will take the function x 1, which is always equal to x naught value x naught is a point in R n which corresponds to your initial condition.

There is one particular function, which is always equal to x naught for all time t that we will take as our initial x 1. Then, we will define x 2 as x naught plus this integral with f evaluated at x 1 instead of x 2. Now, since we have x 1 here, which we know and x 2, which do not know we are allowed to use x 2 defined by this right hand side. In other words, x 2 is equal to p times x 1, so we can similarly define x 3 as p times x 2 and in general x n plus 1 as p times x n.

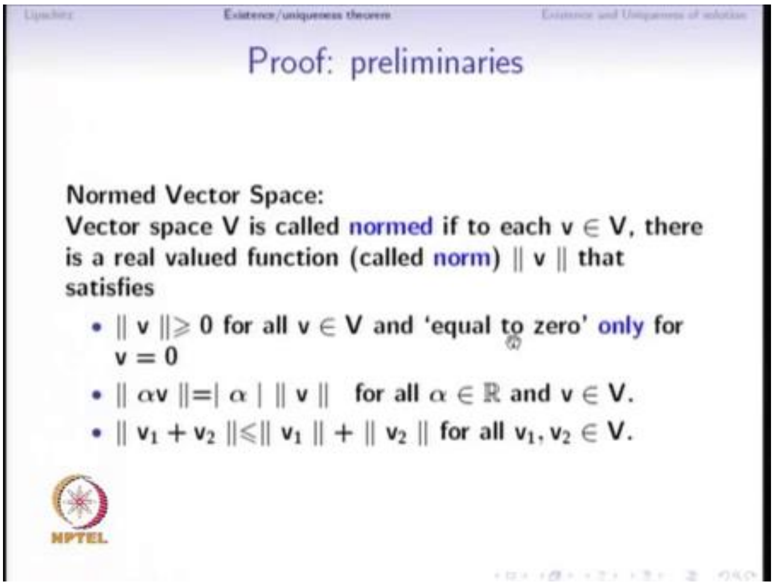
So, the question arises will this converge will this converge to a solution in other words will it converge to a fixed point of the operator p. So, the answer is it will converge for a carefully constructed delta that will be strictly greater than 0 and for ensuring that this delta is strictly greater than 0 and that it exists. We will be using the Lipchitz locally Lipchitz property of f at the point x naught, so this is the hard and reaming part of the proof which we will proceed and do now.

(Refer Slide Time: 36:48)



So, please note that x naught is a point in R n this is the initial condition for the differential equation d by d t of x is equal to the f of x. On the other hand, x 1 x 2 up to x n are continuous functions of time and these are iterates of the operator p. These are no longer points in r, but they are functions, which take their values in R n for different time instance.

(Refer Slide Time: 37:21)

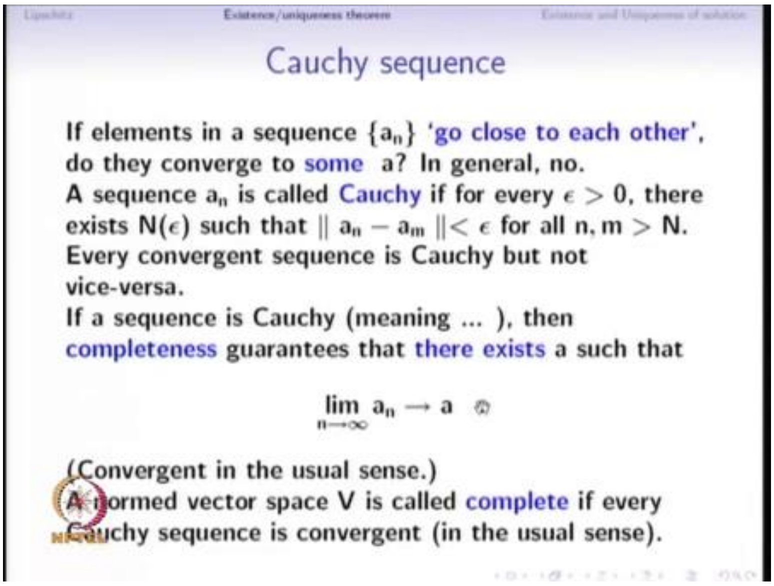


So, for this proof we need some preliminaries, so what for that purpose we will need to use the Banach fixed point theorem and these are some preliminaries for that purpose.

So, we need the notion of a normed vector space, so vector space v is called normed if to each vector v in the vector space capital v. There is a real value function, which we will call as the norm and we will denote it as the norm of v by this notation such that this norm is always greater than or equal to 0. So, norm of vector v will be a real number and it cannot be negative, that is what it says.

So, this is greater than or equal to 0 for all v and also it is equal to 0 only for the vector v equal to 0. This is the first condition required from the norm function; another condition that is required is if we scale the vector v by a constant alpha. Then, the norm also gets scaled by that same number alpha the same number alpha if alpha is positive and absolute value of alpha in general. So, for any real number alpha and a vector v this equality has to be satisfied and finally, the triangular inequality is required to be satisfied. So, for two vectors v 1 and v 2, the norm of v 1 plus v 2 cannot be more than norm of v 1 plus norm of v 2. This triangular inequality is also required to be satisfied for all vectors v 1 and v 2, this is called triangular inequality, this is called the linearity property of norm function, this is the definition that the notion of length of a vector is required to satisfy.

(Refer Slide Time: 39:12)



In the context of convergence, we also need what is the definition of a Cauchy Sequence, so in the context of convergence we like to say that elements in a sequence are going close to each other. So, in general if elements in a sequence a a n go close to each other do they converge to some element a. So, the answer is in general no, this is because elements a n are converging close to each other does not mean they will eventually converge to a number a. So, this converge, then we will call the sequence convergence in order to arrive at that we will define this property called Cauchy.

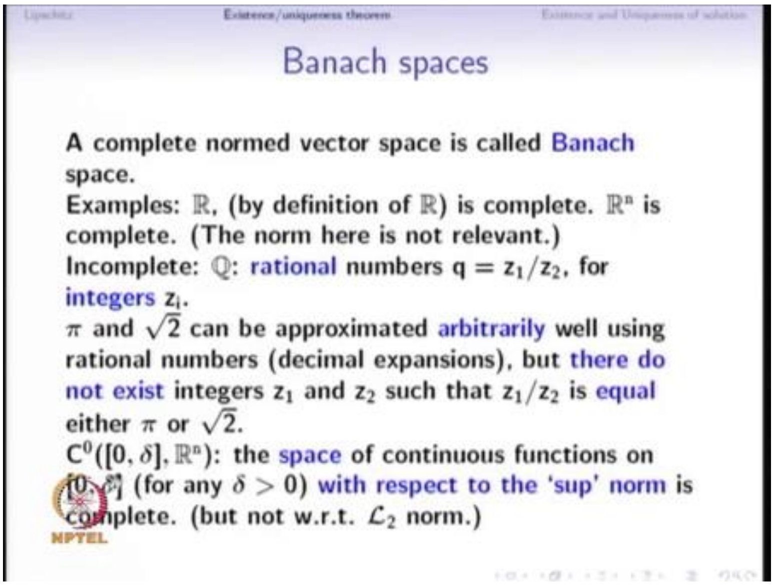
So, a a sequence a n is called Cauchy if for every epsilon greater than 0, there exists some capital n possibly depending on epsilon such that if we want that a n and a m are smaller than epsilon. Then, that will be satisfied for all n, and m that are greater than this capital m, so the importance of this statement is it says that elements a m and a n are going close to each other.

If you want them to be close to each other, closer than a number epsilon, then all we have to do is take m and n greater than some capital n. So, when somebody specifies epsilon greater than 0, we are able to find number n such that this inequality is satisfied for all n and m greater than capital n. So, in general if epsilon is made smaller then capital n might have to be made larger, so this quantifies this makes precise the notion that elements in the sequence a n are going closer and closer to each other.

So, every convergent sequence is Cauchy, but turns out that the converge is not true see if a sequence is Cauchy it just means that elements are going close to each other, but it does not imply that there is a number a to which it converges. So, if we assume an important property called completeness, then we will also be able to guarantee convergence, so what is the definition of completeness see if a sequence is Cauchy. Then, completeness would guarantee that there exists a number such that a n converges to a and this latter convergence is what we will say convergent in the usual sense.

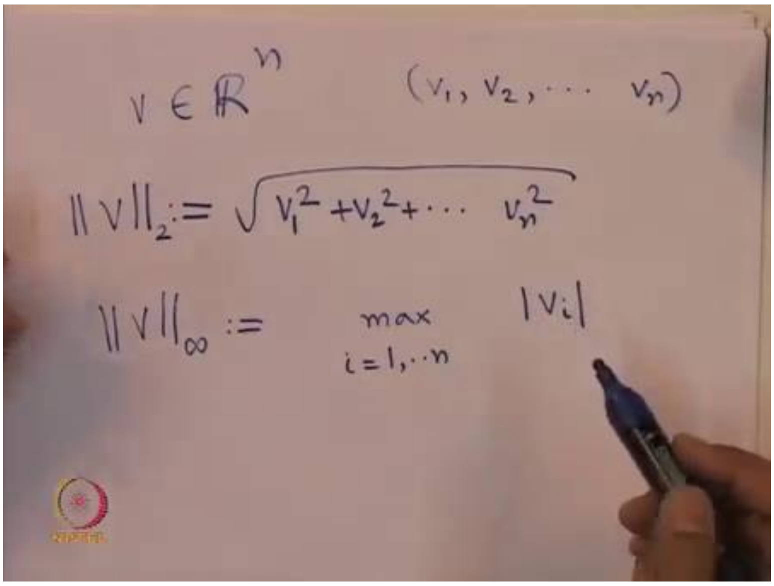
So, this is what we will use for the definition of complete of a norm vector space a norm vector space is called complete. If every Cauchy sequence is convergent in the sense of this limit, so we see that if every Cauchy sequence, in other words a sequence in which all elements are going closer and closer to each other. If we are going to say that it converges to a number in that particular set in that particular vector space v, then this vector space v is going to be called complete.

(Refer Slide Time: 42:26)



Finally, a complete normed vector space is called Banach space, so what is the Banach space, it is a norm vector space which is also complete. In other words, every Cauchy convergence sequence is also convergent some examples of complete spaces a set of real numbers are this is by definition of r. This is complete R n is also complete where n is some finite number, so if we take n tipples of real numbers, then this is also a complete vector space. So, which norm here it turns out is not relevant, we could take for example, the standard we could take the Euclidian norm which is very standard.

(Refer Slide Time: 43:18)



Suppose, a vector v is in R n, then we can define this is one of the definition v 1 square plus v 2 square plus v n square this guarantee that it is positive and when we take square root it is also linear in v. So, this is one of the norms of R n what is R n it is a n triple every element in R n is a n triple in which n is finite. So, this is one definition of norm and this is called the two norm it is also called the Euclidian norm. We could also have taken v, so called the infinity norm as the maximum for i equal to 1 up to n of the absolute value of the i th component, we v 1 up to v n are n components in the vector v.

For each of those vectors, we take the absoluter value and for each of the components, we take the absolute value and we look at the maximum for i varying from 1 to n of this absolute value that is called the infinity norm. So, we will see some more notions of norm, so with respect to any of the norms, it turns out that R n is complete what examples of in complete spaces are. So, the set q of rational numbers this is incomplete what is rational q is a standard notation for the set of rational numbers any number q, small q in this capital Q can be written as the ratio of two integers z 1 by z 2.

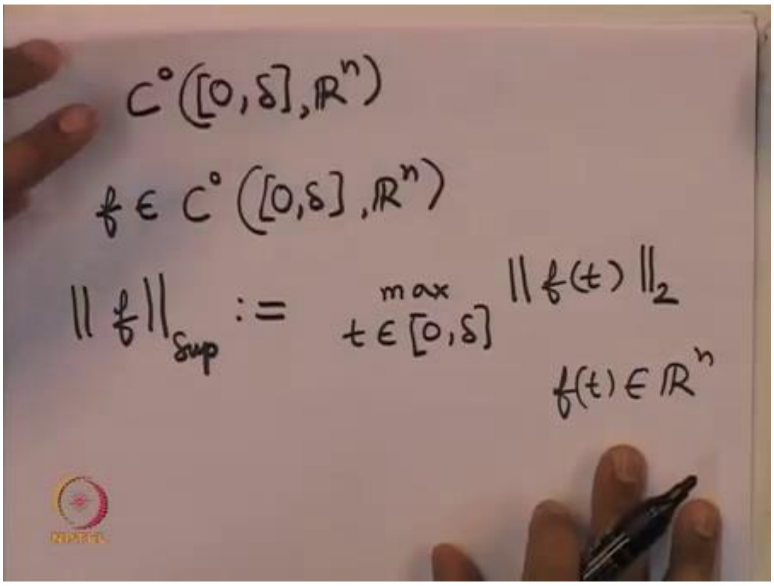
So, if a number q can be written as ratio of two integers, then q is said to be a rational number and capital q is a set of all such rational numbers. So, it turns out that q is not complete for example, pi and square root of 2, these are some numbers which can be approximated arbitrarily well using ration al numbers to be able to approximate arbitrarily well. For example, we know that square root of two can have a decimal expansion, if we take more number of digits in the decimal expansion, then we get a more representation of the number square root of two.

Similarly, for pi, but just because we can approximate square root of 2 and pi by a very good approximation using decimal expansion that does not imply that there are integers z 1 and z 2 such that z 1 by z 2 is equal to either pi or square root of 2. In fact, they do not exist z 1 and z 2 such that z 1 by z 2 is equal to pi, similarly there do not exist integer z 1 and z 2 such that z 1 by z 2 is equal to square root of 2. In other words pi and square root of 2 are so called irrational numbers. However, we know that these irrational numbers can be approximated arbitrarily well this proves that q is not complete.

Another important set in other important space that is complete is the so called space of continuous functions on this interval 0 to delta, where delta is strictly greater than 0 with respect to the sup norm with respect to the sup norm. It turns out that this space of continuous function is complete, so the space of continuous functions on this interval functions form this interval 2 R n. This whole also denoted by c by this notation. So, the zero here means that it is just continuous differentiability is not being guaranteed. So, I should emphasize that it is important that with respect to the sup norm only c 0.

This one is complete, there could be some other norms with respect to which it is complete, but here it is no longer the case that the norm is not relevant, which norm we are taking, that will very crucially decide whether this space of continuous functions is complete or not that statement depends on norm with respect to which we are asking the question.

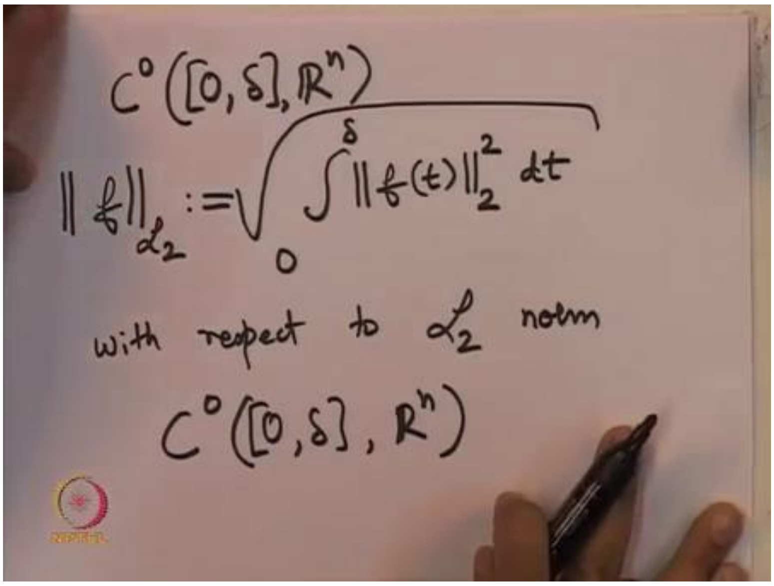
(Refer Slide Time: 47:44)



The space of continuous function from the interval 0 to delta to R n, these are functions which are not necessarily differentiable, but they are just continuous and 0 is being put here. They take their domain is from 0 to delta, it takes it values time varies from 0 to delta and for each time instant in this interval, it gives a vector in R n. So, this is the space of continuous functions from this interval to R n and we are saying sup norm what is a sup norm. If f is an element of this set then the sup norm, we are going to define as the maximum when t varies from 0 to delta of f of t.

So, at each time instant f of t is an element of R n this is the meaning of f is a element of is a element of continuous function from this interval to R n. So, at each time instant t this is some vector and for that vector, we defined already the Euclidian norm. The two norm and this two norm is defined for each time t in this interval and we will look at the maximum of this particular function s t varies from 0 to delta this maximum is said to be the sup norm of the function f. So, sup norm is no longer relevant to a particular time instant even though f of t 2 norm was relevant to which time instant the two norm was being taken. So, the sup norm is a norm on the space of continuous function form this interval to R n. So, with respect to this sup norm, it turns out that this is a complete space, but with respect to which norm is it not complete for example, we can also define the so called L 2 norm.

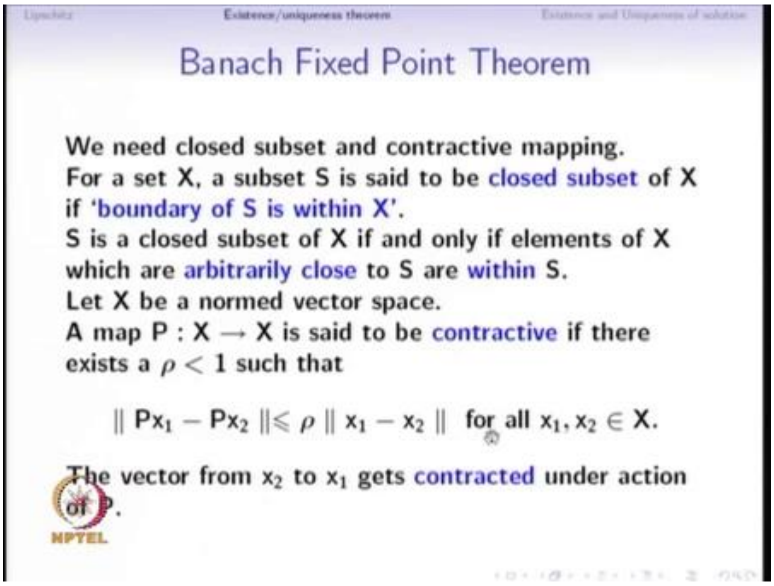
(Refer Slide Time: 50:05)



So, what is the L 2 norm for the same space of functions we can define f L 2 norm as integral from 0 to delta. So, we can take the two norm of the vector f of t at a time instant t and the 2 norm we take the square and we integrate from 0 to delta upon integrating from 0 to delta. We get some number that is no longer dependent on the time t and its square root is said to be the L 2 norm of the function f where f is a element of this L 2 norm is also no longer dependent on t, but f of t before we took the 2 norm. Here is indeed a function of time t and after we integrate from 0 to delta this is no longer dependent on time t.

So, this is a L two norm so with respect to with respect to L two norm this same space of functions is not complete. Just because elements are going very close to each other does not mean that it will also converge to a function that is continuous, and close to each other for that we are using the notion of L 2 norm so with respect to the soup norm. However, two functions if they are going close to each other, they will eventually converge to a continuous function again inside the space.

(Refer Slide Time: 51:58)

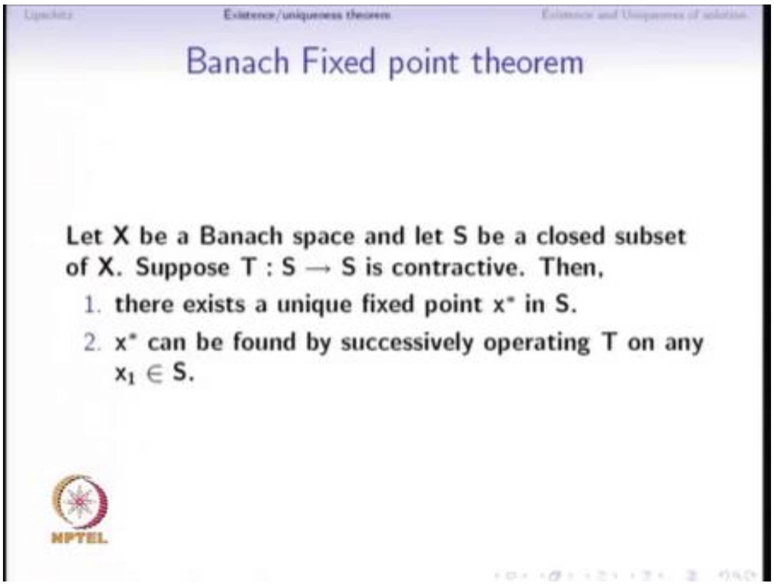


So, finally using these notions, we can prove we can state the Banach fixed point theorem. So, for that we just need the notion of close subset and a contractive mapping so for a set x a subset x a subset s of x is said to be a close subset of x if the boundary of s is within x. This is how we understand a close subset more precisely s is a close subset of x if and only if elements of x which are arbitrarily close to s are also within s. If we take a element of x which is very close to elements of s, which is close to one or more elements of s, then it is not necessary that this elements of x should also be within s, but if it is indeed within s.

Then, s is said to be a closed subset of the set x and what is contractive. So, if x is a norm vector space a map p from x to x is said to be contractive if there exists a number rho that is strictly less than 1 such that this inequality is satisfied for all x 1 x 2 in x. What is this inequality, p x 1 minus p x 2 is at most rho times x 1 minus x 2. So, what is the importance of this inequality x 1 minus x 2 can be interpreted as a vector from x 2 to x 1 and p x 2 to p x 1 is a vector upon action under p.

So, this vector is getting contracted under the action of p why contraction because this rho is some number strictly less than 1. So, we will say this map is contractive if there exist such a number rho such that this inequality satisfies for all x 1 and 2 in x the number. The number rho should not have to be modified depending on which x 1 and x 2 we take in capital x for such a map p, we will see it is contractive if such a number rho exists.

(Refer Slide Time: 54:06)



We are ready to state the Banach fixed point theorem using the notions, we just now defined let x be a Banach space and let s be a close subset of x. Then, let t a map from s to s let t be contractive, then first of all there exists a unique fixed point x star in s fixed for this operator t. Moreover, that fixed point x star can be found quite easily what is easily it can be found by successively operating t on any x 1 in s we take any x 1. Then, we take t times x 1, then we take t times t of x 1 when we do this iteration, we will converge to that unique fixed point. We will use this Banach fixed point theorem for the proof of the existence and uniqueness theorem of the solution to the differential equation in the next lecture.

Thank you.