Nonlinear dynamical systems
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Lecture - 32
Describing Functions for Non-Linearities

So, in the last lecture what we discussed was about describing functions and we saw the calculation of describing functions for two types of non-linearity. And what we will now do is we will look at how one can utilize this describing functions to actually try and find out whether given non-linearity, I mean a linear plan with nonlinearity. If in the given system, there is a limit cycle and how one can utilize this describing function to find out if there is a limit cycle and to find out ((Refer Time: 01:01)) the frequency of this limit cycle is, and the magnitude of the limit cycle is and so on. So let me, let me just recall what we did in the last lecture.

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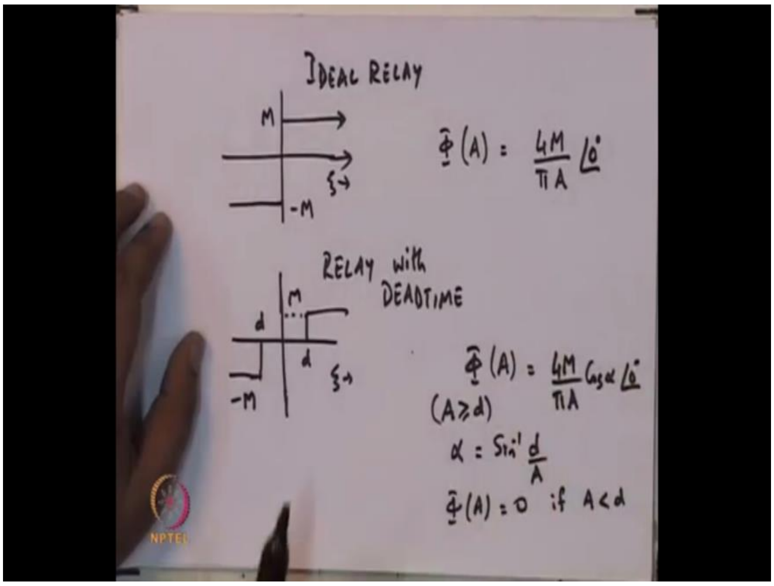

So, suppose you had a non-linearity, which was a sin function, so it gave a gain of minus plus M and minus M, depending on the whether the given input was positive or negative, and we found out that the describing function for this was 4 M by pi A. We also considered the another kind of non-linearity which was relay.

So, this one could think of whether the ideal relay and we could think of another non-linearity, which is relay with dead time. And its characteristics was given by there was a gain of plus M and there was a gain of minus M provided of course, the input was larger than the small DEF. So, this gain is M and minus M and we calculated that in this case also of course, is independent of the frequency of the input signal, the output.

So, the way we are calculating the describing function is that we are computing what is the, what is the primary harmonic and then the gain that we calculate by you know, the output gain by the input gain and the angle also in the same way. So, in this case the angle was 0 and in this case the angle turns out to be 4 M by pi A times cos alpha with angle 0 degrees, where the alpha is given to be sin inverse of d by A.

And this thing is only valid for, this is only valid for A greater than equal to d and phi of A equal to 0 if A less than d. So, this was the describing function for the relay dead time. So, the describing function for the ideal relay is very simple expression where as the, for the relay with dead time there are 2 different cases. So, now what we are going to do is look at, look at the original system that we wanted to analyze.

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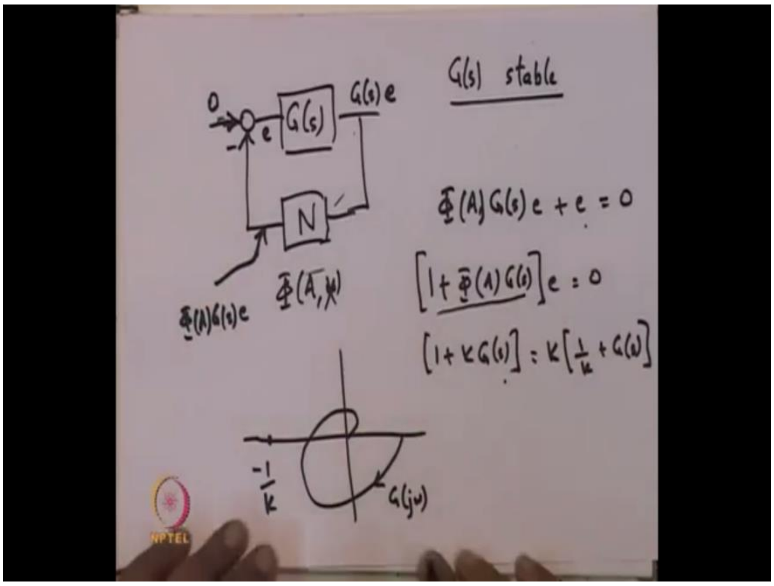

And the original system we wanted to analyze had a linear plant and a non-linearity in a feedback connection. So, and this non-linearity let us assume as A omega. This is the describing function for this particular non-linearity. So, now if you go around the loop, if you go around the loop, that means you start with any signal here, then out here you can think of it has been G S times e, if e is the signal here, then out here it goes through the linear plant then you get G S times e, then you go through N and when you go through the N, you end up with phi A.

So, I will suppress the omega because right now whatever we have looked at the describing function is independent of omega. So, phi A G S times e, that is the signal here and so once you go through this in the negative part. So, what you have is this plus e should be equal to 0. I hope, I hope what am saying is clear. This signal here is phi A G S e and so now if you think of this input here as been 0 input then what we are saying is 0 minus phi A, G S e is equal to this error signal, this signal e and so I can re-write that as this times e equal to 0.

So, if I pull the e out, I will get 1 plus phi A, G S e equal to 0, if this was a not a non-linearity, but it was an example for gain, just a gain k, then this equation essentially would have been 1 plus k G S and of course, we know that this is the characteristic equation for this closed loop system. So, what we have written down here is very similar to the, to the closed loop system equation. The characteristic equation of the closed loop system would have been this, this we can think of as the characteristic equation of the closed loop system, when you are thinking of the linear system with the non-linearity.

Now, what we are going to do is we are going use idea similar to the ideas used for linear systems and so in case of a linear systems, we can make use of this characteristic equation of the closed loop system, and we can say something about whether the resulting system, the closed loop system is stable or not by looking at the Nyquist plot of the linear part. So, how do we do that? Well we look at so this is the characteristic equation, so you look at the point minus 1 by k, what I am really doing is pull out the k and so I have 1 by k plus G S,

So, you look at the Nyquist plot, let us say this is the Nyquist plot of G S and then we can say that the resulting closed loop system is stable or unstable depending upon whether the Nyquist plot encircles minus 1 by k or not so. For example, this is precisely the Nyquist plot criterian that we discussed several times before. So, for example, suppose you have started out with a G of S, which was stable, that means the open loop, the open loop transfer function was stable.

Then in that case, for the closed loop transfer function to continue to be stable, that means for the closed loop things to be stable, what we should have is this being the characteristic equation and you are looking at the closed loop gain k, then G of j omega should not encircle minus 1 by k, in which case it is stable. On the other hand, if it encircles minus 1 by k, it is unstable. So, now what we will do with the, with the non-linear system, is something very similar to this. But before I go into that, let me just look at the closed loop system for the stable, I mean the for a linear case and make some observations.

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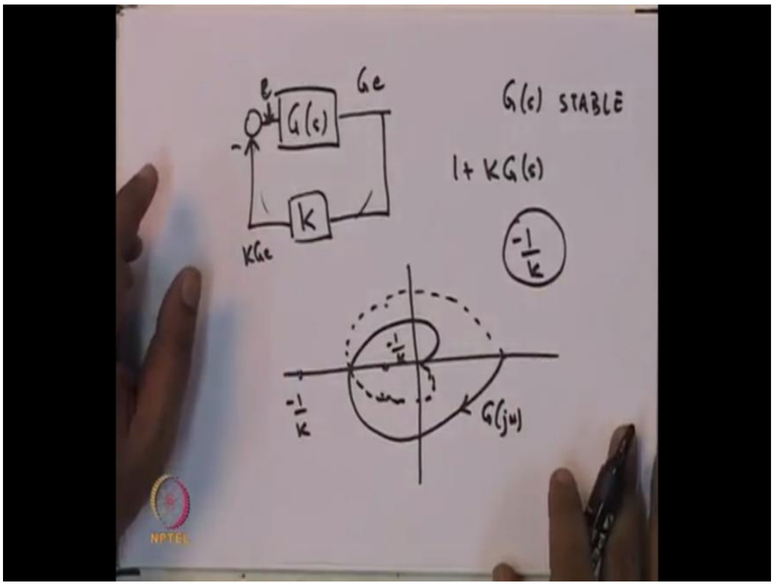

So, suppose you have a plan G s and let makes this standing assumption that G s is stable. If it is not stable, if G s has some right half poles and so on, then whatever interpretation we are doing using the Nyquist plot, you should change it accordingly depending upon you know how much, what is the instability of the G s. But suppose we make the assumption G s as stable and we look at this feedback system with a gain K, then the characteristic equation is 1 plus K G s.

So, now suppose we will look at the Nyquist plot and let us say the Nyquist plot looks like that. Because G s is stable, then this Nyquist plot criterion will tell you that this Nyquist plot, so let me also draw the reflection. So, this is the Nyquist plot. So, we see that the thing that we are interested in this quantity minus 1 by K.

So, if this minus 1 by K is here, that means K is such that you should get minus 1 by K here, then the resulting closed loop system is stable. What does it mean to say that this closed loop system is stable, what does is mean is this suppose the gain is such that minus 1 by K is like this at this point, then you have this closed loop system and suppose by some means, let us say we manage to introduce some noise here at this point, then this noise I mean the linear plant has some this noise and you get some output here.

So, let me just write it as G times e, and this G times e then gets multiplied by this K, so you get K G e and it comes here and there is this feedback by which gets in here. So, it goes around this cycle and what is really happening is, when it goes round this cycle, its magnitude, I mean its magnitude gets attenuated and so what one can expect is that going around this loop, going around the loop, the magnitude of this error gets attenuated until it becomes 0 and when we say that in this linear system if the K gain is such that minus 1 by K, is not enclosed by this Nyquist plot. What we are really saying is that small error signal or error signal cannot exist here, because when it goes round the loop it sort of get attenuated.

On the other hand, if this K was gain, such that minus 1 by K was here. Now, if it is here then you can see that it gets encircled twice, because G s is stable therefore, the closed loop transfer function is not stable. If the closed loop transfer function is not stable, what it means is some signal if the status non 0, then it might trigger the unstable parts of the transfer function and in which case the signal can blow up. And the signal blowing up is equivalent to saying that, when you go around this loop there is a signal which is not getting attenuated, but in fact it is getting magnified.

So, if you assume G s is stable then outside no encirclements would mean that inside this loop signals gets attenuated. On the other hand, if there is encirclements then the closed loop systems is unstable, because what it translates to its there are right half poles of the closed loop transfer functions because we assume that the open loop transfer function is stable, but what that means is this if you have some signals in this closed loop situation, those signals can blow up. So, if minus 1 by K is in here it can blow up, if it is out there it just get attenuated and dies out.

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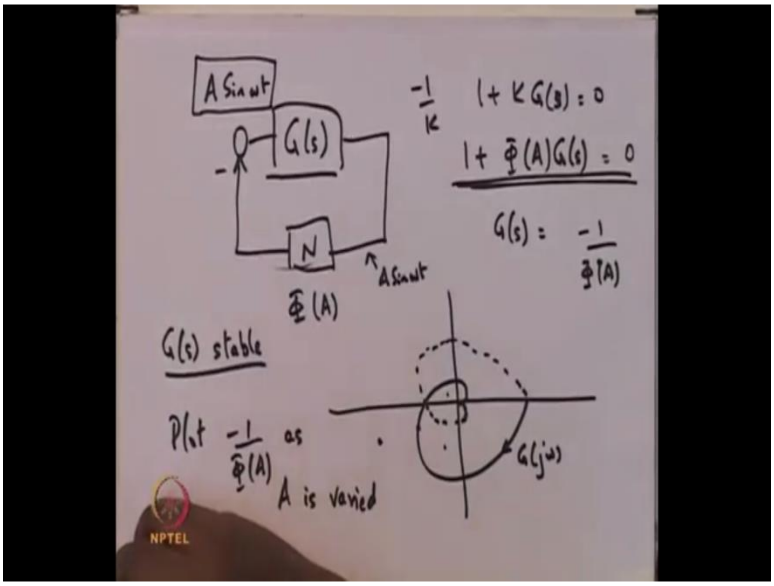

Now let us take, let us take G s and let us take this nonlinearity N and look at this feedback situations. So, we just saw what happens if you take linear gain, but instead if you take non-linearity. So, for the non-linearity, we can represent the non-linearity and especially if we are analyzing sinusoid signals, we can assign the describing function of the non-linearity rather than the non-linearity.

For any specific sinusoidal signal, if you want to study what happens to sinusoidal signal in there. And so just like the characteristic equation that we wrote down for the linear plant, earlier we had written down the characteristic equation for this non-linear system would be this equal to 0, so in the linear system it is 1 plus K G s equal to 0, in the non-linear system that gain K that the constant gain K is substituted by this, by this describing function.

And therefore, given the linear plant of course, one can draw the Nyquist plot of that plant G j omega and may be this is how the reflection looks and we make the earlier assumption that G s stable. So, look at the, look at the characteristic equation of this non-linear system. So what we are saying is the G s, so just like here we were looking here minus 1 by K. Here we could look at minus 1 by phi of A.

Now, if you are going to analyze for some signals, let us say A sin omega t, so we are interested in analyzing what happens if you use this signal A omega t, then we can look at corresponding to that A what this value here of the describing function is. And we can look at G s and see if this point which represents minus 1 by phi, if this point is encircled by the Nyquist plot of G j omega or not.

Now, if this point, now let us assume for this given A we calculate the phi of A, and so we will get some value here and so minus 1 by K is some point, let us say here, then if you think of the phi of A just purely like gain here, then from the earlier discussion what we had, we can conclude that with such a signal A sin omega t here. What is going to happen is this, in this closed loop this is going to get attenuated and therefore, the system is finally, going to settle down with all the signals becoming 0.

On the other hand, if this minus 1 upon phi by A is some point in here or some point in here, then of course there are encirclements of G j omega, if it is a point here one encirclement of G j omega, if it is a point here there are two circles n encirclements on G j omega, but in both those cases the resulting system is going to be unstable. So, if you take signal A sin omega t where A is such that this thing that you calculate turns out to be inside this portion of the Nyquist plot, then the resulting signal is not going to dry out, but it is going to grow exponentially.

So, what we could do, is give this linear plan one can draw its Nyquist plot, and given the describing function of the non-linearity, one could plot minus 1 upon phi by A as you vary A. So, plot as A is varied, now once you have got this plot then all those points where minus 1 by phi A is outside the Nyquist, the Nyquist plot of the linear part will lead to the closed loop system being stable. And if it is inside, then it will lead to the closed loop system being unstable, but if minus 1 by phi A this plot intersects this that some point.

So, in the boundary, then its neither stable nor, I mean its neither stable that means it does not die out, neither is it unstable in the sense it does not blow up. So, what one can expect when it is there, is that it is sought of sustains the sinusoidal motion and therefore, what you can expect to get is a limit cycle in this closed loop non-linear system. Best way to of course, to see this is by using some example. So, let us take an example.

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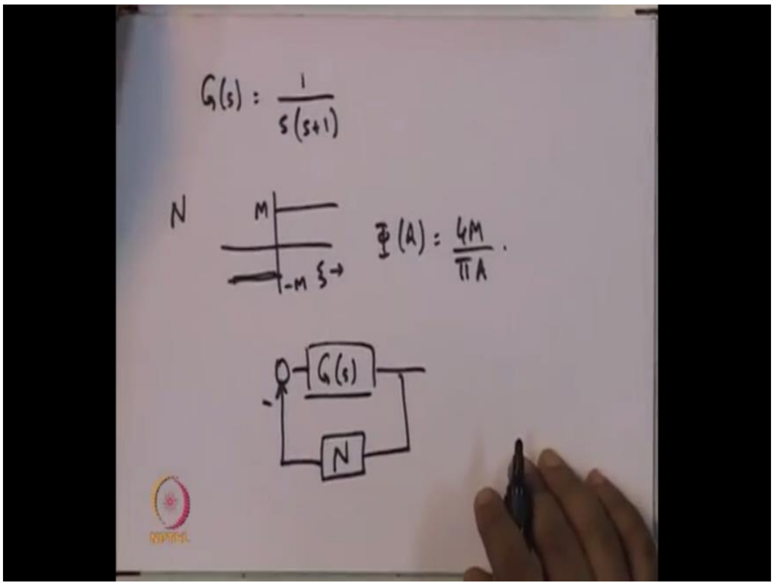

Let us assume, that the linear plant G of s given by, let us say 1 upon 1, and let us say the non-linearity N, in non-linearity whose characteristics are like that. This linearity we already found describing function, for this we see the phi of A is given by 4 M by pi A. And now we are interested in this particular arrangement, where you have the linear system and you have the non-linearity N and this thing is fed back in this following way, and we want to find out whether this given system has a, has a limit cycle and if it has a limit cycle what frequency what amplitude the limit cycle exists. So, for this let us first plot.

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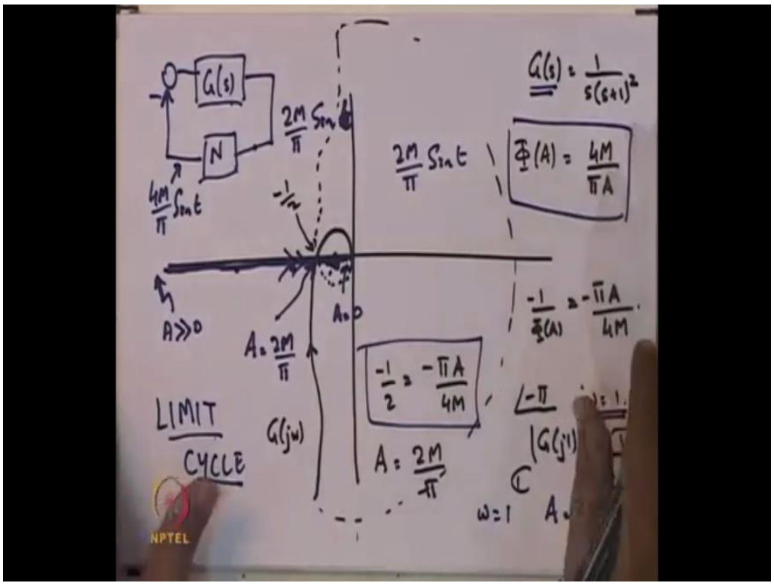

So, the complex plain and we have the G s 1 upon s into s plus 1. So, if you plot this, you are going to end up with, we are going to approach from here. So, let me not take 1 upon s plus 1, but 1 upon s into s plus 1 squared. So, if you look at this Nyquist plot of this, one would expect to be something like that, so we are drawn the Nyquist plot of G j omega I mean at s equal to 0, it is infinity with an angle of minus 90, so it is somewhere here, somewhere down here and then it comes and s tends to infinity this thing goes to 0, but with an angle minus 270 degrees.

So, this is, this the kind of plot that you get. Now, let us the non-linearity part, the non-linear, the non-linear describing function is given by 4 M by pi A and therefore, what we are going to plot is minus 1 by phi A, which is minus pi A upon 4 M. So, what this plot look like, well I let me use blue pen for this when A is zero, it is here and as A increases, you have it going that way.

So, this portion of the, of the plot for minus 1 by phi A corresponds to A equal to 0, and somewhere up there A very large, because of minus sign that is why this side otherwise it would have been, this side because of this minus sign, we are plotting it is this way. So, now what one can see is that for small values of A, so if you draw out the full thing for the Nyquist plot, if you draw the full thing for the Nyquist plot, you are going to end up getting something like that and right round.

So, now you can straight away notice, that small values of A it is as if this point is enclosed by, first of all the plant G s, then open the plant G s is stable. So, for now, for small values of a you are inside therefore, it is unstable, what does it mean to say this is unstable, what it means that in the, in the closed loop the signals are going to blow up. On the other hand, if you have a large A, then the amplitude gets attenuated, which means in the closed loop.

So, when I say closed loop, I mean this particular situation. So, if you had, if you had a A here and you had small amplitude thing here, then it gets blown up. On the other hand, if you had a large amplitude thing here corresponding to the some point here, then from the earlier discussion about gain what would happen is it gets attenuated, that it gets brought down. But if you have a A which corresponds to this point here, then it is precisely on the Nyquist plot. And if it is on the Nyquist plot, what you going to get is sustained oscillations. Again, how do you get this sustained oscillations.

Well, first of all let us use this transfer function to find out what is this point that you cross here, you see that for the Nyquist plot you should have an angle equal to, so the angle must be equal to minus pi, but you can get this angle of minus of pi for the transfer function at, when you evaluate at j omega equal to 1 because j omega equal to 1 s plus 1 both this s plus 1 will contribute minus 45 each, and this as will contribute as minus 90 degrees, this minus 45 each, so the total is going to be minus 180 degrees.

So, at omega equal to 1 that is when, that is when the transfer function, I mean the Nyquist plot when hit the negative real axis and of course, G of j 1 modulus, I mean the magnitude at that point is going to be 1 upon 1 plus 1 the square root of that but then squared half. So, this point here is minus half, and so minus half is equal to minus pi A by 4 M and therefore, equating this, we can say A is equal to 2 M by pi. Now, what we are saying is the following.

In this closed loop system, suppose you had a signal here whose, suppose you had a signal here whose amplitude was 2 M by pi sin t, because we are taking j omega for the, for the, for the linear part, it was at omega equal to 1 that it cross the Nyquist axis. So, the value of A is 2 M by pi. So, we are now looking at an input to the non-linearity which has this particular magnitude 2 M by pi sin t. Now, if you have this input to the non-linearity, the output to the non-linearity of course, is a periodic signal, it need not be a sinusoidal, but its primary component has the value of 4 M by pi sin t.

So, the primary component of the signal here has the value 4 M by pi sin t, this we can see straight away from, the from the describing function that we have written down here. Now, if you assume therefore, that the signal here is minus 4 M by pi sin t, when it passes through this transfer function, there is a gain of half which is the gain that you will have for this transfer function at the frequency 1 and therefore with this gain what you will end up with is this 4 M by pi sin t will now get to be 2 M by pi sin t and so, there would be a sustained oscillation of a signal here. And this sustained oscillation of the signal is the sign, that we have discovered a limit cycle in the non-linear systems.

Now, for the limit cycle, the time period of the limit cycle can be said, time period of the limit cycle is going to be pi because this sinusoid has period pi. And since you can talk about the time period therefore, you also can talk about the frequency, and the frequency is the frequency that we have read off from the Nyquist plot of the linear plant, that means the frequency omega equal to 1. So, the frequency omega equal to 1 and the amplitude is the amplitude you can read off by from this equation, and so the amplitude is 2 M by pi.

So, with this amplitude and this frequency, you can expect a limit cycle to go around in this closed system. Further you can claim that this limit cycle is a stable limit cycle. Now, how can you make this claim that it is stable limit cycle. So, the way that the limit cycle is operating is that there the frequency is fixed and the amplitude is fixed by looking at the intersection of the Nyquist plot of the linear part and the describing function of the non-linear part and it is out here.

Let us make some small changes. So, let us assume that this amplitude of the signal here by some disturbance increases. Now, if the amplitude increases what that means is, on this amplitude thing as far as the describing function is concerned, it has moved away from the existing point. If it moves away from the exiting point this side, then from the earlier discussion, we know that the system is, such that this system is stable in the sense that the signal will die out, it will attenuate.

So, what does it mean to say that this system is attenuating, that means the amplitude which had increased is going to come down, but the amplitude coming down means that along the, along the plot of the describing function had you gone up there and you will come back. So, if the amplitude had increased it will start decreasing. On the other hand if the amplitude had decreased, gone down below 2 M by pi, you would have been inside, and if you are inside you know that the resulting system supposedly unstable, which means you will, you will be pushed back and so along the, if you look along the line of the describing function, if you go that side you get push back in here, if you go this side you will be push back in there, which essentially means that in the phase plane, if you looking at the phase plane you have this limit cycle.

If you get an amplitude larger that means you have pushed out, then you were some where here and you get push back, so you get back to the limit cycle. If on the other hand from the limit cycle you get pushed inwards that means you get into a point here, then again you get, you know because it is resulting system is unstable and this grows and the signal grows and you get right back to where you have started from. And therefore, in this particular case, the limit cycle that you have got here is a stable limit cycle. We can now analyze the non-linear system that you will get by taking the same linear plant as before.

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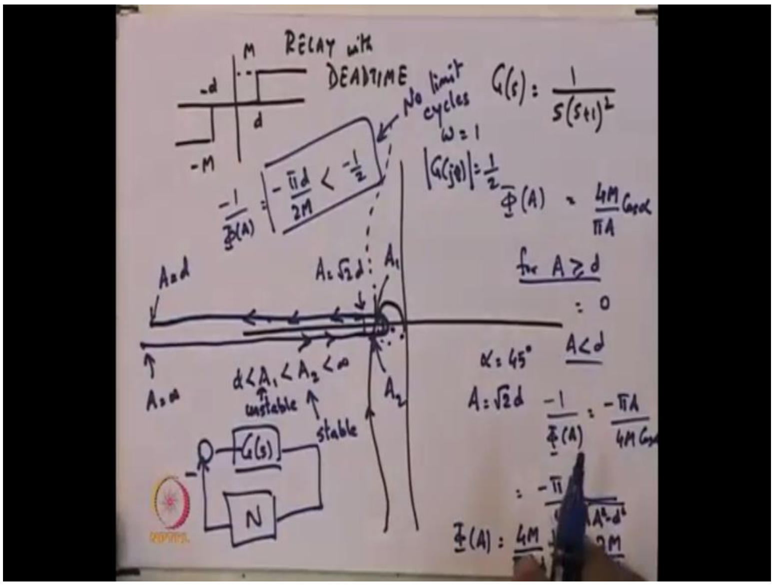

So, let us take the same linear plan which is 1 upon s into s plus 1 squared, but this time the non-linearity that we consider is the other, the other non-linearity, which is the relay with dead times, so that has characteristics like this. This is d, this is minus d, so when it is larger than d then we will get plus M and minus M. So, this is relay with dead time, so this is relay with dead time. So, now again the Nyquist plot of this is exactly the same as before, so the reflection is going to go like that.

So, it is exactly the same as before, the only difference is that you are going to get, is from the, from the characteristics, from the describing function. Now, the describing function we saw, was 4 M by pi A cos alpha, for A greater than equal to d and it was equal to 0, for A less than d. So, for a less than d and of course, we have to plot minus 1 by phi A.

So, what is the plot of minus 1 by phi A look like, so minus 1 by phi A is minus pi A by 4 M cos alpha, and let me write down cos alpha in terms of d and M, so that this ((Refer Time: 37:02)) unknown quantity, so let me get rid of that. So, it will be pi A by 4 M square root of A squared minus d squared, this is squared. Now, when A is 0 or when A is less than d, of course this formula is not applicable, but this formula which is applicable. So, minus 1 by phi A is going to be at minus infinity, then once A becomes larger than d.

So, when A has just become larger than d, then if you look the denominator is going to be a very small number. As a result this whole thing, this whole thing is going to be a very large number. So, minus is somewhere near this thing, and then as A increases, this minus 1 by phi A decreases. Of course, they all real values, so I am not drawing along the x axis because I wanted to be clear what this is. And this will keep decreasing until a certain point, and then when A become even larger than the certain point, then what you are going to get is it keeps increasing because if you think of A close to infinity. I mean A is very large number then this thing, a square root of A squared minus d squared, this can be approximated as being A and you have pi A squared upon 4 M A.

So, 1 A goes, so you have another A, so you again go back to minus infinity. So, this is the case where A is equal to d and it is increasing and it comes up to some maximum value, and then it starts decreasing until this is A equal to infinity. So, A as A is increasing first it comes down in this way and it goes back to this way. Of course, all these numbers are just real numbers along this negative real axis, but I have drawn this way just for the sake of clarity.

What is this value where this is going to be a minima, where I mean the magnitude is going to be a minima, well that you can calculate by just doing a bit of, I mean because you have varying A, all you have to do is you take the derivative of this function and find out when this function either phi A attains the maxima, or 1 upon phi A attains the minima. So, we could just look here and try to find out what is the value of A, where this phi A attains the maxima, and it turns out that would be the case when this alpha is 45 degrees, and you get alpha to be 45 degrees.

If A is equal to root 2 times d, so when A is equal to root 2 times d, then what you have here cos of alpha is 1 by root 2. So, this whole thing phi of A can be calculated to be 4 M 1 by root 2 upon pi and that is again root 2 d, so root 2, root 2, so you get 2 M by pi d. Observe that this quantity here only depends on values that comes from the non-linearity here M and the d. So, this is the case where A is equal to root 2 d of course, one should actually do the calculus and find out, that this is the value for which phi of A gets minimized.

But trust me on that, that is the value for which it should get minimized, so therefore, you have the situation. Now, as a result now in this particular, in this particular case where you have this G of s, the linear plant which is given by this transfer function, with this particular non-linearity in the feedback loop. When you plot the Nyquist plot of the linear part and when you plot the describing function of the non-linear part, this is what you get.

And now, something interesting happens you see if you have the situation, if there is to be a limit cycle, then this Nyquist plot should intersect this thing, but that is only possible if you see out here we have already calculated, that the gain margin, not gain margin, but you know the cross over frequency omega is 1, and G of j omega you know is half, which is what we calculated sometimes back G of j 1 is half magnitude, this half.

So, if minus 1 by phi A which is this quantity, if this quantity here for A equal to root 2 d phi of A turns out to be this, which means minus 1 by phi A, minus 1 by phi A, the minimum value for minus 1 by phi A or rather the modulus, the minimum modulus value for minus 1 by phi A or the maximum value for the minus 1 by phi A. This is going to be minus pi d by 2 M, and if this minus pi d by 2 M turns out to be, so it depends on d M, correct, and if this turns out to be less than minus half, then it is exactly the situation that I have drawn, and if it is this situation, then the closed loop system is such that for all values of A, the system is stable.

So, what you are going to get is the any signal gets attenuated and so therefore, there is no limit cycle. The only hope of a limit cycle is, if this not less that minus half that is greater than minus half. If it is greater than minus half, then this is the kind of thing you can expect. Which means there are two values of A, so let me call this A 1, let me call this A 2. There are two values of A for which the describing function and the, and the Nyquist plot intersect.

Of course, where the Nyquist plot intersects, that point is already decided because there is only one point that cuts the negative real access and as far as the Nyquist plot is concerned. So, for A there are 2 plot 2 points, one would be A 1 and one would be A 2. Both of which go through this point minus half, and now if you analyze these two points, if you take this point, if you take the first point A 1, what is going to happen, corresponding to this point A 1, so A is d here as A is increasing, then you first hit A 1, then you hit A 2, and then A increases to infinity.

So, we can say d is less than A 1 is less that A 2 is less than infinity. That is the relation between A 1 and the A 2, the two magnitudes that you have, for which the evaluation of this function minus 1 upon phi A turns out to be exactly minus half. Now, for this point A 1, let us see what happens. If you have a signal in here with A 1 and it gets perturbs, so that, so that the signal magnitude becomes smaller, that is equivalent to being out here, but being out here essentially means that the system, the system will attenuate the signal, but if the system attenuated the signal, what it means is this A becomes even smaller and so it goes further out, so you move off that way.

On the other hand, from A 1, if there is a perturbation which increases the magnitude, then you come inside, and when you come inside, then you know that the resulting closed loop thing is unstable that means that the magnitude grows. So, magnitude grows means you go from here and you go right round to A 2. So, this A 1 corresponding to A 1, you get a limit cycle, but that limit cycle is an unstable limit cycle. So, A 1 corresponds to an unstable limit cycle.

On the other hand, A 2 correspond to a stable limit cycle, and one can argue in the same way because if A 2 suddenly the magnitude decreases, so you are inside. But once you are inside, then the resulting closed loop system is stable, is unstable, which means the signals get magnified and so it gets back to A 2. On the other hand, if it increases more than A 2, you come out here, but then the system is stable so it attenuates, so you go back.

So, there will be two limit cycles when you use a relay dead time in this plant, and this linear plant, then provided this I mean the condition that minus 1 by phi A, minus pi by d minus pi by d, minus pi by d by 2 M less than minus half. If this condition is true, then there are no limit cycles. On the other hand if, minus pi by d by 2 M is greater than minus half, then there are two limit cycles, and one of the limit cycles corresponding to the smaller magnitude is an unstable limit cycle, and the one corresponding to the larger magnitude is a stable limit cycle.

And therefore, now using describing function and the Nyquist plot, one can say a lot of things about the closed loop system with the non-linearity and the linear part. While looking at the describing function of the non-linearity and the Nyquist plot of the linear function and doing analysis similar to this. So, I guess I am out of time for this particular lecture, so let me stop here now.