Nonlinear Dynamical Systems

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Lecture - 3
Classification of Equilibrium Points

Welcome every one, today is the third lecture on non-linear dynamical systems. This is between Madhu Belur that is me, and Harish K Pillai. So, we had just began with face portraits of second order systems last week. So, consider this differential equation x dot is equal to A x in which A is a 2 by 2 matrix, now we are trying to see various situations, various situations that arise depending on whether the Eigen values are A are real or complex, whether they are repeated or distinct whether A is singular or nonsingular.

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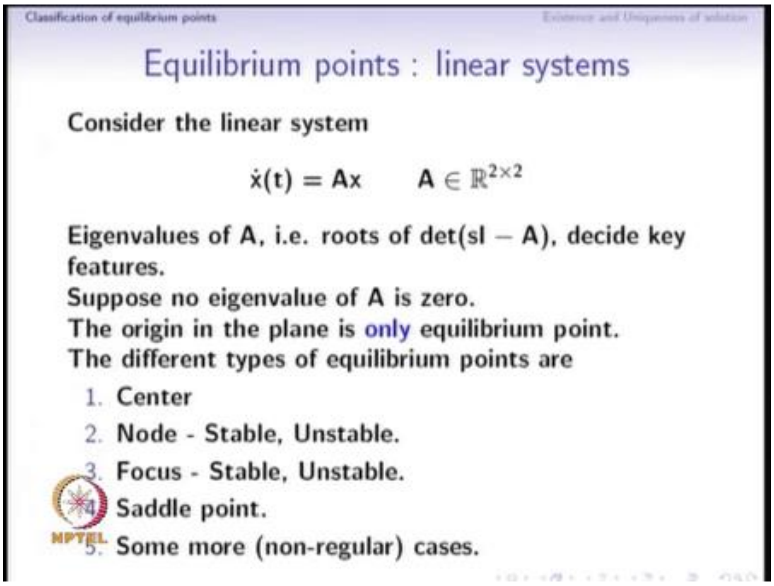

So, the Eigen values of A, let me recap are the roots of the determinants of s I minus A and this decides Eigen values decides the key features. So, we begin assuming that A has no Eigen value at 0, which means A is non-singular. In such a situation the origin in the plane is the only equilibrium point, different types of equilibrium points for this situation are center, which we had just began seeing, the node in which case it can be stable or unstable node. Then there is a focus a stable or unstable focus a saddle point and some other situations, which for example, when there are repeated roots and when there is one or more Eigen values at 0, those are the situations we will see separately. So, a stable node a node can be stable or unstable. So, what is a node, it is a situation when A has 2 distinct Eigen values and both are negative.

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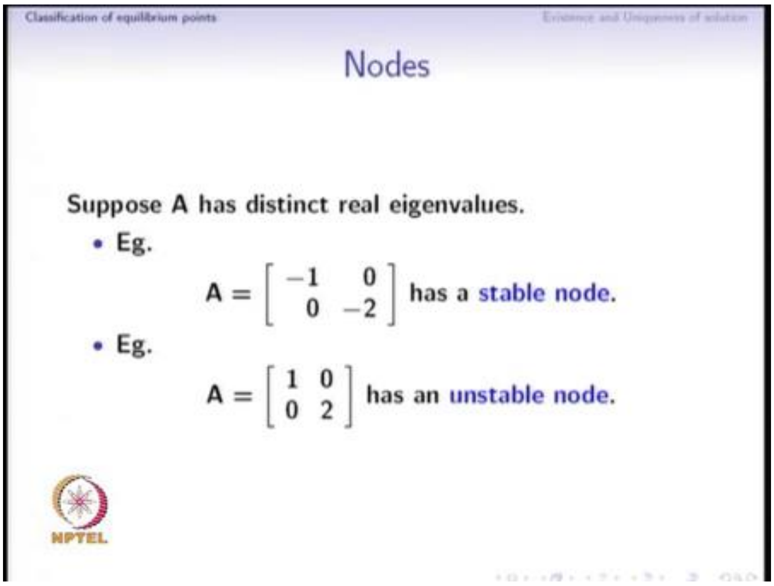

In such a situation it is called a stable node. The other situation when A has both real Eigen values and positive is called an unstable node. To analyze this, we will quickly see how the vector field looks for this particular a. So, look at this figure.

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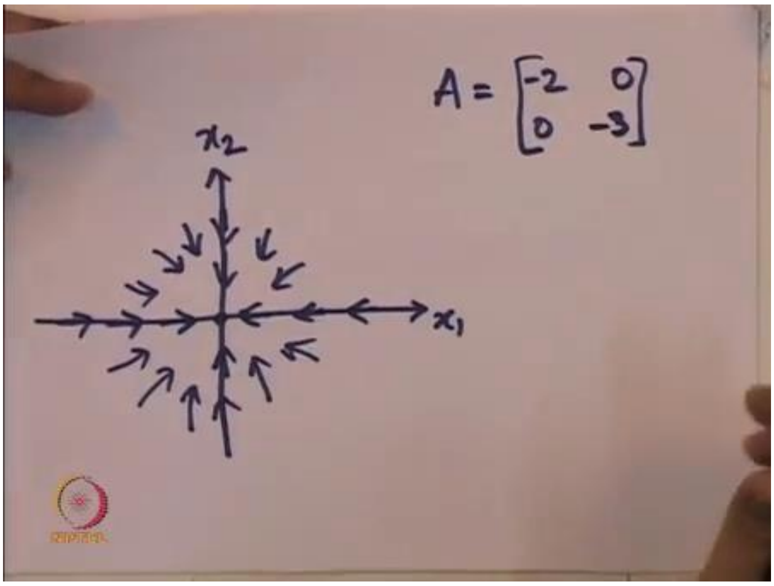

This is not the same example that is there on the slide, but it explains, what is a stable node? This is the x 1 axis this is the x 2 axis, what this says is, if we are along the x 1 axis then because x 2 component is 0 when A acts on such a vector again the x 2 component is 0. That is the significance of a diagonal matrix A.

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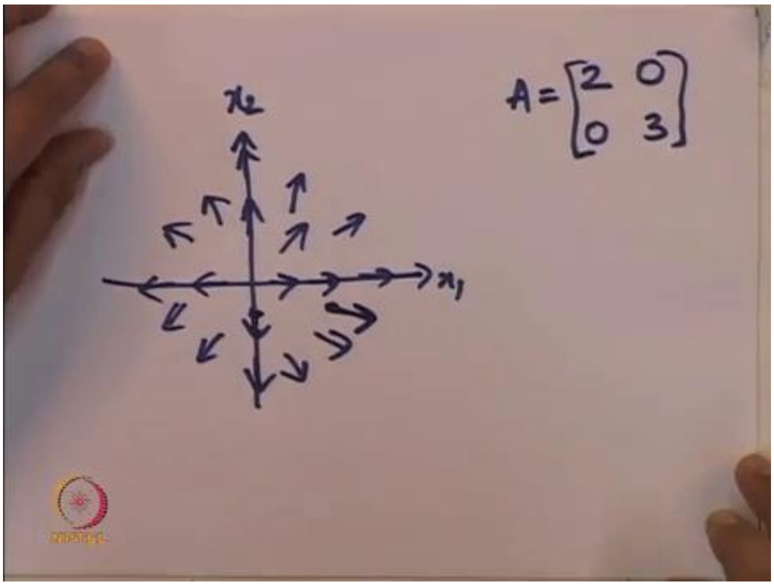

Similarly, along the x 2 axis, x 1 component is 0 and this diagonal entries being negative, imply that that is also along the x 2 axis, the arrows. The relative distance the relative length of the arrows certainly depends on the x 1 and the x 2 components, but then as far this picture is concerned as far as qualitative study is concerned this explains how the various arrows are.

So, the origin the unique equilibrium point appears to be a stable node. It is a node, all arrows are directed towards it. There is no rotation involved because the off diagonal elements are equal to 0 and all arrows are directed towards the origin. This is what we saw as a stable node. We will later see that it is a asymptotically stable, in the sense of Lyapunov. Let us quickly see what an unstable node is.

Take the same A except that the, diagonal elements have sign opposite. Again because of the diagonal nature of A along the axis the arrows are parallel to the axis to themselves, with careful attention to the arrows, whether they are in a positive direction of x 1 or negative direction of x 1. It will be away for the origin because of the positive sign of the diagonal elements. For points, which are not along the x 1 axis, by just super imposition, because this is a linear vector field, by super imposition.

For example, at this point the x 1 component of this arrow can be obtained by this point. The x 2 component of the arrow can be obtained by the arrow at this point. This is the net arrow. So, this is what we can obtain by a super imposition because A is a linear math because we have a matrix that decides the vector field at different points. So, before we go to stable and unstable focus, we will quickly see what diagonal has got to do with what we are studying. So, if we are given with a general A let us say 4 5 6 7.

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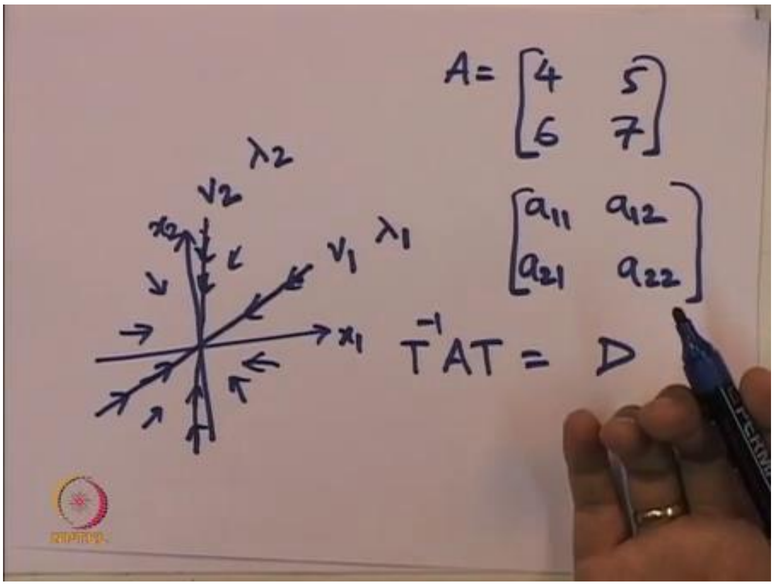

So I am just guessing the various elements. Suppose the entries are such a 1 1 a 1 2 a 2 1 a 2 2, suppose the entries are such that, this matrix is diagonalizable it may not be diagonal itself. In other words a 1 2 and a 2 1 might not be 0, but if it is such that there exists a non singular matrix T such that T inverse A T is equal to a diagonal matrix. Then by choosing the columns of T as a basis we still have this decoupled vector field.

Decoupled vector field like we saw for x 1 x 2, we can see it is not along x 2 x 1 x 2 axis any more, but suppose this is one column of T 1 and suppose the other column of T 2 is like this. In general the two columns need not be perpendicular to each other. Suppose, this is Eigen vector v 1, this is Eigen vector v 2 and suppose this Eigen vector corresponded to the Eigen value lambda 1 and this corresponds to Eigen value lambda 2. These are the x 1 x 2 axis these are not the Eigen vectors. More generally Eigen vectors are vectors v 1 and v 2 which may or may not be perpendicular to each other.

These Eigen vectors are corresponding to Eigen values lambda 1 and lambda 2. So, if lambda 1 is negative, then we can draw the arrows just like we had drawn for a stable node and if lambda 2 is also negative these arrows also can be drawn, towards the origin. Other places arrows can be filled again as I said by super imposition. So, more generally if A is diagonalizable we have 2 directions called Eigen vectors along which we can draw the arrows either towards the origin or away from the origin, depending on whether lambda 1 is negative or positive respectively. In which case again we are able to decide whether the node is a stable node or unstable node. Our assumption till now has been that, both the Eigen values are of the same sign. When they are of different sign that is the next thing we will see. Before we see the situations when the Eigen values have the opposite sign, we will start with what a center is.

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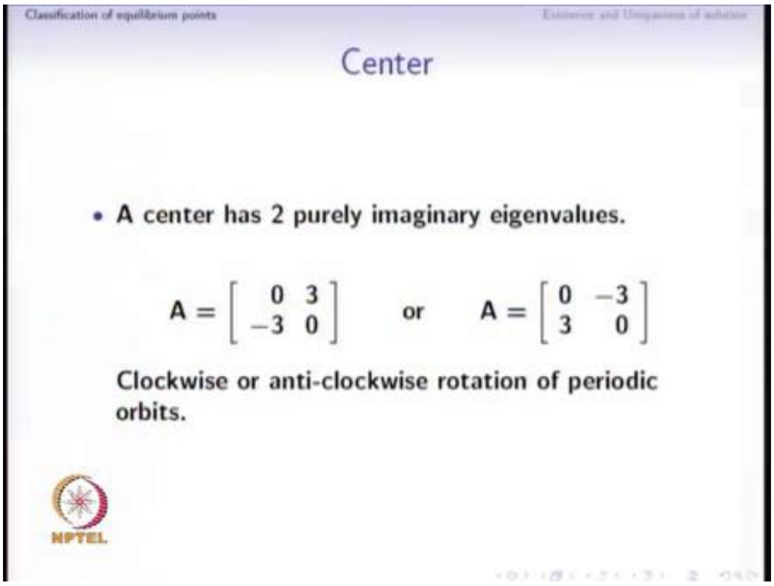

This is a situation when A has purely imaginary Eigen value.

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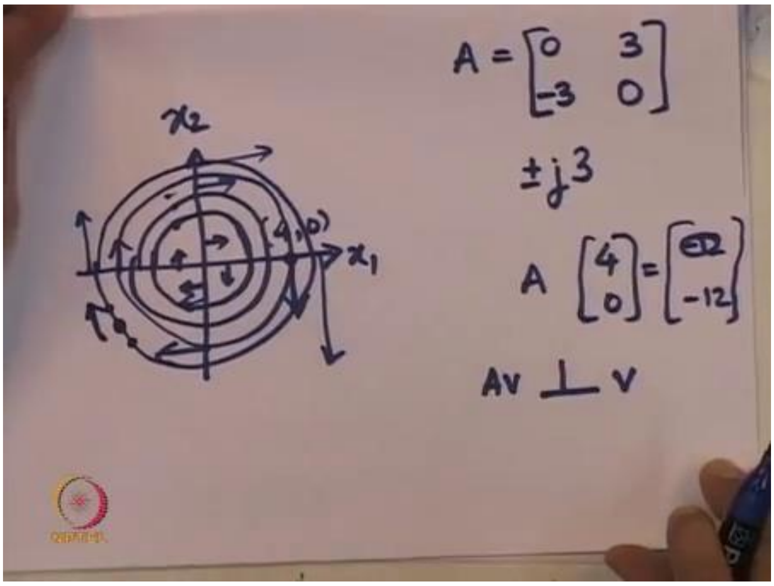

Take for example, this because of this particular form and in which the diagonal elements are equal to 0, the Eigen values are plus minus 3 times j. So, this corresponds to as I said rotation about the origin either in the clock wise or the anti-clock wise direction, which we will decide very quickly. So, take a point along the x 1 axis, suppose this point is equal to 4,0 the point 1 the x 1 axis has x 2 component equal to 0, when matrix A acts on this we get minus 12 sorry 0, minus 12. We see that we get a vector, which is parallel to the x 2 axis and in the negative direction.

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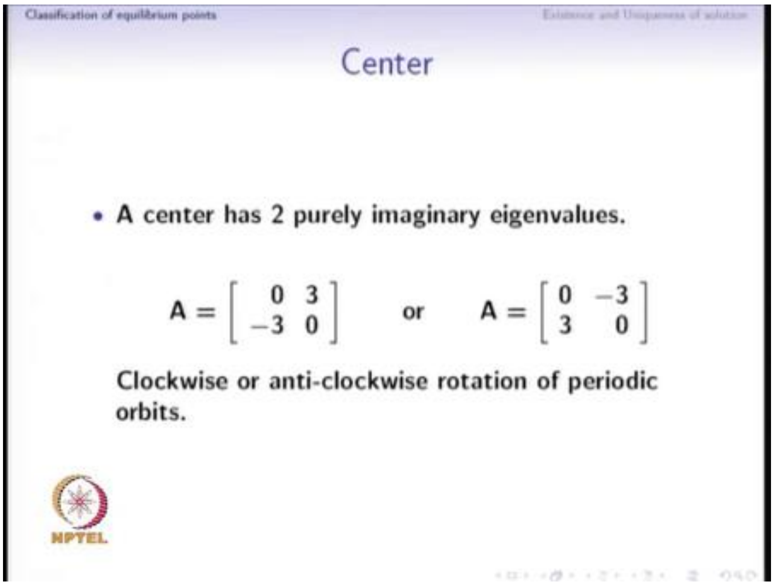

So, this is the arrow at the point 4, 0. Similarly, when we draw these arrows at different points, we see that we have a rotation in the clock wise direction. Every point except the origin, if we start at any point, then we are continuously rotating and it turns out that the vector A times v is perpendicular to the vector v itself. So, if are at any point v, then the arrow at that point A v that is perpendicular to this. We see that this is nothing but what corresponds to pure rotation in which the velocity is perpendicular to the radius vector.

The clock wise or anti clock wise just depends on whether the sign whether we have a plus sign here or a plus sign here. So, this other examples that is there on the computer corresponds to an anti clock wise rotation because we have a negative sign here and a plus sign here, we have a anti clock wise rotation for the second example of A and both the A corresponds to periodic orbits. With the number 3 indicating the frequency, but since we are interested in a qualitative study, the precise value of the frequency is not significant.

Another important point to note here is, we have a collection of periodic orbits. For each initial condition the radius, the distance from the origin decides which periodic orbit it is. The x 1 x 2 space itself is made up of periodic orbits, which are all very close to each other, which form a continuum. From each form each initial condition x 1 x 2 there is a periodic orbit, unique periodic orbit going around it. If we go a little away or little closer to the origin, then we have another periodic orbit.

So, for the situation that A has imaginary axis Eigen values, we have a continuum of periodic orbits and for a liner system it is not possible to have isolated periodic orbits. As you saw in one of our introductory lectures, that we can have isolated periodic orbits for a non-linear system, but for a linear system when we have periodic orbits, it appears that we have we have a continuum of periodic orbits. In other words if we start from a slightly different initial condition, then it is very unlikely to be on the same periodic orbit.

If we are on this periodic orbit, starting from this initial condition unless we are perturb unless we perturb the initial condition to another point on the same periodic orbit, the periodic orbit is going to be different. If it is from this initial condition, then this initial condition corresponds to a different periodic orbit which means that different amplitude even though it is a same frequency. So, this is an inevitable situation with linear systems when we have periodic orbits.

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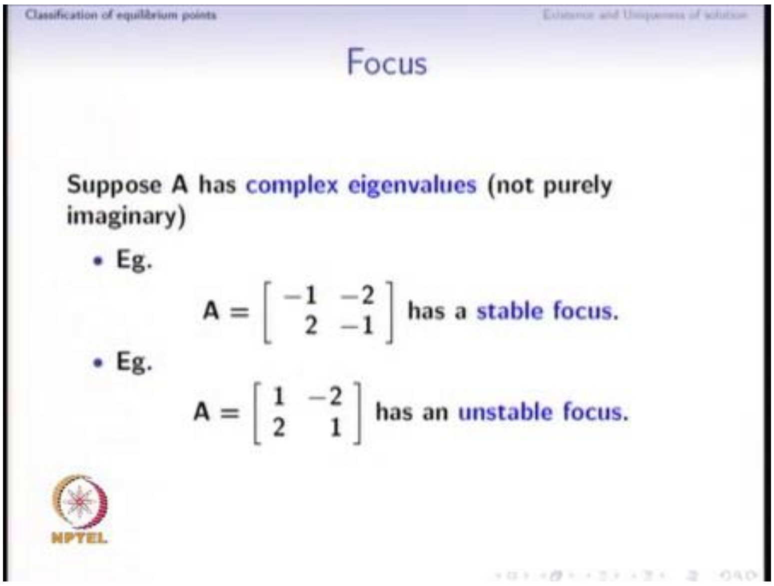

The next type of equilibrium points we will see is when a have complex Eigen values. These Eigen values are not purely imaginary. Take for example, A equal to. So, in which the diagonal elements are equal to minus 1 and the off diagonal elements have opposite signs 1 is plus 2 1 is minus 2.

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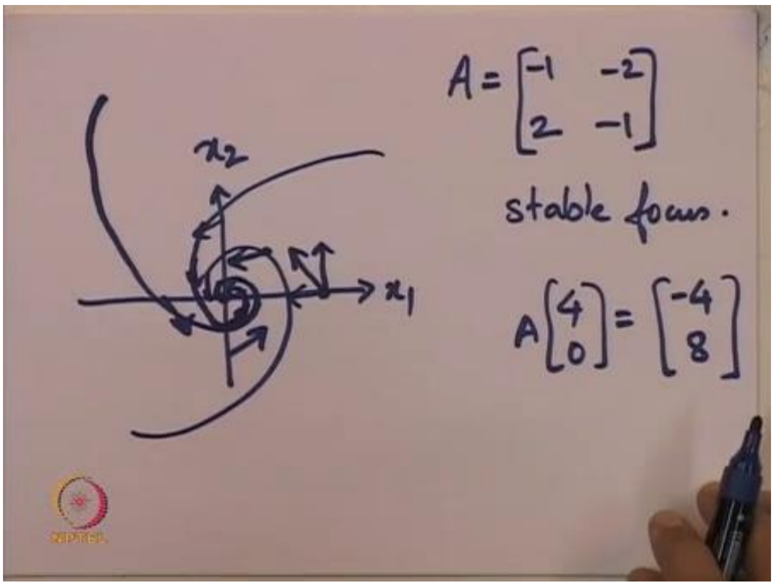

So, this we will call is a stable focus. As I said the off diagonal elements cause rotation about the origin. Each of these cases A is non-singular, hence the origin is the unique equilibrium point. So, let us take an example of a particular point and decide where the arrow is when we are at this point. So, this point for example, is 4, 0 when A acts on this we get minus 4 and below we get plus 8. So, this is a vector which is like this.

There is minus 4 component towards the origin and 8 component along the positive x 2 direction because of which we have this. When we take different points, we see that it is no longer perpendicular to the radius vector, but it is directed inwards. So, every point it turns out that we have some rotation and eventually the trajectories come to the origin. For example, if this point if we draw arrows at different points, all trajectories seem to be approaching the origin even though they do not approach the origin in finite time. Each trajectory, these trajectories do not intersect, but they all approach the origin and they reach the origin only asymptotically. So, this is the stable focus and unstable focus is also very easy to see.

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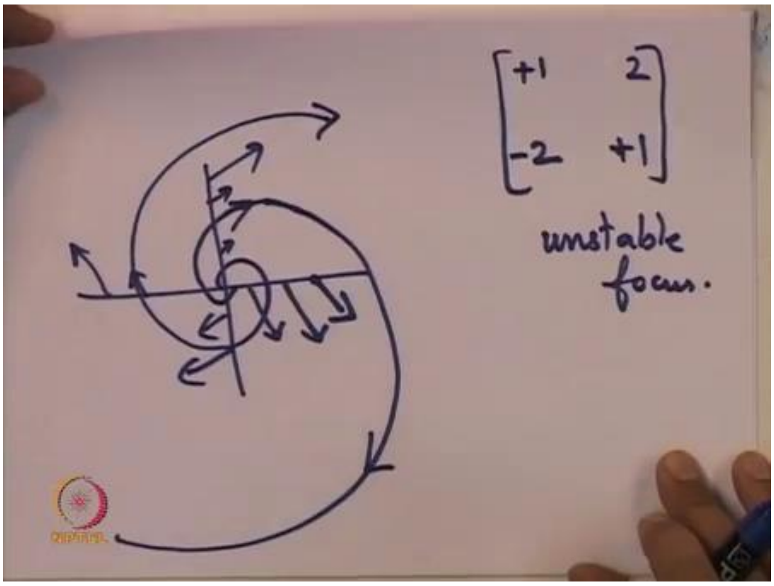

Only that the diagonal elements have positive sign. Now, all the arrows are directed away from the origin, also the rotation has been reversed because the signs of this. The previous example have been interchanged. So, here is an example where the arrows are all directed outwards. So, we have at any point we have trajectory that is going away different points are all going away from the origin. So, this is what we will call an unstable focus. Finally, we will see what is a Saddle point?

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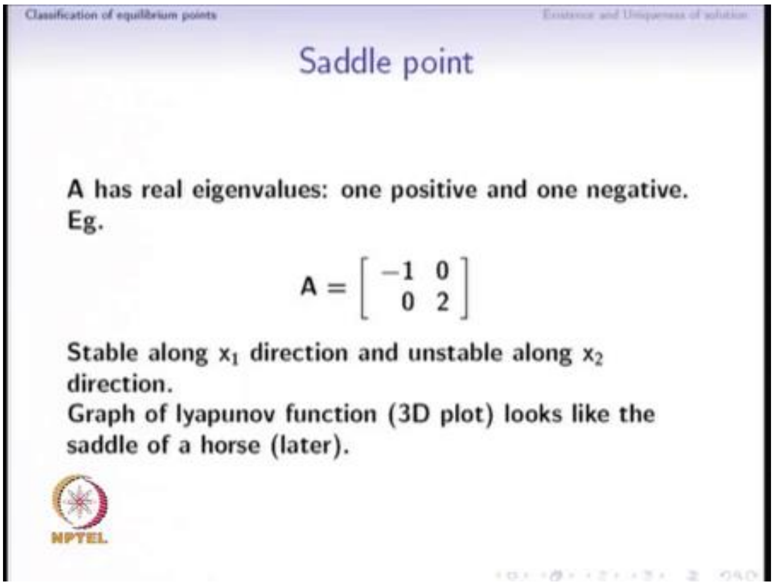

So, the situation when A has real Eigen values 1 positive and 1 negative. Again for simplicity, we will start with the diagonal case. That time because it is diagonal again we have a decoupled nature of the face portrait. So, we see because A is equal to minus 1 0 0 plus 2 along x 1 direction its approaching the origin.

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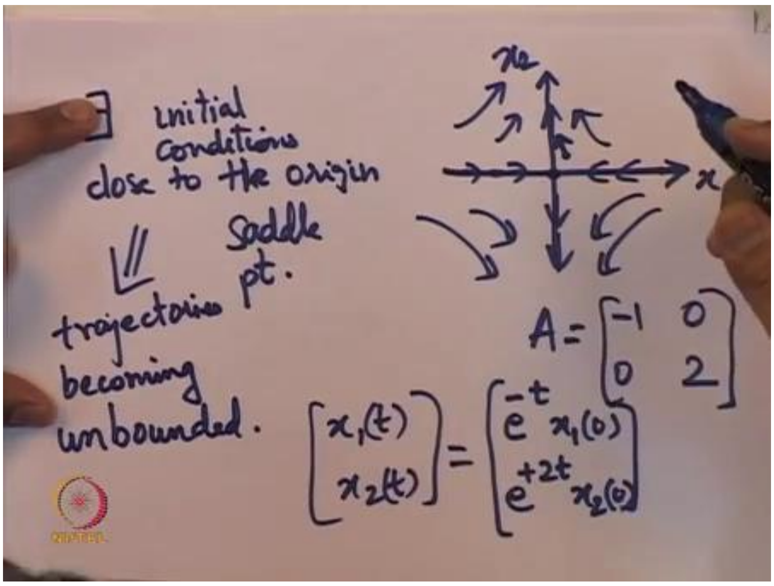

While along the x 2 direction it is going away from the origin. Any other point is a super imposition of these 2 features. So, we see that unless the x 2 component is 0 which means we are in the x 1 direction all trajectories are coming towards the origin. Any other point where the x 2 component is non 0 while the x 1 component is still decreasing the x 2 component is going to blow up, why because the solution to the differential equation x 1 t x 2 t, because it is diagonal, can be easily written as e to the power minus t times x 1 0 e to the power plus 2 t times x 2 0.

So, unless the initial condition has x 2 component equal to 0, the x 2 as a function of time is going to grow exponentially. On the other hand if the x 1 component is non 0, it is going to decrease and eventually become close to 0 asymptotically. So, this is what we will call a saddle point. The question arises is the saddle point stable or unstable equilibrium point. We see that while the origin is in equilibrium point for very small perturbations about the origin trajectories either come to 0, if they are along the x 1 axis or they do not come to 0, if they are not along the x 1 axis.

In any case there are very small perturbations, such that the trajectories when they begin from the perturbed initial condition do not approach the equilibrium point. So, in other words there exists. So, this is the symbol for there exist, there exist initial conditions these initial conditions are close to the origin, what is significance of the origin it is an equilibrium point, such that the trajectories are not coming back to the origin. So, we have in fact the trajectories are growing trajectories are becoming unbounded. This is precisely the property that decides that the equilibrium point the origin is an unstable equilibrium point. So, the saddle point is an unstable equilibrium point it is not an unstable focus nor an unstable node that equilibrium point is just an unstable equilibrium point. So, what is saddle about this.

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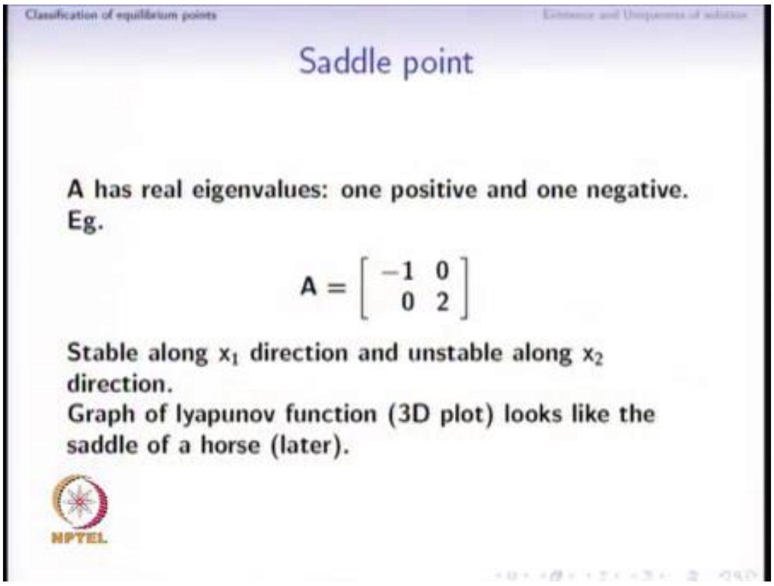

So, the graph of the lyapunov function, we will come back to this later. This graph in the 3 d plot looks like a saddle of a horse that is the reason that this equilibrium point is called a saddle point.

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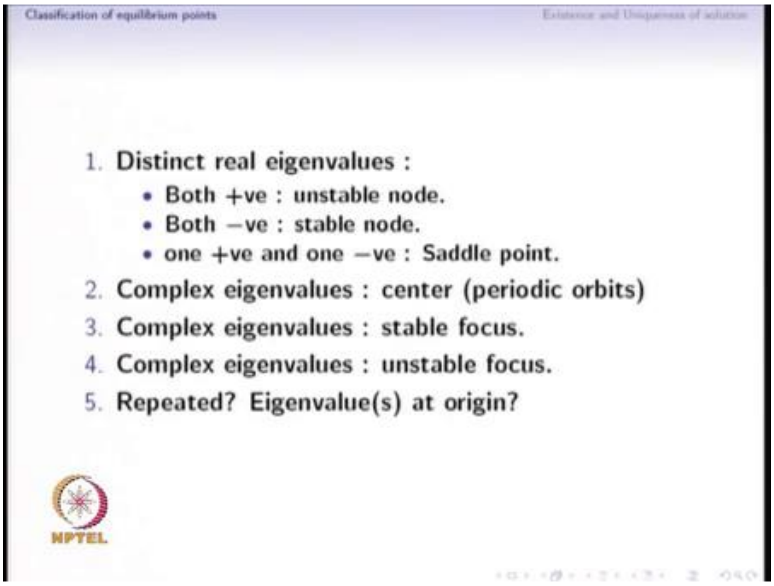

So, before we go to the other situation where there are repeated Eigen values or 1 or more Eigen values except the origin, we will just quickly recap what was done. So, we have seen the situation when there are distinct real Eigen values when both are positive or both are negative or when they have opposite signs. Then we saw the situation, when the Eigen values are both complex in which case, if they are on the imaginary axis we call it as center this is the one that corresponds to periodic orbits. We saw that we will have a continuum of periodic orbits for this situation at the Eigen values are complex.

If they are on the imaginary axis, then it is called as centre when the real part is negative we call it as stable focus and when the real part is positive we call it an unstable focus. So, whether it is stable or unstable depends on the real part of the complex Eigen values. Now, the next situation last situation that is remaining to be seen is when there are repeated Eigen values.

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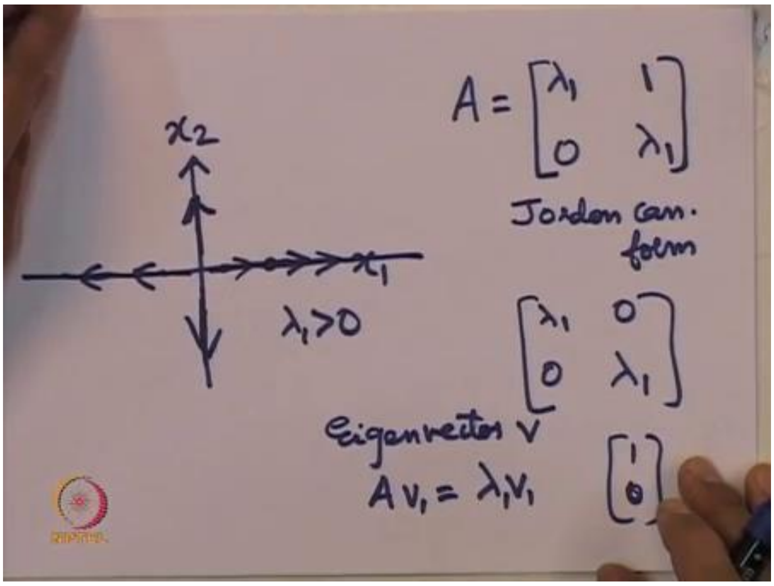

Also the situation when one or more Eigen values are at the origin. Coming back to the matrix A. When there are repeated Eigen values, that time the matrix a may or may not be diagonalizable. Say suppose we have a repeated Eigen value lambda 1 and if A is diagonalizable that is when we will like to put a 0. Here and if A is not diagonalizable, then we put a 1 this is called the Jordan canonical form, for the case when there are repeated Eigen values and a is not diagonalizable.

We are restricting ourselves to the 2 by 2 case and this is the Jordan canonical form. For the case when Eigen values of A are repeated, but A is diagonalizable. So, what is the significance of a diagonalizable matrix we saw that the Eigen vectors are. So, called invariant directions. In this particular example x 1 and x 2 directions are themselves

Eigen vectors, if we are along an Eigen vector then lambda depending on lambda 1 being positive or negative the arrows are directed either away or towards the origin.

So, this is the case when lambda 1 is greater than 0. Let us restrict our study for that situation. The x 2 direction is also an Eigen direction is also an Eigen vector and because lambda 1 was positive, it is again directed away from the origin. So, we see that the Eigen vectors are the invariant directions, what is invariant about it. If the point starts along an Eigen vector because the arrow is also directed along the Eigen vector, we continue in that direction. So, we there is no tendency to move out of an Eigen vector.

Let me repeat Eigen vector v is a non 0 vector such that A v is just a scaling of the vector v. So, we are interested in the first Eigen vector v 1 which is nothing but Eigen vectors are not unique in magnitude. We can scale this vector to any number by any number and also get an Eigen vector. So, it is a non 0 that satisfies this equation. So, this v 1 if we are along this direction if we are at a point v 1. Then the vector is parallel to the vector v 1 because of this particular equation. Hence the trajectory will remain along that particular direction.

If we start here then there is no reason to out of the x 1 axis. Similarly, if we are here we will remain along the x 1 axis similarly, here x 2 also being A invariant direction being an Eigen vector it continues to be along the x 2 axis. So, we see that there are this particular complex plane contains certain invariant sets, what are those invariant sets.

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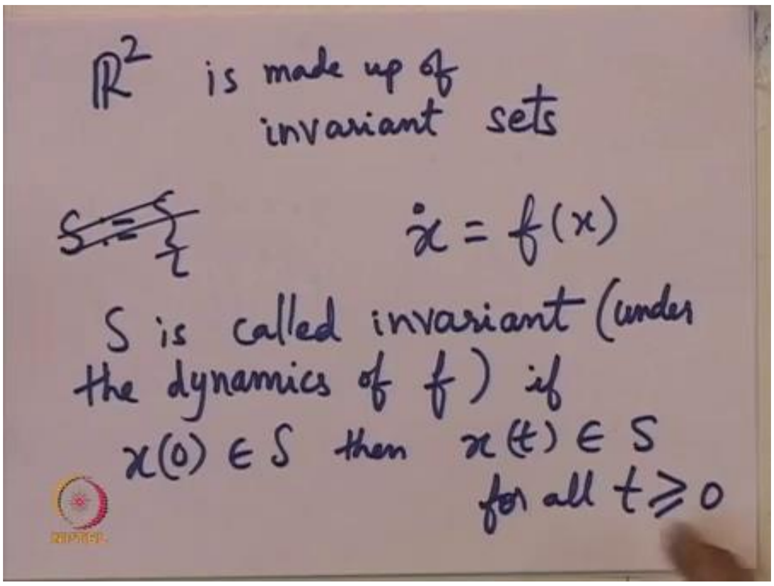

So, we will define the plane R 2 is made up of invariant sets, what is this invariant set. A set S is called invariant, in this case it is invariant under the dynamics. Dynamics of the differential equation x dot is equal to f of x. If we start inside this set S then we will remain inside this set S for all future time is called invariant. Invariant means under the dynamics. Under the dynamics of f, if we start inside S then x of t is also going to be inside S for all for all t greater than or equal to 0. So, that is the significance of an invariant set. That a set S which could be a subset of the plane R 2 or it could be the plane R 2 itself if is called it is said to be invariant if, the initial condition is inside S then the entire trajectory is inside S for all future time. Hence this is also called a positively invariant set, what is positive about it because we are interested only for positive values of time T x of t is inside S. So, what is what are the invariant sets inside R2?

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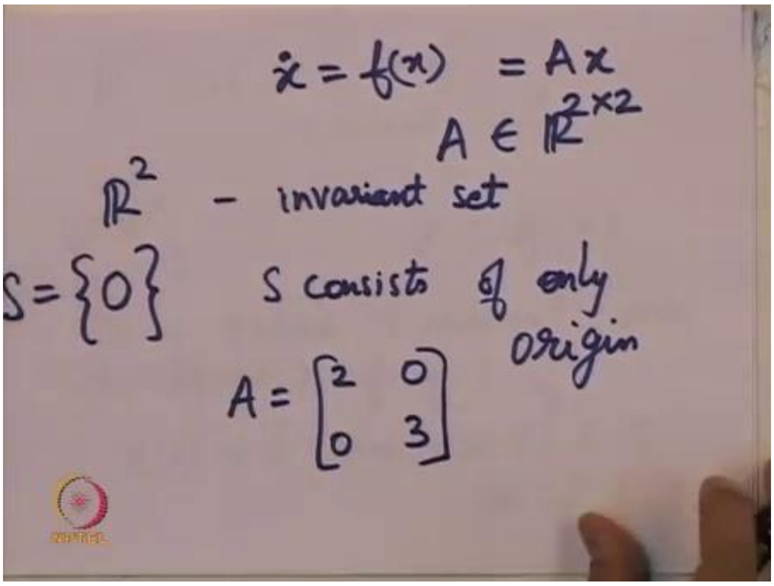

If we have this differential equation and let us take this special case when matrix A acts on the vector x and A is 2 by 2 which means x has 2 components. So, of course, R 2 plane itself is an invariant set why because if it begins inside the set R 2 there is no reason it will leave the plane R 2. If the origin is an equilibrium point 0 the set S consisting of just the origin, S consists of only origin. This is also an invariant set why because if it begins inside this set S because it is an equilibrium point, it will remain at the equilibrium point for all future time. Hence, the set S is also in equilibrium is also an invariance set. So, all equilibrium points is an invariant set. For this particular case when A is a diagonal matrix. For this particular A there are some more invariant sets.

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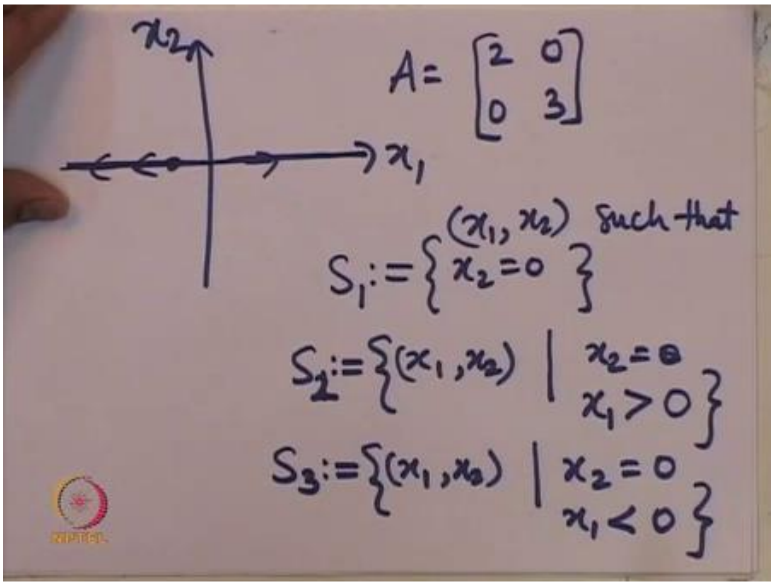

So, take the set S which is defined as all the points, where x 2 is equal to 0. The set of all points x 1, x 2 such that x 2 is equal to 0. This set is also an invariant set why because if we are along the x 1 direction because A was a diagonal matrix, x 1 axis itself being a Eigen vector. We see that the set S 1 which is defined to be the x 1 axis is also an invariant set. Of course, S x 1 axis itself contains a origin which is also an invariant set in other words another set.

Let us call the set S 2 defined as all points x 1, x 2 such that x 2 is equal to 0 and x 1 equal to 0 which is nothing but the equilibrium point is an invariant set, but we are interested in some non trivial invariant sets. For example, we could take x 1 positive. This particular situation is along the positive x 1 direction excluding the origin. This is also an invariant set if it is once inside the set S 2, it remains inside the set S 2. Consider S 3 which is the same x 1, x 2 except that, now x 1 is negative.

This is another invariant set which corresponds to the negative x 1 axis. If the point starts here, then it is going to always remain on the negative x 1 direction. So, these are different invariant sets. So, we are usually not interested in the equilibrium point as an invariant set, we are also not interested in the plane R 2 as an invariant set because these are the trivial invariant sets.

We are interested in some more sets, which are larger than the equilibrium point and smaller than this set R 2, which are invariant under the dynamics of f. The Eigen vectors are examples of such invariant sets. Eigen vectors, the entire null, the entire direction except the origin is also an invariant set and the 2 sides of this Eigen vector, one on the positive side, one on the negative side of the origin also form invariant sets.

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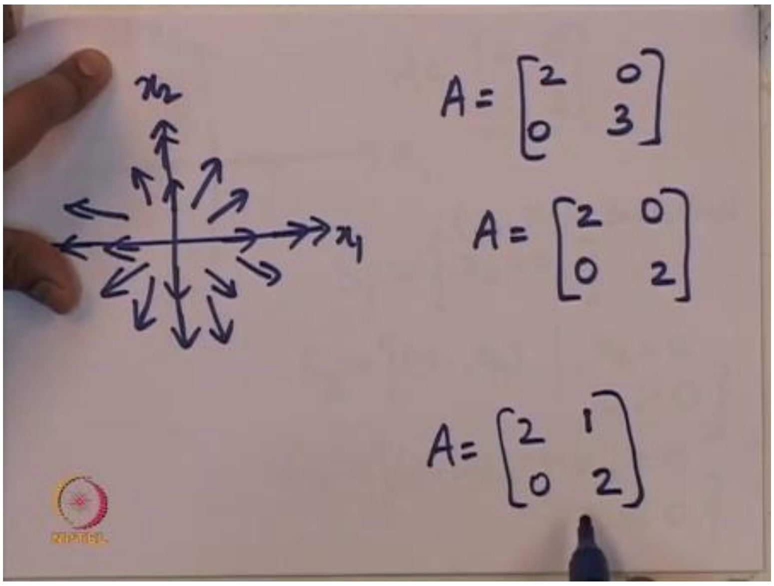

So, coming back to the case, when A is diagonalizable for that situation as we saw in some basis A already looks diagonal. So, we have x 1 axis which is an invariant direction x 2 axis which is also Eigen vector. Hence that is a invariant direction and it turns out that this invariant this 2 directions being invariant is not particularly related to the Eigen values being distinct for the case when a has repeated Eigen values, but if it is diagonalizable. It still is a unstable node of course, in this case every direction is an invariant direction is every line through the origin is a invariant set because the 2 Eigen values are repeated, but for the situation when A is not diagonalizable. So, let us consider the case when a is equal to 2 0 1 here.

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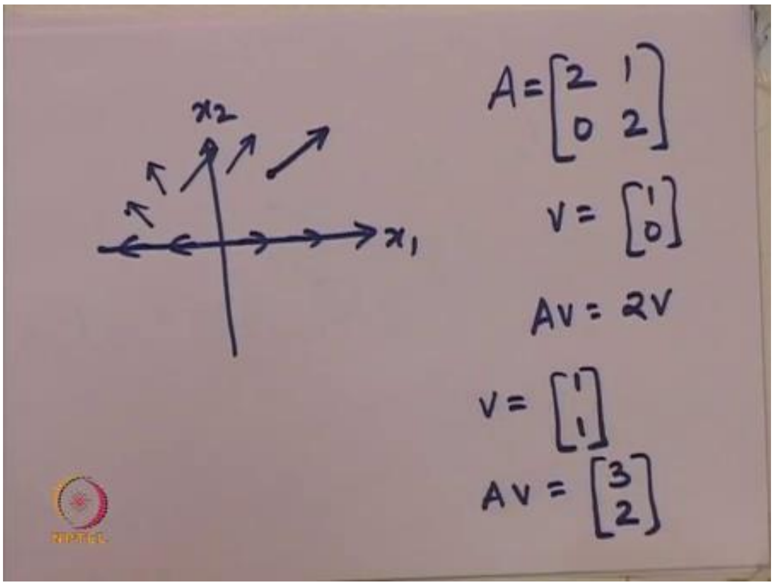

This example of A has only one Eigen vector. The other Eigen vector is what we want to call a generalized Eigen vector. This A, which A are we dealing with now we see that the x 1 axis, if we take the vector v equal to 1, 0 A v is nothing but 2 times v. So, the x 1 axis is an invariant direction. All arrows are directed away from the origin, but there is no other invariant direction, there is only one independent Eigen vector. Hence if you take an example let us see v is equal to 1, 1 when A acts on v we get 3, 2.

Let me check this. So, for this particular vector at 1, 1 the vector has, it has both x 1 x 2 components of that arrow non 0. So, we see that because there is only one independent direction x 2 axis is no longer an Eigen vector, but there are these other arrows that cut. How exactly they cut, they depend on the particular form of the Jordan canonical form, but along the independent axis there is only one x 1 direction. So, this is the significance of a non-diagonalizable A. That there is only 1 Eigen vector x 1 and everything else is emanating out of this x 1 direction. If it is very close to x 1, but if it is along the x 1 axis then x 1 axis being an Eigen vector is an invariant set under the dynamics of f and hence it does not leave the x 1 axis.

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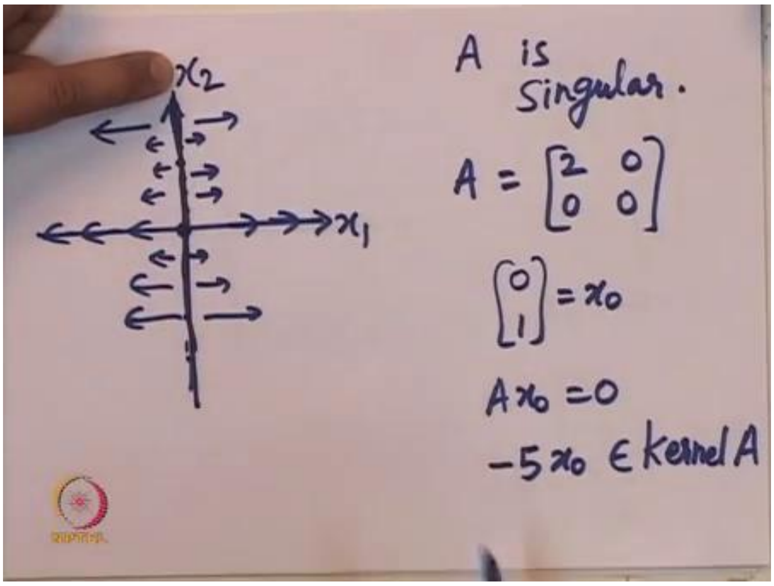

So, this brings us to the final case when A is singular.

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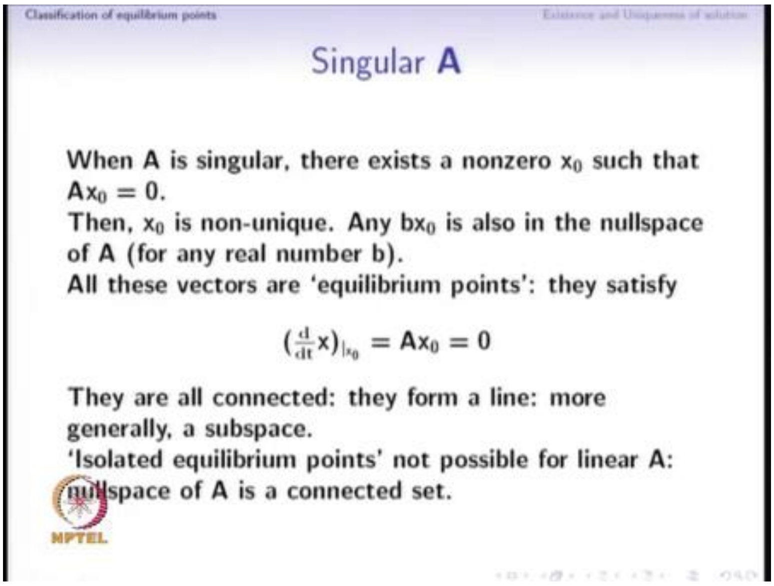

When A is singular there might be 1 or more Eigen values at the origin. So, let use the case when there is only 1 Eigen value at the origin first. So, when A is singular it means that there exists a non 0 vector x naught such that a x naught is equal 0. This x naught is also is said to be in the null space of the matrix A. The origin is always there in the null space, but when A is singular there are some non 0 vectors also sharing the null space. Such a non 0 vector x naught is non unique why because if we are given with x naught, then we can multiply x naught by a real number B and also get B x naught to be in the null space of the matrix A.

So, all these points x naught B x naught any scaling of the vector x naught are all equilibrium points, why because they satisfy the derivative of x at that point evaluated at the point x naught is obtained by A acting on x naught which is equal to 0. So, we see that in this case all the equilibrium points are connected. They form a line the null space which is a linear sub space. In general they form a subspace and in our case because A has only one Eigen value at the origin they form a line. So, as we have seen in the beginning of this series of lectures, we saw that isolated equilibrium points is not possible for a linear system.

For a linear system the equilibrium points as we have seen happen to be in the null space of the matrix A. If there are some non 0 vectors in the in the null space then they are all connected, they form a line. So, the isolated equilibrium points is possible only when we have a non-linear dynamical system.

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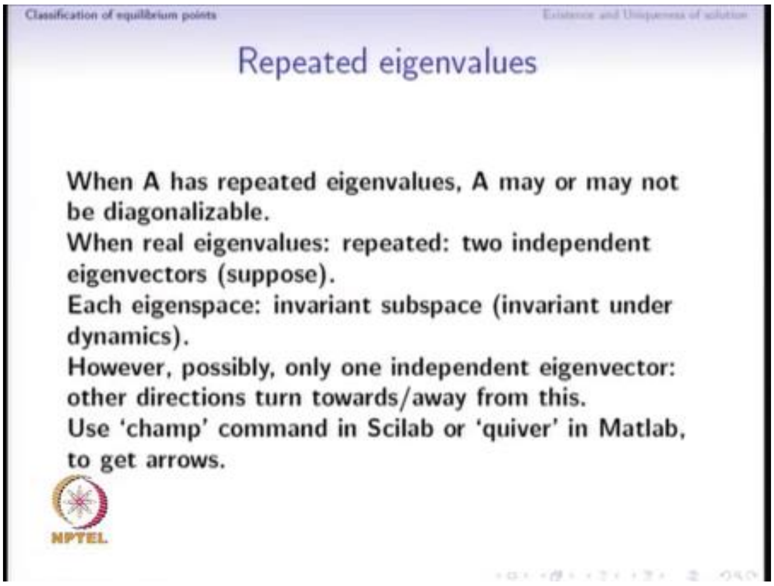

So, we have just began seeing the repeated Eigen values case. When A has repeated Eigen values A may or may not be diagonalizable, we will quickly review this part. So, when the Eigen values are repeated, then they have to be real for the case that A is 2 by 2 matrix. If they are, if the matrix A is diagonalizable then we have 2 independent Eigen vectors. Then each Eigen space is an invariant sub space, invariant meaning it is invariant under the dynamics of the system, but it is also possible that we have only one independent Eigen vector in which case other directions either turn towards this or turn away from this, depending on whether the Eigen values is positive or negative.

So, one can have a look at how the arrows look using champ command in psy lab or quiver command in mat lab when A is singular. So, take for example, A equal to, in this case this is a example, such that A x naught is equal to 0. This is of course, not the only vector x naught that satisfies a x naught equal to 0 because any constant minus 5 times x naught also is in the null space. The null space is also said to be the kernel of the matrix A.

So, what is the significance of this, we see that the x 1 axis is a Eigen vector, but correspond to Eigen value 2 and hence we will draw the arrows away from the origin, but the x 2 axis are all equilibrium points. So, each of the arrows have length 0. So, if the x 1 component is non 0, then we see that the trajectories are having the x 1 component increasing as a function of time increasing with exponent equal to 2, but the x 2 component is always going to become equal to 0 when a multiplies to it.

Hence, we see that these arrows are all parallel to the x 1 direction first of all. Secondly along the x 2 axis because x 1 is equal to 0 along the x 2 axis all these points are equilibrium points, they form a connected set. The origin is not the only equilibrium point for this example, but each of these points are equilibrium points. So, this is what we see for the case when A has one Eigen value at 0. The next example is when A has both Eigen values at 0.

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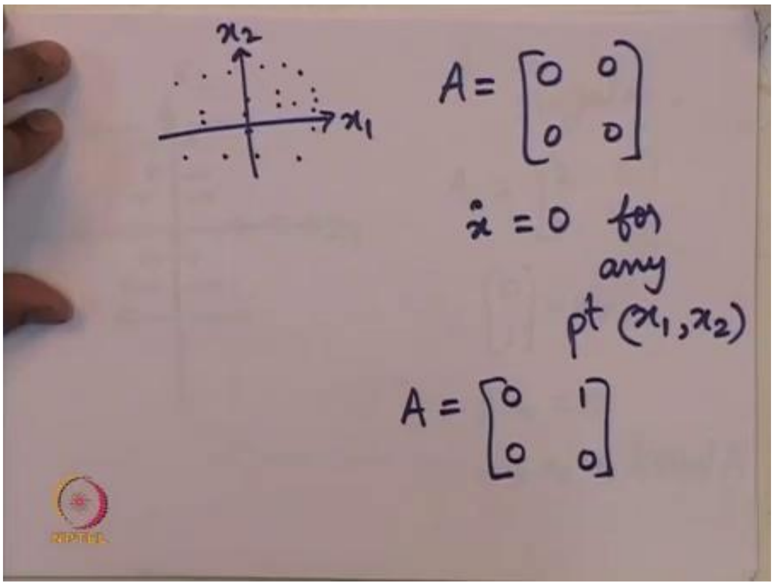

This is again an example of repeated Eigen value. So, let us first take when A has two Eigen values at 0. When A is diagonalizable, that is when we have 0 here. So, A is a 0 matrix. So, the entire R 2 plane is made up of equilibrium points any point x 1, x 2 is an equilibrium point, why because what does this matrix say x dot is equal to 0 times x which is equal to 0. For any point x 1, x 2. So, this is the less interesting case, but still this situation is likely. The other situation when A has repeated Eigen values at 0, but A is not diagonalizable is when we have this for example.

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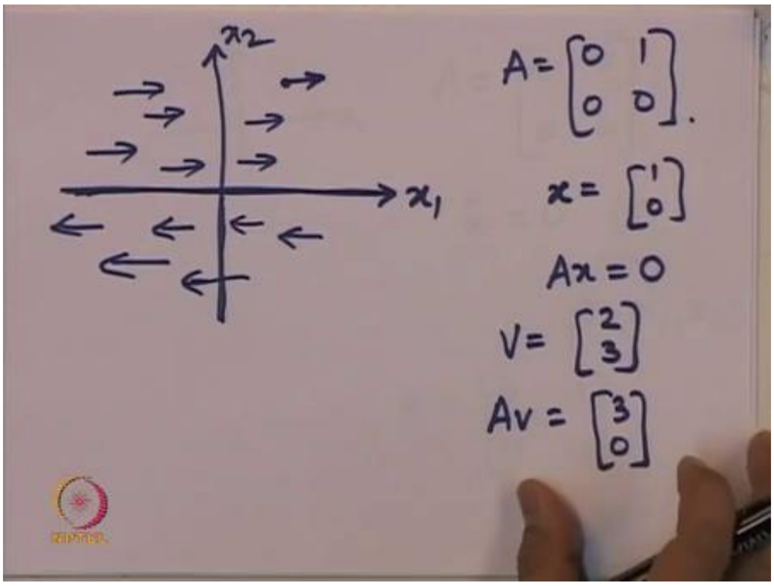

So, in this case we see that, if we have vector x equal to 1, 0 then A x is equal to 0. So, the vector 1, 0 and all linear multiples of this x are in the kernel of the matrix A, they are in the null space of the matrix A. Hence the x 1 direction is a set of equilibrium point what is important about x 1 direction, they all have x 2 component equal to 0, but if we take a vector v which is equal to 2, 3. In particular the second component x 2 component is not equal to 0 this particular vector here, when A acts of v we get something that is parallel to the x 1 axis. So, we see that the arrows look like this.

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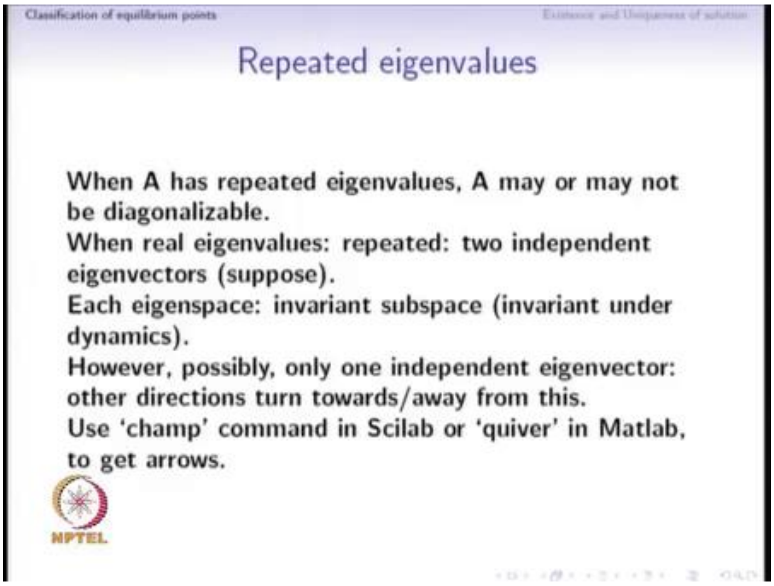

They are all in increasing direction of x 1 when x 2 component is positive. They are along decreasing direction of x 1, if the x 2 component is negative why because A is this matrix and when A acts on a vector v it gives us the second component of v as the first component of A times v. This is an example where we have only 1 x 1 axis which is the equilibrium point set of equilibrium points and every other vector is being turned towards either positive direction of x 1 or negative direction of x 1, depending on whether x 2 component is positive or negative.

So, this completes our study of equilibrium points for second order systems. We have seen the case when A has repeated Eigen values distinct Eigen values and when A has real or complex Eigen values. So, the next important question we will start studying now is, when does there exist a solution to the differential equation.

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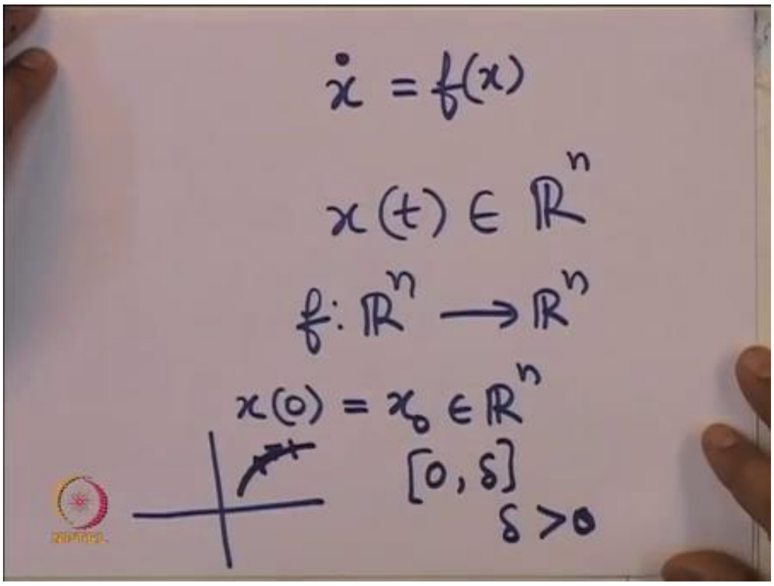

If we are given with a differential equation x dot of t is equal to f of x in which now x has n components at any time instant t x has n components. Hence f is a map from R n to R n. For this situation, suppose we are given with the initial condition x naught, x times t equal to 0 is some vector called x naught, which is an element of R n we are interested in the question. Suppose, this is our space R n, this is our point x naught, then the direction is given here by f evaluated at the point x naught. We are interested in answering the question when does there existed trajectory that starts form the point x naught at t equal to 0 and there is a unique trajectory, for some time duration for a time duration 0 to delta.

In which delta is some positive number possibly very small, but for this duration of time we have a unique solution to the differential equation x dot is equal to f of x. So, this is the question we will answer in the next few lectures starting from now. So, let us look at this differential equation.

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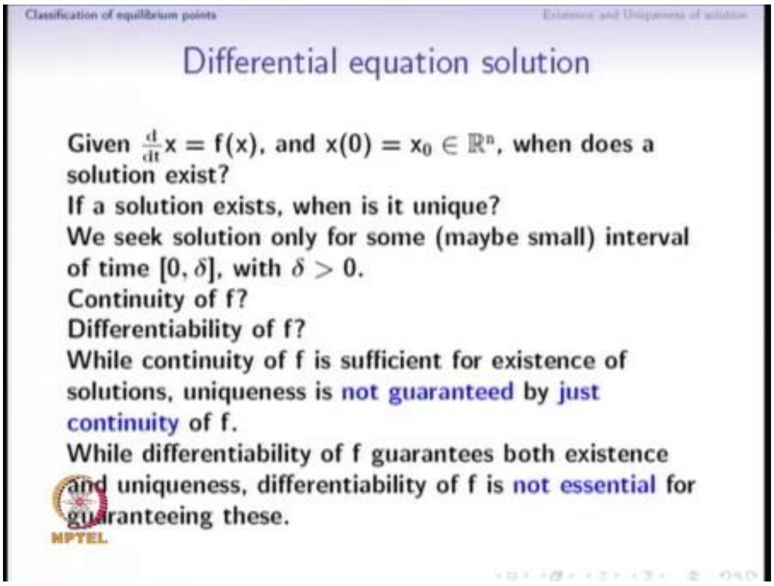

So, given d by d T of x is equal to f of x and the initial condition x 0 is equal to x naught an element in R n, when does the solution exist. Then we will ask if a solution exists when is it unique. Under what conditions on f at the point x naught do we have a solution and when is it unique. So, please note that we are interested in a solution possibly for a very small interval of time. It might be difficult to guarantee existence and uniqueness of solutions for a large duration of time, but we are interested only for an interval 0 to delta in which delta is greater than 0 possibly quite small.

So, we ask is continuity of f, the important property here or is it differentiability of the function f at the point x naught that is required here. So, it is important to note here that while the continuity of the function f is sufficient for existence of solutions, uniqueness of the solution is not guaranteed by just continuity of the function f. On the other hand while differentiability of function f guarantees both existence and uniqueness of the function f, both existence and uniqueness of solution to the differential equation x dot is equal to f of x. this differentiability of f is not essential for guarantying existence and uniqueness of the solution.

So, keeping note of this we can ask, what is the important property required for existence and uniqueness of a solution to a differential equation, it appears to be a property that is a slightly more strong condition than continuity, but might not be as strong as differentiability of the function f at the specified initial condition x naught.

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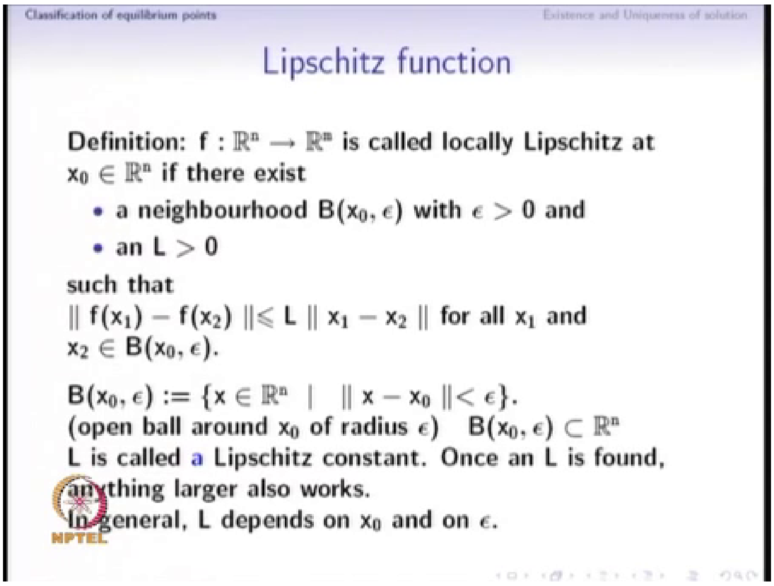

So, it turns out that this property is an important property called Lipchitz condition, on the function f. So, what is the definition this definition is valid for a function f from R n to R m, even though in our case f is always from R n to R n, we will define this definition if Lipchitz for a case when f is a map from R n to R m. So, it is said to be locally Lipchitz at a point x naught, if that exists a neighborhood B of x naught of radius epsilon, we will see a precise definition of a neighborhood very soon.

A neighborhood B of x naught, epsilon with epsilon greater than 0 and a constant L greater than 0. Such that an inequality is satisfied what in equality, f at x 1 minus f at x 2 norm, this distance is less than or equal to L times x 1 minus x 2. This in equality is required to be true for all x 1 and x 2 in the neighborhood, in that neighborhood of the point x naught. So, this neighborhood is being called as a ball B centered at x naught and of radius epsilon. So, this is the precise definition of the ball. So, B x naught, epsilon is defined to be the set of all points x such that distance of this point x from epsilon is strictly less than epsilon.

It is not more than epsilon away from the point x naught. Even equal to epsilon away we are not including into the ball B x naught, epsilon and hence this is called an open ball around x naught of radius epsilon. Around which point the ball is centered, that is centered around the point x naught and what is the radius that is epsilon. We are saying it is an open ball because this distance is strictly less than epsilon. So, this ball is contained in R n because we are taking all points in R n that satisfies this condition.

So, for this for some ball around the point x naught with a radius strictly greater than 0 we should be able to guaranty that, this inequality is satisfied for all x 1 x 2 inside this ball. So, this number L positive number L is said to be a Lipchitz constant. It is not unique because if we have found a constant L, such that this inequality satisfied for all x 1 x 2 in the ball B x naught, epsilon then you can take a number larger than L.

For that larger L also this inequality would be satisfied. Hence, we see that this Lipchitz constant is not going to be unique, but in general this Lipchitz constant L will depend on x naught and on epsilon. It will depend on the point x naught itself, and also on the radius epsilon, radius epsilon radius epsilon of the ball of the open ball around x naught. So, using this definition of Lipchitz function it is possible to specify under what conditions solution to a differential equation exists and when it is unique.

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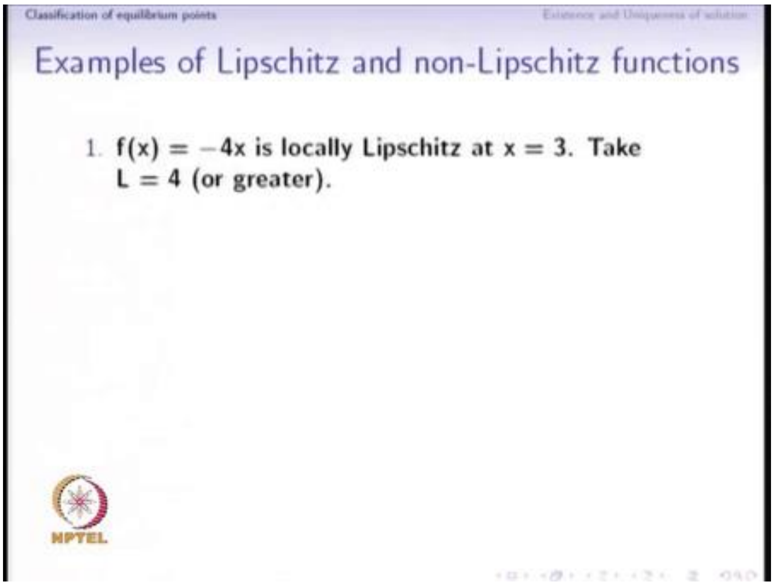

So, we will see some examples of a Lipchitz function and of some non Lipchitz functions. So, the line f x is equal to minus 4 x is locally Lipchitz at the point x equal to 3. If it is Lipchitz then we are we should be able to give a number L such that that inequality satisfy and here we can take L is equal to 4. So, notice that we can take the slope of the function f absolute value of the function f or we can take something larger.

To understand the Lipchitz function we will take graph of a function f for the situation that f is a map from R to R.

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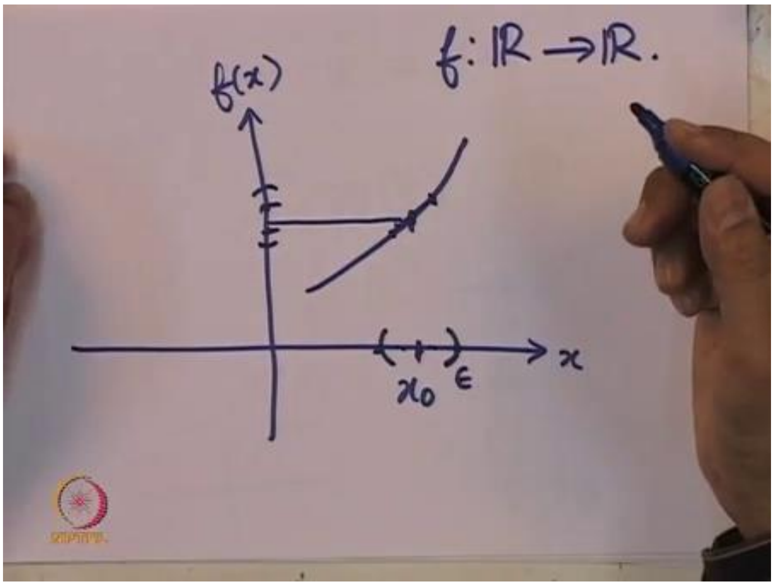

Suppose, this is our point x naught and this is a graph of the function. So, what does this say, f is said to be Lipchitz at the point x naught if, there exists a ball of radius epsilon which means that this point is x naught plus epsilon. This point x naught minus epsilon and both these points are not included in the ball because it is an open ball. In other words this interval is an open interval. So, for this particular ball we require some inequality to be satisfied. So, we take all the points take any 2 points x 1 and x 2 in this ball. We look at the corresponding distance between them, and when we connect. So, it is required to draw a larger figure to be able to see what the Lipchitz function is specifying on the function f.

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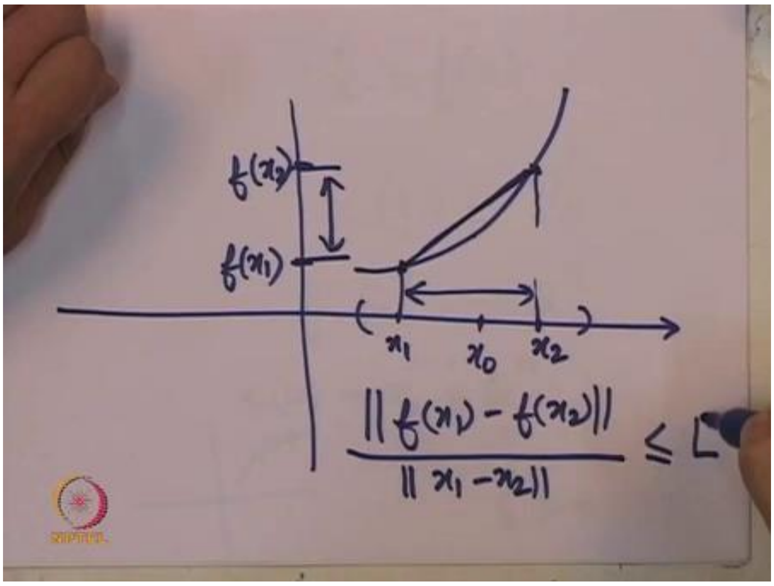

This is the point x naught, this is the ball in this case it is an interval of width 2 epsilon, an open interval of width epsilon in which the center is x naught. Suppose, we take x 1 here and x 2 here they do not have to be on opposite sides of the point x naught. So, what is being specified is this is the value at x 1. This is the value of f at x 2 and the distance between f x 2 minus f x 1 the distance between f x 1 and f x 2 that distance is nothing but this gap. This gap divided by this gap this ratio in absolute value should not exceed capital L f x 1 minus f x 2 in absolute value.

In this case it is just absolute value more generally it is a norm should not exceed L. They should exist in number L such that this in equality is satisfied for all x 1 x 2 in the ball around x naught of radius epsilon an open ball in this case it is just an open interval. So, this particular ratio is nothing but absolute value of the slope of this line that connects this point. This point which point the point with x 1 f x 1 here and x 2 and f x 2 here when we connect these 2 by a line then the slope of this is precisely this, but without the absolute values.

Once we take the absolute values, then it is absolute value of the slope of this line and the Lipchitz condition on f at the point x naught says that there should exist a ball around the point x naught of radius epsilon and a number L, such that the line has slope of absolute value at most L. There should exist one number L, such that this slope is bounded from above by L, the absolute value of the slope. So, this property of Lipchitz condition is a key property. We will see examples of Lipchitz and non Lipchitz conditions functions. It will play a key role for existence and uniqueness of solutions to a differential equation. This is what we will see in detail from the next lecture.

Thank you.