Nonlinear Dynamical Systems
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Lecture - 29
Describing: Optimal Gain

Welcome to this next lecture about describing functions, we are going to continue describing functions and in particular we will use describing functions for finding amplitude and periodic and frequency of periodic orbits. Of course we should again note that this is only an approximation method, we will also see some conditions under which describing functions provides us a guarantee of existence of periodic orbits. And some other conditions which has a guarantee for nonexistence and necessary conditions for existence and sufficient condition for non-existence. Sorry, I might have gone wrong in this we will see a sufficient condition for existence of periodic orbits and a sufficient condition for nonexistence of periodic orbits.

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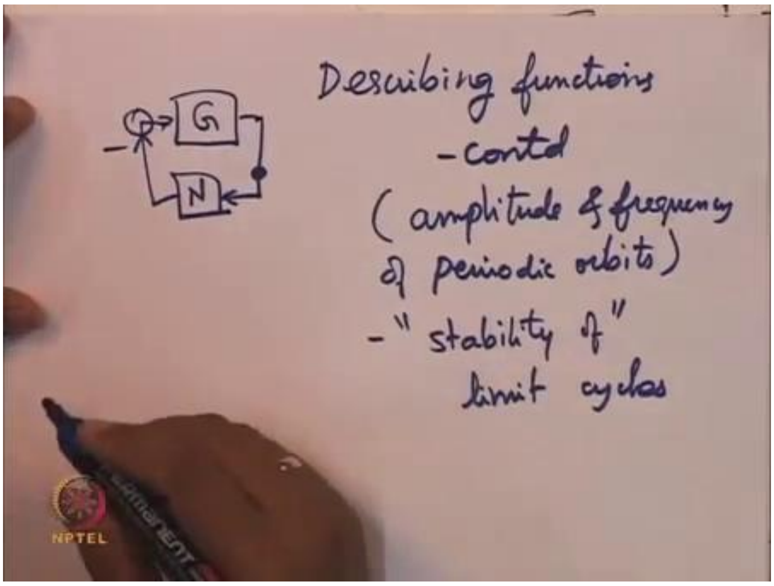

So, we the last thing we noted was that if we have a feedback loop like this, we had we had reviewed a describing functions, we also derived a describing functions for the jump instances. That was an example of a of a non-linearity with memory the memory aspect about it also made brought in an imaginary part of the describing function. Because the other non-linearities that we had considered namely the dead zone, the signum non-linearity and the saturation non-linearity they were all memory less. They were odd that is why they happened to be real they had only a real part.

But, then the jump instances also depends on the rate of change the sin of the rate of change of the input signal input sinusoid that is what brought in the imaginary part also. So, today we are going to see how to find amplitude and periodic and frequency of periodic orbits. What we will not see in this course is stability of the limit cycle of course this periodic orbit is already a limit cycle stability of the limit cycles. What this means is of course we are just seeing we are seeing some conditions for existence of these periodic orbits whether for closely for initial conditions that are perturbed from the periodic orbit if it is an initial condition close to this orbit.

But, not on the periodic orbit those trajectories also converge to this periodic orbit in that case we will call this periodic orbit. We will call this limit cycle stable limit cycle or do they go away. So, let us see what happens in the case of linear systems there we have already noted very early that we have continuum of periodic orbits. And that is the situation that we will just verify again using the describing function method also.

So, please come back to this figure, so we started with some point here it is important to note that the amplitude and frequency of the periodic orbits that we obtain. The frequency of course is the same throughout anywhere in this orbit. But, the amplitude that we obtain by calculations is amplitude of the sinusoid at the input to the non-linearity it is not the amplitude at this point. It is the amplitude in the sinusoid at this point, but the frequency of course remains the same everywhere inside this loop. So, let us take an example.

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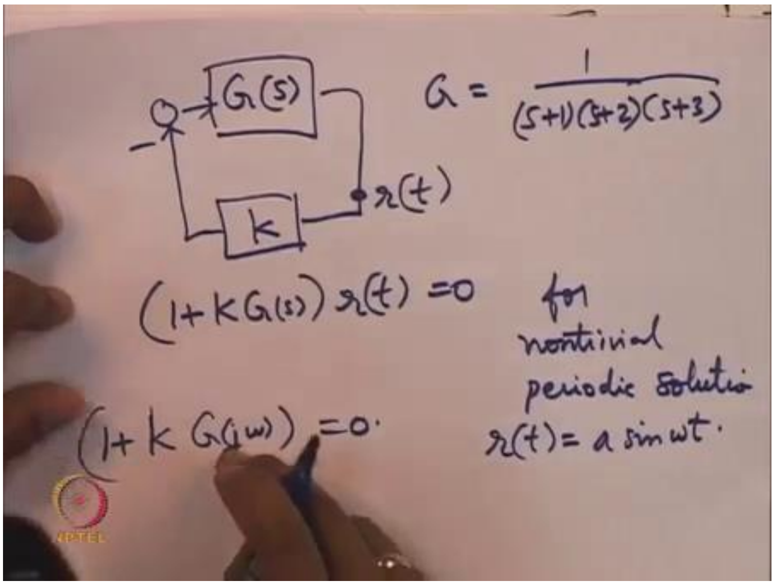

So, consider again G of S with G equal to 1 over S plus 1 S plus 2 S plus 3 connected with constant k in the feedback. Now, we this is this is like solving for 1 plus k G of S r of t equal to 0. Notice the close relation with Nyquist plot, we start from a point here we call this r of t it goes through gain k and minus k after going through this point. And then minus G k that acts on r is same as r that equation gives us this for non-trivial periodic solution this should be a non-trivial periodic solution. Let us assume that this r of t is equal to a sine omega t that gives us 1 plus k G of j omega equal to 0. This is like asking for what value of k the k has to be real we allow amplification only by real numbers positive or negative that is a minor issue.

Is there some value of k such that this becomes equal to 0 the fact that k has to be real means that the Nyquist product of g has to intersect the real axis. If k has to be positive then it also means that j of j omega real part of j omega has to be negative. If k has to be positive for this equation to be satisfied, for any omega if this should be satisfied for some omega then these conditions have to already be true.

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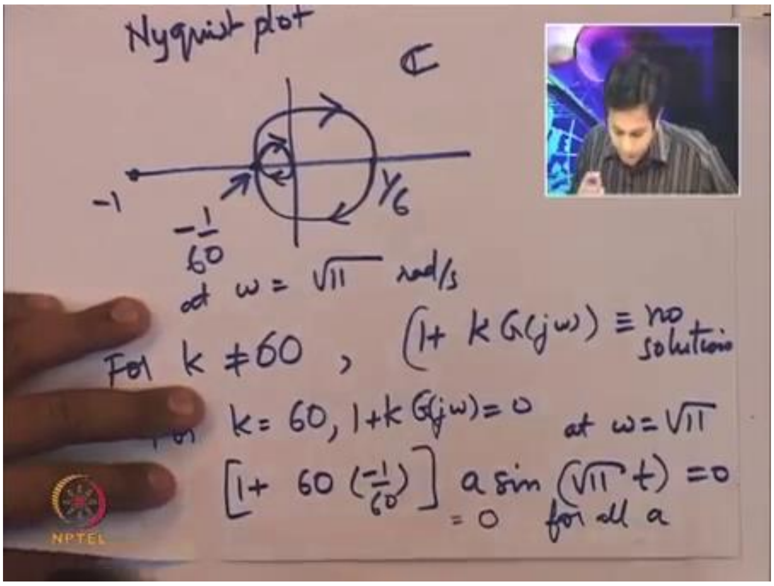

So, plot Nyquist plot of this transit function you already noted was equal to this is a point minus 1 and this is 1 by 6. And this we already noted is minus 1 by 60 and that happens to be at omega is equal to square root of 11 radians per second. Of course plus minus both plus square root of 11 and minus square root of 11 both then indeed Nyquist point intersects at two points here.

Now, for k not equal to not equal to 60 1 plus k G of j omega no solutions for k equal to 60 1 plus k G of j omega equal to 0 indeed equal to 0 at omega is equal to square root of 11 . Let us take only positive frequency of course minus 11 also, but so what it just means that the periodic solution we have two solutions, so that the signal eventually is real. To get a real signal we need two independent complex sinusoids which when we add by a real combination we get a real signal. That is the reason we have minus square root of 11 also that is besides the point when dealing with real signals.

Now, at k equal to 60 alone this equation has a solution for some omega when k is not equal to 60 this particular thing does not it does not have a solution for any omega. Now, let us see at k equal to 60 at k is equal to 60 we get at k equal to 60 and omega equal to square root of 11 we obtain 1 plus 60 times minus 1 by 60. This acting on a sine square root of 11 t equal to 0 this is a differential equation if there is a dependence on omega. Here one should interpret as differentiating this what we are writing here is a differential equation when applied to this particular sinusoid. When applied to sinusoid is when the frequency very conveniently comes as evaluated at j omega. So, this we get is equal to 0 for all for all a and for any amplitude. So, if you ensure that k is equal to 60 and the frequency omega is equal to square root of 11.

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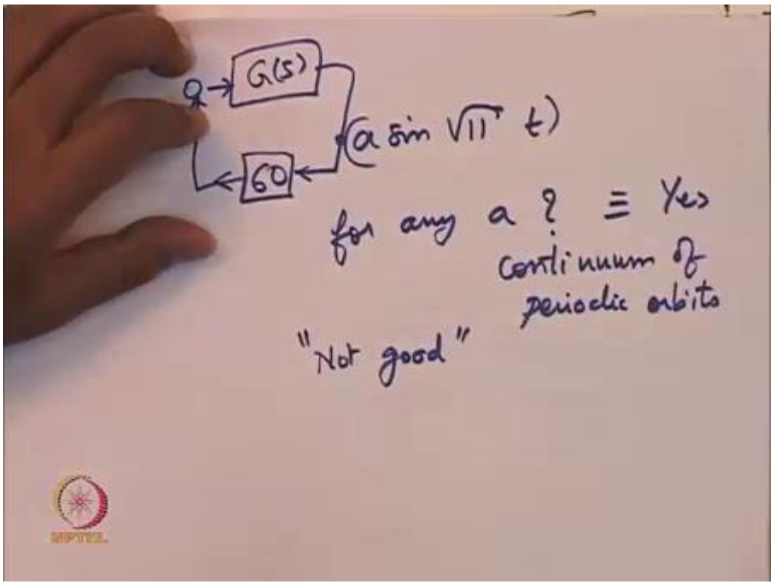

Then the periodic orbit inside this particular figure, so in the a sine square root of 11 t is a periodic solution here when k is equal to 60 and this transfer function this system. For any a yes answer to this is yes which means continuum of periodic orbits it is not isolated it is not for just for very a specific value of a that you have a periodic orbit you have a continuum. This is in a way this is not good what is bad about this if you building an oscillator in the laboratory after sometime you want that it stabilises to a fixed period. We do not want that any period is possible of course another important another worse situation is for arbitrarily small perturbations we might have for arbitrary small perturbations in the plant in the system information. It might turn out that we have instability we do not have a sustained oscillation.

But, we have instability or the sustained oscillation or the oscillation just die down to 0 depending on whether the point whether this gain 60 turns on to stabilise or causes instability. Depending on that these oscillations might die down to 0 or they might blow up this is inevitable because this is in a linear time invariant system.

What about for k larger than 60 we know that again we cannot have periodic solutions in fact for any initial condition we have instability for any non zero initial condition we have instability of the closed loop . So, the next thing to ask is we are going to replace this block by the saturation non-linearity.

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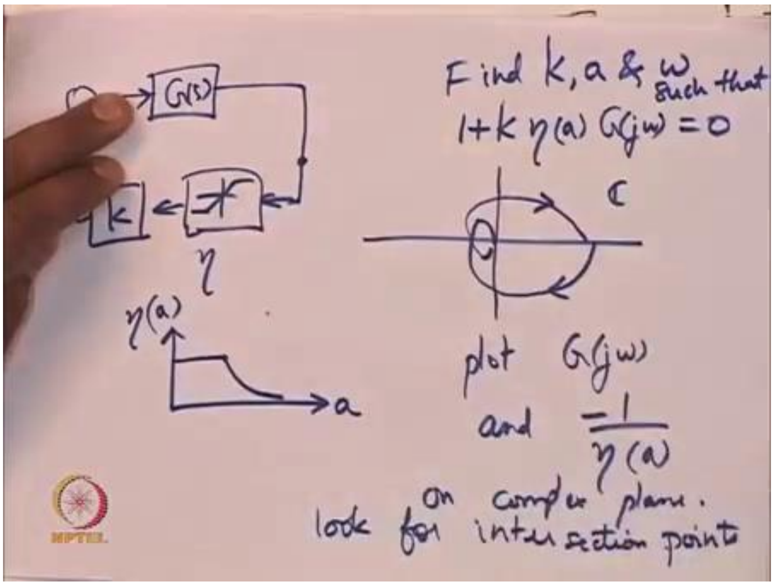

So, we have a standard saturation non-linearity here we have a block k notice that as soon as we have this we will in fact be able to set what amplitude we want. So, this is eta this is a non-linearity which we will try to plot we already saw last time that this is how it looks. Now, we want to evaluate 1+k eta a G of j omega equal to 0 find k a and omega. Such that for the time being why do not we just take k equal to 1. Let us take k equal to 1 this is our complex plane Nyquist plot of G. We already plotted like this what is conventionally done is that plot G of j omega and minus 1 over eta a on a complex plane.

Luckily these both are kind of independent what is independent about this this depends only on omega and this depends only on a because it does not because this non-linearity is memory less it does not depend on frequency explicitly. Hence this is some locus of points for different values of a and it is some curve in the complex plane that is parameterized only by the parameter a amplitude. So, we can plot these both independently and check where they intersect look for intersection points.

Intersection points to say that k is equal to 1 in this equation to say that 1 plus eta a time G of j omega equal to 0 means G of j omega is equal to minus of 1 by eta a of a. This means that these two curves intersect that particular complex number when they intersect you look at what amplitude this curve has reset point at what frequency omega. That curve has reset point and that gives you the a and omega of the periodic orbit only that. We should note that the amplitude is amplitude of this sinusoid at the entry to the non-linearity.

Please do not write this again an approximate analysis why because the describing function has only taken first order harmonic one might say that if G is has low pass characteristics. If the higher harmonics all have very small amplifications then that results in pretty good accuracy of this periodic orbit. If G has a good roll off if difference in the degrees of the numerator and denominator is 2 or more then this indeed is a valid approximation. But, it should have low pass characteristics for all this arguments to work and also the frequency omega should intersect should be well within the cut off. These are only just some thumb rules, but then Vidyasagars book makes this more precise more rigorous. So, just this argument we will see in more detail and then we will return to this saturation non-linearity example.

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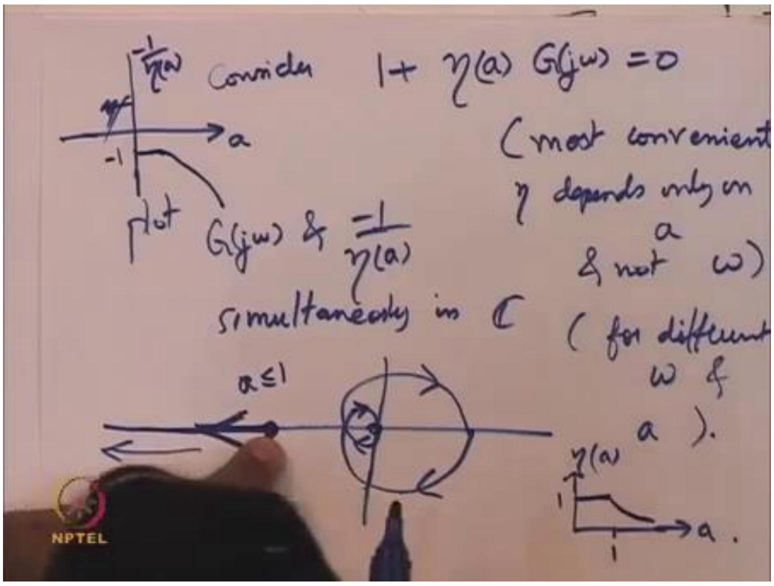

So, consider 1 plus eta a G of j omega most convenient what is what is the convenient aspect about this eta depends only on a and not omega. Plot G of j omega and minus 1 by eta a simultaneously in the complex plane for different omega and a. So, in a example that we have been seeing this is G of j omega for different values of omega what about minus 1 by eta of a. Notice that for our example we had this as eta of a, so minus 1 of eta of a. So, this is a point of minus 1 for this value is equal to 1 until amplitude is equal to 1 because we are dealing with the standard saturation non-linearity.

The saturation non-linearity that causes amplification one exactly 1 for all amplitudes upto a equal to 1 for all a sine omega t inputs upto a equal to 1 beyond that saturates to 1. So, for all these points minus 1 over 1 turns out to be this point and for a larger than that for a larger than that eta of a is decreasing. So, minus 1 over eta of a this is some curve like this. So, this is for a less than or equal to 1 and then this is for a increasing a greater than 1 increasing.

What have we plotted this is a curve of eta of a it is a curve of not eta of a, but minus 1 over eta of a starting from a equal to 0 upto a tending to infinity. This is a locus of points where curve is stuck to this point until a is equal to 1 for a larger than 1 it starts moving in this direction. How do we conclude this we have plotted this eta of a versus a while we can also plot minus of 1 eta of a versus a that we do on this part here sorry this is how minus over eta of a looks like. So, it stays at minus 1 until here for values lower for values larger values of a it goes on decreasing like this it is the reciprocal of this multiplied by minus 1.

So, that you see its all real important point to note is it is always real and always negative and the magnitude is going on increasing. The distance from the horizontal axis is going on decreasing distance from horizontal axis is nothing but distance from the origin here that is why we have plotted the curve like this. So, what is it that we have concluded from here that this Nyquist plot for different values of omega and the curve minus 1 over eta for different values of a positive do not intersect at all do not intersect these two curves. This particular equation this equation for the saturation non-linearity with k equal to 1 has of course no intersection. So, no periodic orbits not too surprising we will see this expected using the Popov criteria and the Circle criteria G of S saturation standard.

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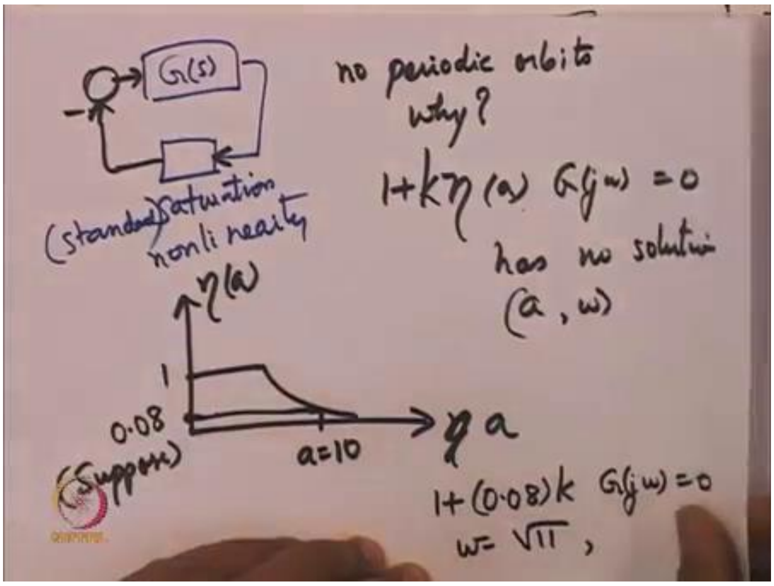

Why do not we call this standard because we assume, so if we have this then no periodic orbits why because 1 plus eta of a G of j omega equal to 0. And has no solution a comma omega here one solution is one pair a comma omega there is no pair a comma omega for which this equation is satisfied, so that is the reason we had a k there.

Now, come back to this we have eta of a, so why do not we pick why do not we pick a suppose we want a equal to 10 as the amplitude it turns out that if we have this one equation and two variables. Then there might be a degree of freedom, but the degree of freedom is infact does not have any solution. But, even if there exists a solution the degree of freedom still may not be there might be a unique point that might also happen because that is in this case happening because this intersects the real axis only at one point. And this is always real you can have intersection only on the real axis and that may not be a continuum in that sense there might not be a degree of freedom.

Even though we have 1 equation and 2 unknowns a and omega on the complex plane they are supposed to intersect and eta of a is not arbitrary complex number. It is only a real number G of j omega intersects the real axis only at omega is equal to 0 omega is equal to plus minus square root of 11. These situations has first of all caused that there is no solution here, but for larger for different if we have a parameter k which is again forced to be real perhaps we can get whatever amplitude we want that is the part we are going to see next.

So, we will look at this standard saturation non-linearity suppose we want a equal to 10 and suppose for a equal to 10. This value turns out to be 0.08 this value here suppose what is suppose about it ideally. We are supposed to use that particular closed form expression that we saw in the previous lecture which we if time permits we will perform on the ((Refer Time: 22:08)) lab and indeed show that suppose for a equal to 10 the value here is 0.08. Notice that this curve has nice property that for any value larger than 1 for every amplitude there is a unique gain at that has that intersects here . So, in other words this map is eta of a 1 to one for all values of a larger than 1 it is a 1 to one map suppose 0.08 is a value here now we can plug in this.

So, 1 plus 0.08 times k times G of j omega equal to 0 we also know that k is real all this point is real for this to have a solution G of j omega also has to be real . We know that this can happen only at omega equal to 0 or at omega equal to square root of 11 omega equal to 0 is not interesting. Because it is a periodic orbit with frequency equal to 0 is not really a periodic orbit it is a DC gain. So, we are going to put omega is equal to square root of 11 by which we are now going to solve this equation for k 1 plus 0.08 times k times minus 1 by 60.

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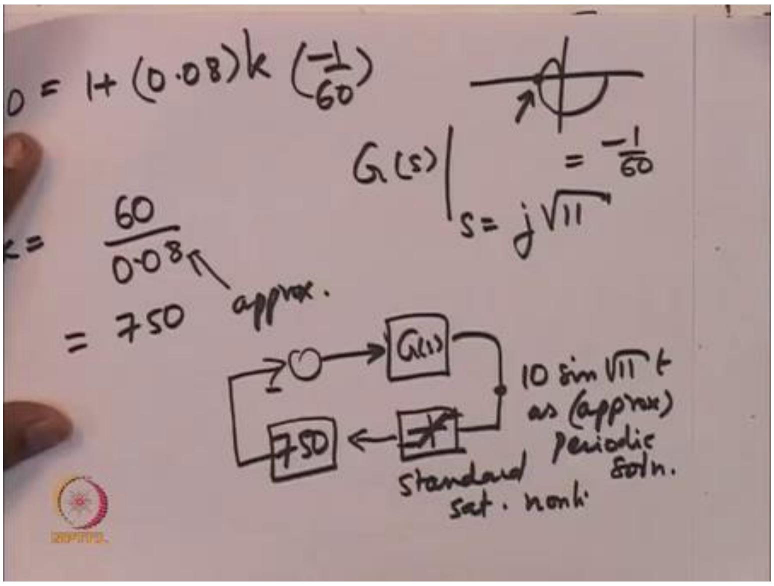

Why because G S evaluated at S is equal to j square root of 11 is nothing but our real axis negative real axis intersection which we saw was equal to minus 1 by 60. This is nothing but this point is equal to minus 1 over if 60 this should be equal to 0 this gives us an equation in k , so k is nothing but equal to 60 divided by 0.08. Let us just calculate this calculation this turns out to be 750 it is little more than 10 times of this because if this is 0.1. This would have been 600 less than 0.1 and little more than 600 this turns out to be 750 of course noting that this is an approximate. So, we expect that for gain as large as 750 is when we get amplitude equal to 10 G of S is saturation non-linearity standard.

And with k equal to 750 there is a minus sign here then we have 10 sine square root of 11 t as approximate periodic solution. In this way we can set the frequency we can set the amplitude frequency we are not able to set because G of G omega intersects the negative real axis intersects the real axis only at omega equal to 0. And omega equal to plus minus square root of 11 there is no choice of frequency for the standard saturation non-linearity. But, the amplitude can be set it may not be exactly 10 because the describing function method is only a approximation procedure.

But, there is an isolated amplitude only for a very particular value of a will this have a periodic orbit that is what makes this method to be useful as for building a oscillator of a fixed amplitude unlike linear systems. Where we end up having a continuum of periodic orbits one can also check that this is pretty robust to perturbations inside the system parameters. Also for nearby initial conditions it comes back to this that of course argued for using that for oscillations smaller than this it ends up blowing up because this range 750 happens to be making it unstable. But, it is chopping it to value one that is why it cannot become unstable that is what makes this limit cycle as a stable limit cycle.

Before we see another example in which the frequency also can be set we will proceed a little further with this example. And we will relate how for k less than 60 we already know that there cannot exist periodic orbits using the Circle criteria using the Circle or the Popov criteria. Both these criteria’s are relevant in this particular case, so coming back to this particular figure G of S.

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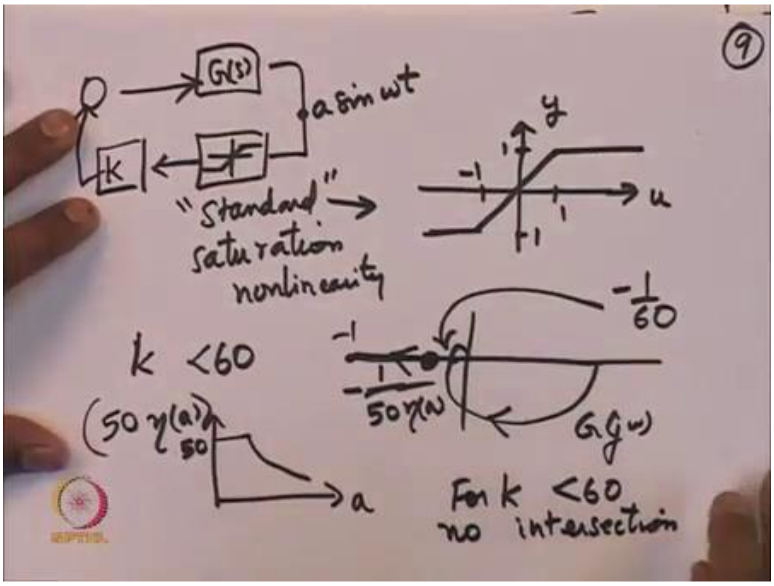

The saturation non-linearity is pretty easy to visualize, but we have defined a word called standard that is actually not. So, standard across the literature that is why I will just define this again input u output y in which upto 1 became minus 1 and one it has amplification by exactly 1. This is line with slope one beyond that it has slope 0 in other words it has saturated this is what we refer to as a standard saturation non-linearity .

Now, we can ask the question this minus sign is also important that is we have 1 plus k eta a times G of G of S G of j omega equal to 0. One might ask for k less than 60 for k less than 60 this describing function you see one can directly multiply k with the describing function also. We already have a here eta of a we can multiply this by say fifty times eta of a. So, that this is a describing function of this non-linearity as a whole these all these 2 blocks together this time starts from 50 and comes down so that. So, that minus 1 by this, so Nyquist plot is like this G of j omega.

What is this point this point here is minus 1 by 60 on the other hand it is here it starts and this is minus 1 over 50 times eta a . The plot of this whole block is this plot of 50 times eta a has been plotted here. So, minus 1 over that starts from a little closer point minus is here it has come to minus 1 by 50 it has come much closer than what it was. But, point of intersection is minus 1 by 60, so we can conclude that for k less than 60 no intersection no intersection what is the meaning that there is no intersection.

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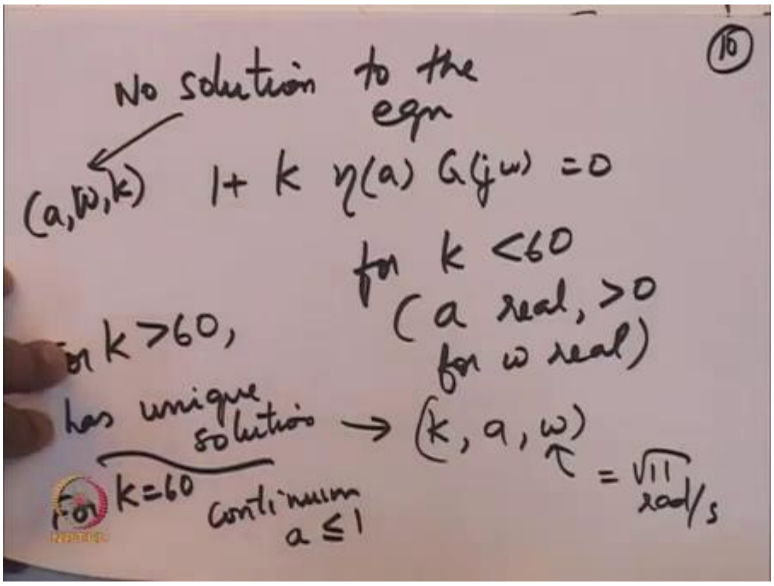

No solution for no solution to the equation 1 plus k eta a G of j omega by solution we mean here a omega k triple there is no k a. And omega value that satisfies this for k less than 60 of course for a real greater than 0 for omega real we are also working under this constraint what is the meaning that omega is real and positive. It means that it is a sinusoid plus or minus omega is less important, but it is required to be a sinusoid a is positive. Because it is amplitude and its real number and k less than 60 is what this condition says from the Nyquist plot intersecting at minus 1 by 60 and eta of a eta of a by itself starts at minus 1.

And only decreases after that this equation has a solution only for k larger than 60 k greater than 60 has unique solution what about for k equal to 60 has unique solution k a. And omega of course omega is always fixed equal to square root of 11 radians per second. What is the amplitude that you can get that amplitude depends on the value of k there is a one to one corresponding between amplitude and k for each amplitude that you want you have a value of k. If you want amplitude less than 1 there is a non uniqueness then if the amplitudes are larger than 1 you have some uniqueness coming from the gain k.

Why because for k equal to 60 continuum of periodic orbits and all these periodic orbits have a less than or equal to 1 . So, this is not hard to check I think that this is a very good exercise to pursue that how for k equal to 60 it is indeed expected that you can have many periodic orbits or continuum of periodic orbits. All of them having amplitude at most 1 what is continuum about it a equal to 0.65 works as an amplitude a is equal to 0.6500. One also works as an amplitude other words it is a continuum why because for k equal to 60 there is a range for the saturation non-linearity. In the linear range itself already has a continuum the amplitude cannot be larger than 1 because that is where non-linearity starts having its effect and then it will clip. So, this is something that one can check as an exercise, now we are going to use the Popov criteria to say that for k less than 60 of course no solution is expected all this periodic orbits can exist only for k larger than 60.

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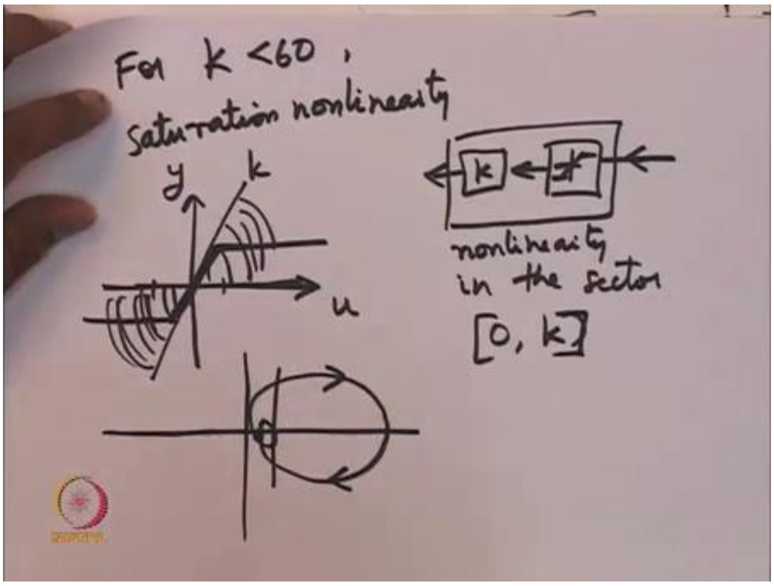

So, for that purpose we are going to say for k less than 60 saturation non-linearity which saturation non-linearity not the standard one. Now, we are going to say the standard one together with this block this whole non-linearity. We can think of as non-linearity in the sector 0 to k. This is some non-linearity inside the sector 0 to k is of course positive we expect k is positive in this arguments input u output y there is our sector this is a line with slope k and of course this line with slope zero is horizontal axis. This is our sector 0 to k does the saturation non-linearity lie inside the sector very much for each k it is equal to this.

So, the saturation non-linearity indeed lies inside this, in fact it lies inside this because for some range it is equal to this line for some range of the input for the range plus minus

1. Now, we want to ask upto what value of k will you have stability for all non-linearities inside this range inside this sector for what value of k will you have stability absolute stability means that there for every initial condition the trajectory equal to 0. And what is absolute about it for all non-linearities inside this sector all non-linearities. One might further distinguish time invariant or time varying which is a distinction between the Circle criteria and the Popov criteria, so Popov criteria applies to only time 0 to k. So, this is how the Nyquist plot is the Circle criteria I will draw more correct figure where it is tangent more properly consider.

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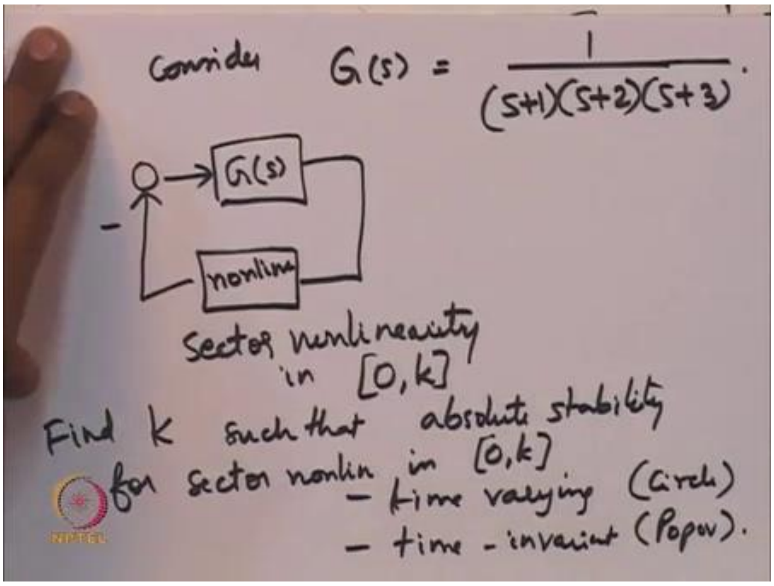

G of S 1 over S plus 1 S plus 2 S plus 3 what is the problem that we are trying to solve. Now, for this example non-linearity specifically what non-linearity are we talking about sector non-linearity in 0 to k g is fixed. Find k such that absolute stability for all for sector non-linearities in 0 to k first time varying time invariant for here we will apply the Circle criteria for here we will apply the Popov criteria we saw the two things that we will do now. So, for this we will plot the Nyquist plot in little more detail.

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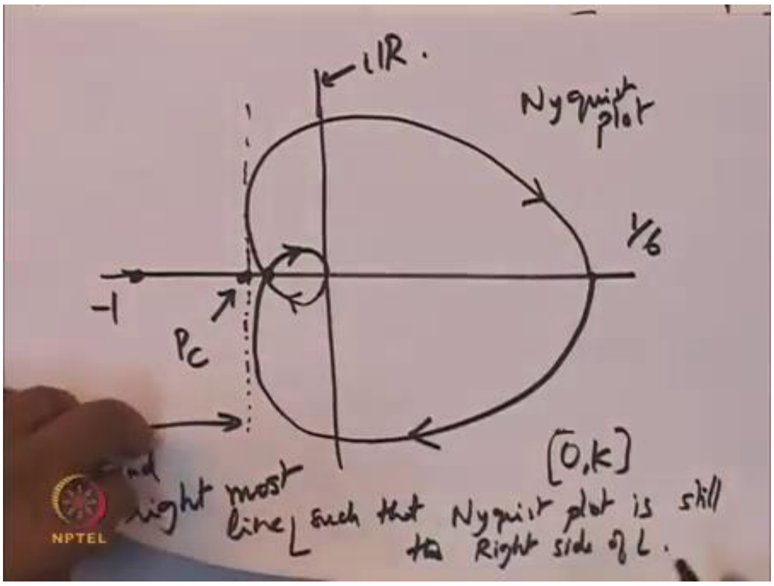

This is where 1 by 6 is this is Nyquist plot this particular axis is the imaginary axis to apply the Circle criteria for the sector 0 to k means of course this requires that the open loop transfer function G is already stable. Why it requires that G is already stable because the non-linearity the particular linearity with slope 0 is also inside the sector. And if 0 has to cause stability that means open loop also has to be already be stable point minus 1 is somewhere there. So, the Circle criteria says find right most find right most line right most line such that Nyquist plot is still to the right of this.

We cannot put we cannot put the rightmost line here or here the Nyquist point has to be to the right of this particular line find rightmost line L. Such that Nyquist plot is to the right to right side is still to the right side of L. Now, this says that this point this real axis is suppose this particular point is some point P, P we will call as P C, C for Circle criteria is the one that tells us the condition suppose P C is point. Then this k max that we can get from the Circle criteria is nothing but minus 1 over P C. The real part P C we are looking at the real axis intersection of this vertical line.

So, we push this line as much to the right until it touches then P C will extract out the real axis intersection it will be a negative point for this example importantly minus 1 by 6 is further here. So, we will get that by the Circle criteria P C suppose it is approximately equal to minus 1 by 50.

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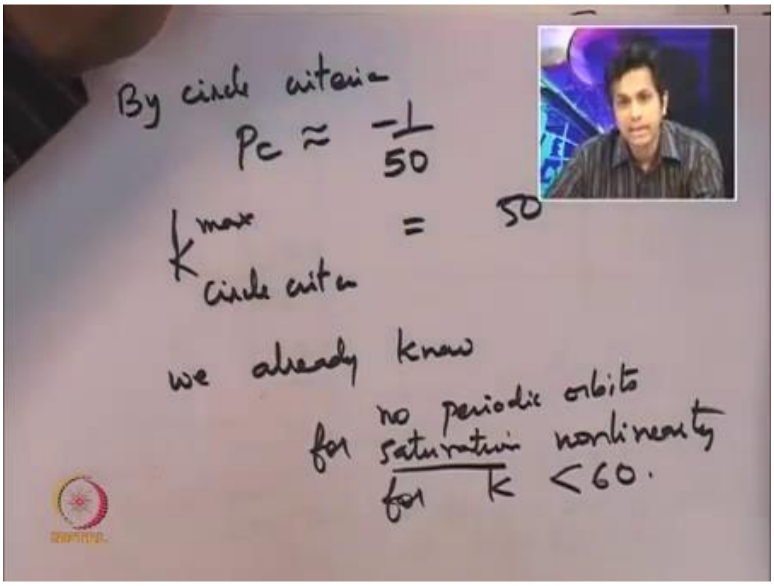

Why do we know that this point minus 1 by 50 it should be some point to the to the left to the left of this point. This point is minus 1 by 60 and the Nyquist plot of course this point cannot be to the right here. It would have intersected already it should be to the left because this point this curve is intersecting at an angle here. If we draw a vertical one then it will indeed be tangential here. But, here it is be coming and cutting and going in it is not coming in vertically here, but going inwards. Hence it would have come further out and then gone in hence the point that is tangential to this would be tangent at some point.

Here it would not have been tangential at the real axis intersection which would have happened here. So, we know that this point minus this p c point is to the left of minus 1 by 60 that is why I have taken minus 1 by 50. Then k max by Circle criteria Circle criteria equal to equal to 50 then. And we know we already knew we already knew no periodic orbits for saturation non-linearity we already knew that there are no periodic orbits for saturation non-linearity for k less than 60. We have got a number which is going to be less than 60 we have got just 50 why is it expected that we will get a lower number in Circle criteria.

And also the Popov criteria are not conservative in some slightly more generalized sense they are necessary and sufficient. So, this number 50 stays because we have to guarantee absolute stability for all sector non-linearity in the range 0 to k max and these nonlinearities can be time varying also. This criteria gives us absolute stability for all nonlinearities inside the sector which could be time varying also. On the other hand here we have fixed the non-linearity to be a saturation non-linearity. It is an extremely conservative very fixed non-linearity and hence we can get a number as large as 60.

Here on the other hand we are supposed to guarantee absolute stability for all nonlinearities inside that sector both time varying and time invariant. In time invariant there are many other non-linearity which are invariant other than the saturation non-linearity for them also the guarantee for absolute stability has to be provided. Of course time varying non-linearity are another big class of nonlinearities again inside that sector hence we are getting a smaller number 50. So, up to 50 no periodic orbits we already are able to get from Circle criteria upto 60 no periodic orbit for the saturation non-linearity.

We can next investigate what the Popov criteria tell us what is the number we get from the Popov criteria and before we investigate the Popov criteria. We can check upto what range of linearities upto what sector largest sector of linearities what is that number it will turn out that number is also exactly 60.

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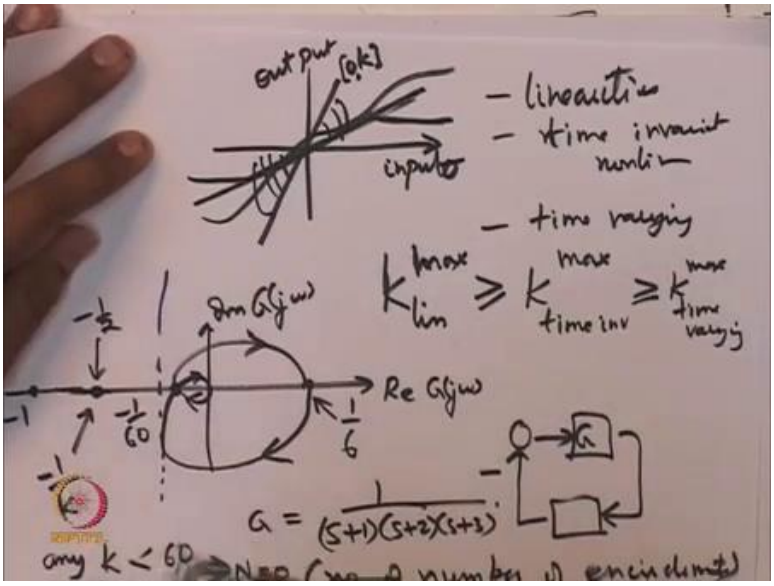

So, we are going to see next what is the maximum value of k we get for taking a sector of the type 0 to k . This is 0 this is a line slope k this is a sector, but inside the sector we can include either linearities are nothing but the lines inside the sector all passing through 0 of course lines with slope any slope between 0 and k nonlinearities. If we say time invariant time invariant nonlinearities and time varying linearities are of course small set. These are lines these are just lines of course they are invariant also the input output lot does not depend on time because it is a line time invariant non-linearity atleast it may not be a line.

But, atleast the dependence of output on input does not change with time the output of this non-linearity this is the input dependence of output on input. If it is a line then it is a fixed line then it has no option this slope only that is a linear system. The next option is dependence of output on input does not change with time. But, it is a non-linearity for example saturation non-linearity the last option is time varying at some point it could be like this after some time it could become something like this again inside the sector. So, inside the same sector when you allow the linearities you can vouch for a larger value of k we expect that k max for linear the max is the maximum sector of the type 0 to k.

For which you have absolute stability will be greater than or equal to k max for time invariant and that will be larger than k max for time varying. Why is this true because for a given value of k max the time varying non-linearity set will include time invariant also that will include linear also after all linearity is also time invariant non-linearity is also time varying non-linearity. In time varying aspect is negligible infact it is equal to 0, so these because of those one set sitting inside another set. If you make the set of nonlinearities allowed within a sector to be very small then you can vouch for a larger sector that is the meaning of a robust.

How much how much large type of nonlinearities you want to allow within a sector if you want to allow a time invariant nonlinearities. Do you want to allow time varying also depending on that the sector will have to become small. So, that you have absolute stability for that larger sector, now coming back to this Nyquist plot we will first find k max lin this point is minus 1 by 60 point minus 1 somewhere here. And this is equal to 1 by 6 real part of G of j omega and this is imaginary part of G of j omega each line inside the sector. We are speaking with respect to very specific type of feedback configuration in which this is where the non-linearity is the negative sign because of negative sign this minus 1 assumes its importance.

Now, any point here suppose this corresponds to minus 1 by 2 it means that it is a line corresponds to a line with slope two different lines inside the sector correspond to just points on the negative real axis. Because it is a point which is not encircled by the Nyquist plot this is the. So, G is our famous transit function it is an open loop stable number of poles to the right of plane is equal to 0 if you want the number or poles to also be equal to 0 the n had better be equal to 0. So, this point minus 1 by 2 instead of instead of checking the number of encirclements of the point minus 1. We have now started conveniently started asking the number of encirclements of this point minus 1 by 2 because this corresponds to minus 1 by k point k, where k is the gain that we introduced to g or to this non-linearity anywhere in the loop. So, magnification by k and k has to be positive for us to be considering the points on the negative real axis.

So, you can take any point to the left of this minus 1 by 60 and it will not be encircled or in another words it will be encircled 0 number of times. They will all correspond to linearities that cause stability. So, any k less than 60 will not be encircled will imply N equal to 0 what is this N this N is number of encirclements. Number of encirclements of which point of the point minus 1 by k corresponding to gain k we have 1 by k as a point to check instead of minus 1, so this is our encircle. So, as long as N is equal to 0 P is also already equal to 0 thus by Nyquist criteria Z is also equal to 0 and hence we have closed loop stability. Closed loop stability we can use Nyquist criteria precisely because we are interested in checking the points on the negative real axis points correspond to linear lines and hence linear system and Nyquist criteria is indeed applicable for linear system.

Now, what does the Circle criteria say Circle criteria said that the . So, what is this value now k max lin it is nothing but equal to 60 we cannot have k max we cannot achieve for 60. We already have periodic orbits which is instability marginal stability for linear systems cannot achieve this max we call it k lin sup k linearities sup why sup instead of max. Because we cannot achieve k lin sup equal to 60 at 60 we have marginal stability we do not have asymptotic stability in other words we do not have absolute stability.

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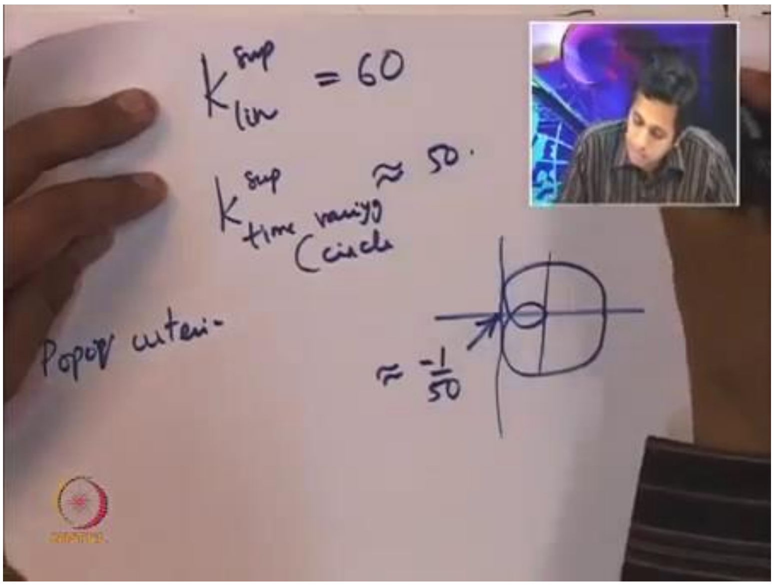

What did we get for k time varying which means Circle criteria sup again that because approximately 50 that requires to requires us to take the vertical axis that is tangential to this. This tangent point we said is to the left of the point minus 1 by 60 we said is approximately minus 1 by 50 that corresponds to 50 as a sector.

Now, our next we are going to use the Popov criteria and we might get some number that is between this might be strictly less than 60 or might be strictly more than 50 or it could be equal to 1 of these. That we are not sure at this point for that purpose we will plot the Popov plot. We will sketch the Popov plot for this particular transit function. So, recall that the Popov criteria involved sketch Popov plot which amounts to plotting like the Nyquist plot the real part G of j omega. But, on the vertical axis plot omega times imaginary part of G of j omega this particular quantity is a even function of omega because a part is a odd function.

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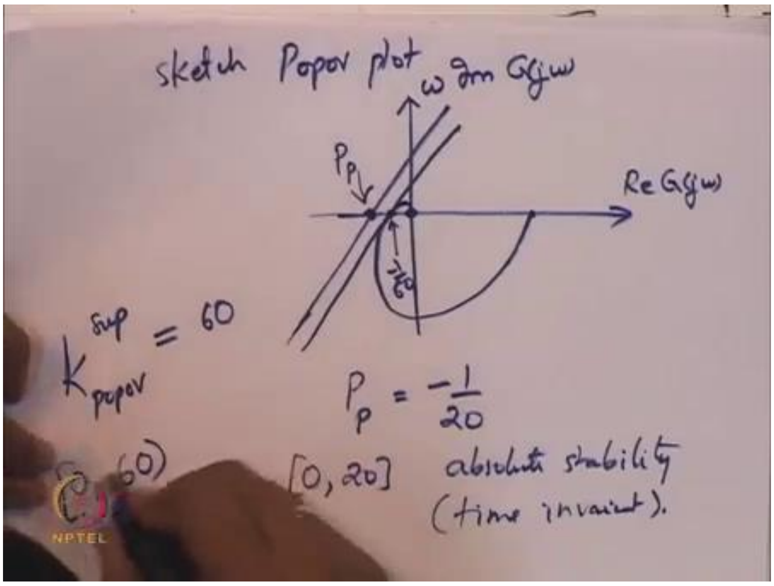

So, we multiply by omega plot this for only positive values of omega for this particular transit function Nyquist point was coming down. So, this means that we are taking the Nyquist plot and just distorting it along the vertical axis only along the imaginary axis. We are doing some distortion on the horizontal axis there is no because we are still plotting that real part, so perhaps it looks like this it still goes to 0. In this case because imaginary part of G of j omega is 0 pretty fast and multiplying by omega cannot change at the rate at which it goes to 0 requires a careful check you can verify this after some time.

So, what does the Popov criteria say take any line with positive slope that this line should have positive slope and this Popov plot should be to the right of this line because it has positive slope to its right is unambiguous. The real the real part of the every point on the Popov plot should be greater than this should be greater than this line with positive slope should be above and to the left of the Popov plot. In other words the Popov plot should be to the right and below of this particular line.

Now, we can ask as long as this line has positive slope it is fine, but then this intersection of the negative axis gives us the range for which we have stability absolute stability for that sector 0 to k for time invariant nonlinearities only . Suppose this corresponds to point P P this P for point and this subscript P for Popov then suppose P P equal to minus 1 by 20. Then we have by the Popov criteria from 0 to 20 absolute stability absolute stability for this sector of nonlinearities, but they have to be time invariant. Now, comes can we make this sector larger can we take this line just to have positive slope, but further to the right.

So, this point the intersection on the negative real axis does not change because the real axis does not change at all the real axis part of the plot. That we are plotting has not changed only the imaginary axis has been imaginary part of G of j omega has got scaled by omega it has become larger for omega larger than 1 smaller for omega less than 1.

So, how much to the right can this point come for this particular example it can come as much right until it intersects this rightmost line rightmost line would be tangential it is allowed to be tangential at this point that is what makes us. So, k sup again it is sup because we cannot achieve this by Popov, which is same as time invariant will be equal to 60. Because the real axis intersection of the line that comes as much to the right as possible and still the Popov plot is further to the right the rightmost. We can bring it when it is tangential at that point the precise value of slope is not important at that time the real axis intersection will again be equal to minus 60.

So, this what Popov of course 60 is not achievable because at that point there will be an intersection we are not going to allow intersections of this line with the Popov plot. So, the Popov criteria also tells us that for all time invariant nonlinearities in the range from 0 to 60 open bracket, because 60 is not allowed for all time invariant nonlinearities.

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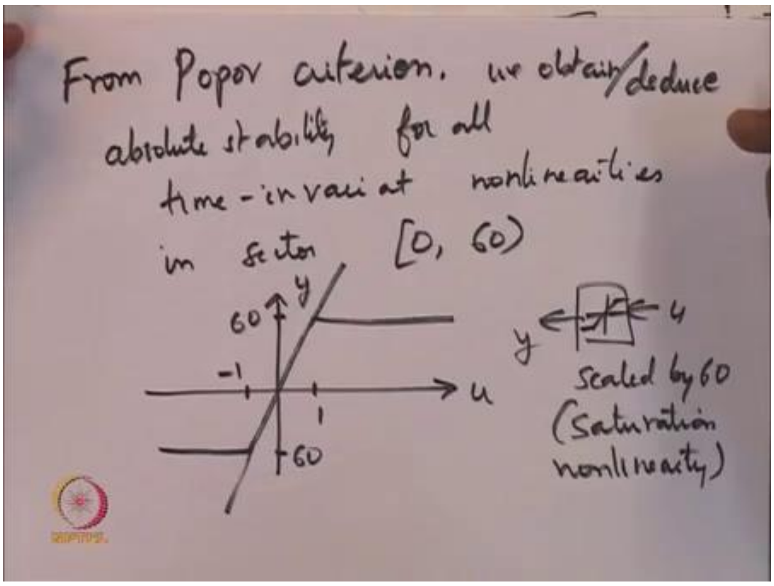

So, let us just write this conclusion from Popov criteria absolute stability for all for all time invariant non linearites in sector 0 to 60 from Popov criteria we obtain we obtain. We obtain deduce that we have absolute stability for all time invariant nonlinearities inside this sector is that saturation non-linearity scaled by 60 inside the sector no it is just outside the saturation non-linearity scaled by 60 saturation non-linearity.

If it is scaled by 60 then the slope of this is 60 and hence it is not strictly contained inside this sector 0 to 60 it lies on the boundary. So, if we take square bracket here of course it will be inside the sector, but if you take scaled by 59.99 then it will be strictly inside the sector. And we have only verified using the describing function that there is no intersection and hence we will not have periodic orbits we will have periodic orbits only for k 60 or more for k equal to 60. Of course we end up having a continuum, but for k strictly larger than 60.

The Popov criteria already said that perhaps it is not going to work Popov criteria was again only a sufficient condition after some after allowing a slightly larger class of nonlinearities. It becomes necessary and sufficient condition for absolute stability. So, for time invariant nonlinearities in the range from 0 to 60 describing function also told us we do not have periodic orbits. Popov criteria guarantees absolute stability which also kind of which automatically rules out periodic orbits for any value of k strictly less than sixty this is the conclusion that we got for k and for linearities.

We of course already know that for k larger than 60 for k equal to 60 we have continuum of periodic orbits because amplitude will not matter for a linear system for k equal to 60. And for k larger than 60 we have instability for the linear system, but for the saturation non-linearity we have a unique amplitude, that we obtain from the describing function and that is precisely when the Popov criteria fails to guarantee anything. In this sense we are able to see that the describing functions is able to give us for what amplitude and frequency. We have that intersection of minus 1 by eta a plot and the G of j omega plot and the Popov criteria is not able to guarantee absolute stability for good reason these are all complimenting each other in that sense.

So, we will continue in the next lecture with another example where we are able to set both the frequency and the amplitude by choosing the non-linear parameters carefully. In particular, we will worry we will look into the jump instance non-linearity this ends today's lecture.

Thank you.