Nonlinear Dynamical Systems

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Lecture - 27
Describing Function: 2

Welcome to this next lecture on describing function we just saw some motivation in the previous one and we also saw that for linear time invariant systems real part and imaginary part correspond to the sinusoid part and the co sinusoid part. So, the co sinusoid part of the output for a input, so we are going to use that to define the describing function in this lecture.

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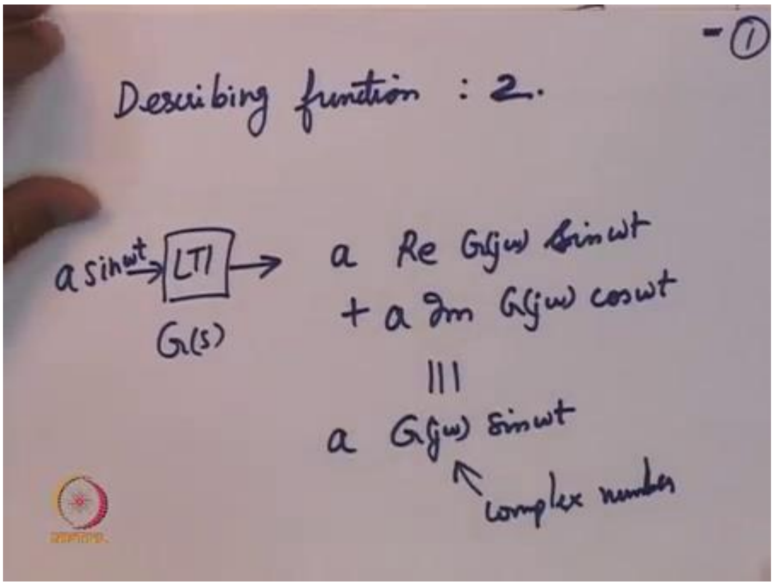

So, recall that for L T I systems, L T I systems let us say we transforming form G of s we should give a sin omega t the output has a transient part. But, where are not going to worry from now on we are going to have the output a real part of G of j omega times sin omega t plus a imaginary part of G of j omega.

Hence, cos omega t which we are going to write this together as a G of j omega times sin omega t, this one is a real signal sinusoid part co sinusoid part cos omega t times j we are going to write as nothing but sin omega t. So, G of j omega has is now complex number, complex number this making it complex has helped us eliminate this cos omega t part does that mean that everything in the output is in phase with sin omega t. Now, there is a imaginary part hidden inside G of j omega that imaginary part does not mean that the sinusoidal is complex, we measure only real sinusoidal in a real world.

We do not have any oscilloscope that measures imaginary part and that is just notation, so the imaginary part is to be understood as the co sinusoidal part. The imaginary part inside G of j omega should be understood as the co sinusoidal part, the part that is 90 degrees out of phase with sin omega t, 90 degrees leasing or 90 degrees lagging. So, that is extremely important, that depends on indeed whether the imaginary part is positive or negative, so we are now going to say.

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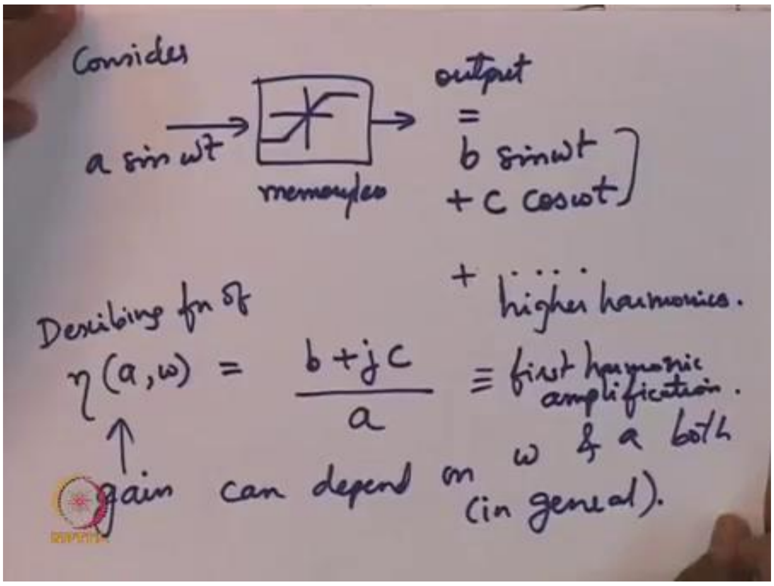

Consider on what class of systems, can we do this we will worry very soon consider saturation non linearity and we give a sin omega t output. So, output equal to some b sin omega t plus c cos omega t plus may more, what are these many more are these transients no this is memory less.

But, we can have transients only when the system has memory where it takes some time for the output to stabilize to the input. But, this is I mean suddenly you start this and suddenly the output will come to something may not be this it has something more plus higher harmonics. This is also missing in linear time variant systems in linear time variant systems the forced part has only the input frequency. It does not have higher harmonics or sub harmonics in our very first lecture on non linear dynamical systems.

So, we say that non linear systems can have harmonics that are different from the input sinusoidal can have higher harmonics also. But, we are should describing function describing function of the saturation non linearity a, omega equal to b plus j c divided by a, yes why because you see as I said in our previous linear time variant system. This situation also there was a factor that is within a and if the, we are going to think of this as a gain because it is gain they are going to divide by a.

So, of course dependence on a for linear time variant systems b will have a factor a, c will have a factor a, they will all get canceled neatly and the net gain is independent of a. But, even for linear time variant systems we already saw that there is dependence on omega, can depend on omega and a both in general. But, of course we will see that for this situation of saturation non linearity there is no dependence on omega, in fact this c will also turn out to be 0. It will turn out to be real, the imaginary part will be missing, in other words the cos omega t will never be there.

So, the input when you give all the higher harmonics will all have only sinusoid terms, there will be no cos 2 omega t, there will be no cos 4 omega t, etcetera. So, how do you find b and c, one finds the Fourier series of the output and then finds out the first harmonics coefficients the sinusoid part we put into the real. But, here the co sinusoid part we out as with multiplied by j and we divide this is nothing but first harmonic, first harmonic amplification this defines the deciding function.

So, for saturation non linearity we what we will do first we will do this calculation for the lesser the sine, for the sin non linearity which non linearity it just gives the sin of the system. Then we will analyze in a more detail a readymade formula for this memory less for this saturation non linearity and then we will also see for what class of non linearity one can define this describing function this way. So, what are it properties for example when is it going to be real, when is it going to be independent of omega, all this we will explore in the rest of today’s lecture.

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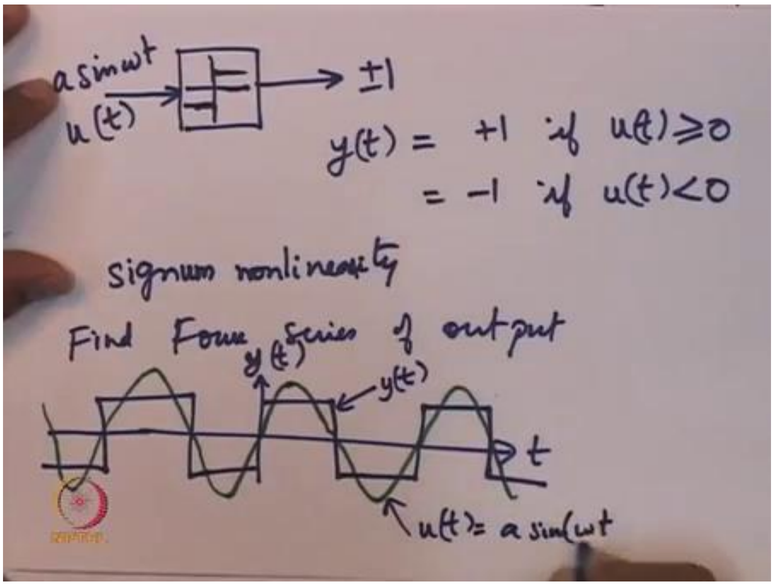

So, first consider the non linearity that takes a input and gives you plus or minus 1, what does it do it just gives you the sin of the input at that time instant output y of t equal to plus 1. If this is u of t, if u of t at that time instant greater than equal to 0, equal to minus 1, if u of t is strictly less than 0, of course one might say why is it that at t equal to when u of t is equal to 0. Now, you are biasing towards plus 1 that does not matter, you see u of t is equal to 0 for just one, for just one instant most of the time it is either positive or negative.

So, at only a 1 instant at only 0 0 2 pi by omega 0 2 pi by omega pi by omega at only these instants it is equal to 0. So, at such a, for a set of measure 0 what value you define y of t that does not make any difference. So, this we will call sign non linearity one some people also call sinusoidal nonlinearity sinusoidal non linearity same as sin non linearity its graph looks like this. So, it gives a output plus 1, if the input is positive it gives output minus 1, if the input is negative, so find Fourier series of output how does the output look for the sinusoid input.

Let us plot the output first, output is like this for what value of the input, let us also plot input when the input is like this the way I have drawn this figure a is greater than 1. So, value of a is greater than 1 that is why this peak is greater than plus 1, but it is in phase, so we can find the Fourier series of this periodic signals this is, this is y of t. This is of course the input u of t equal to a sin omega t, so first important thing to note is the output is periodic. The output is, the output y of t is a periodic signals, periodic signals are the ones which have a Fourier series, let us find out this Fourier series co efficient.

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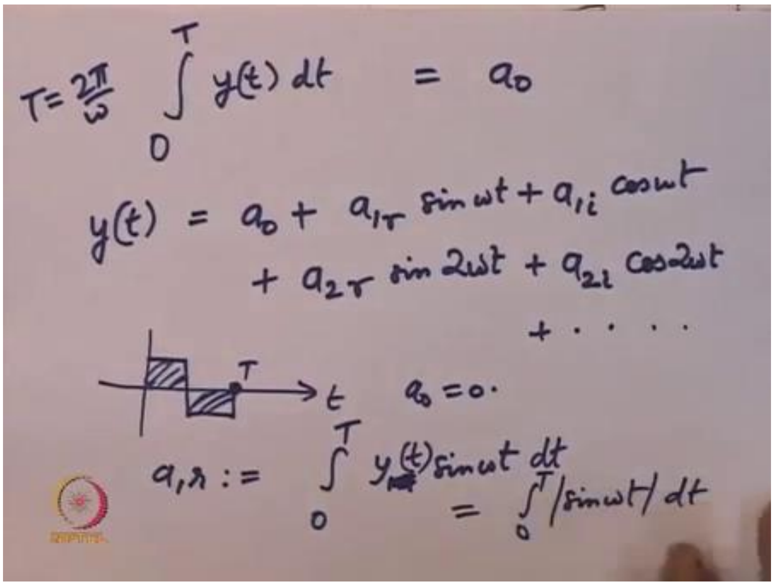

So, we are going to integrate over the period 0 to T where t is the period which is 2 pi over omega of this is the output y of t d t this will give us so called a of y of t. So, we write as a 0 this is a d c term plus a 1, let us call this real times sin omega t plus a 1, imaginary for cos omega t plus a 2 real. So, for sin 2 omega t plus a 2, imaginary for cos 2 omega t etcetera, now we are interested in finding a 0 to find a 0, one has to just integrate from 0 to t.

Since, y of t is over one period this is value of t, so over one period this is 0 add up to 0, this is equal to 0 this is equal to 0. So, a 0 equal to 0, now how will you find a 1, r a 1, r is nothing but v can be obtained by 0 to t y 1 r sin omega t, so that we these all basis function sin omega t cos omega t are an orthogonal basis.

Then ortho normal basis, so that in order to get the component of a 1 r along this direction all you have to do is project a 1 y 1 on to this particular sinusoidal and then find this integral. So, this particular integral evaluates to just integral from 0 to t sin omega t why because y 1 r is nothing but equal to plus 1. Now, this is y 1, y of t, y of t is equal to just plus 1 when sin omega t is positive is equal to minus 1, when sin omega t is negative.

So, this evaluates to this, so we will just evaluate this quickly we will just evaluate this integral that will give us this coefficients except that we have missed some factors. So, this a 0, if we define a 0 like this then what comes here is a 0 by 2 and a 1 r, we need some normalizing factors that we have missed. So, this is 2 by t, here 2 by t, here I just now checked that these factors will not gives us without these factors we do not get the correct describing function.

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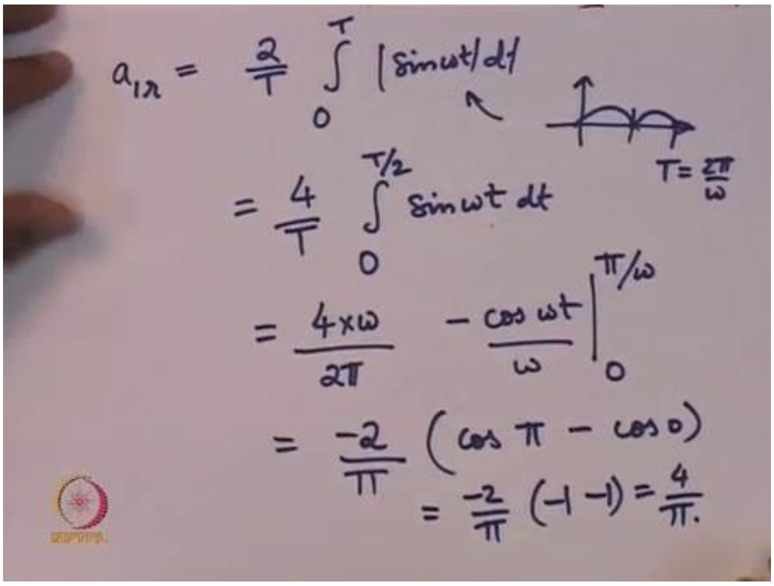

So, what we are going to calculate is a 1 r is equal to 2 over t integral from 0 to t of sin omega t d t, why did we put this absolute value, because we are multiplying sin omega t with y and y changes its sign. If sin omega t changes its sign y is equal to plus or minus 1 only, say in other words when we multiply this by y we get exactly this which is equal to now 4 over t.

We multiply integrate from 0 to only half the period why because this particular sinusoidal, here is something like this is this is t, t is equal to 2 pi over omega. We are going to integrate only up to here, so 0 to t by 2 of sin omega t over this interval sin omega t has not changes sign. So, that is equal to 4 divide by t is equal to 2 pi by omega, so we are going to get 2 pi here into omega of integral of sin omega t integral of sin omega t is minus cos omega t divided by omega.

So, evaluate from 0 to t by 2 t by 2 is nothing but pi over omega t is 2 pi over omega t by 2 is pi over omega. So, this gives us omega, omega cancels 4 we have 2 minus 2 over pi times cos of pi minus cos 0. So, this gives us cos of pi is minus 1 and cos of 0 is plus 1, so we get minus 2 over pie, hence minus 1, minus 1 that gives us 4 over pi this is a 1 r.

Now, we have to also calculate, recall this formula in this formula we have calculated a 1 r, this is the part that comes a sin omega t, a 1. So, it is the part that comes with cos omega t and similarly these all higher harmonics of course we have decided to not calculate. But, because it is having function requires only these both, now we are going to find the reason why a 1 i is equal to 0.

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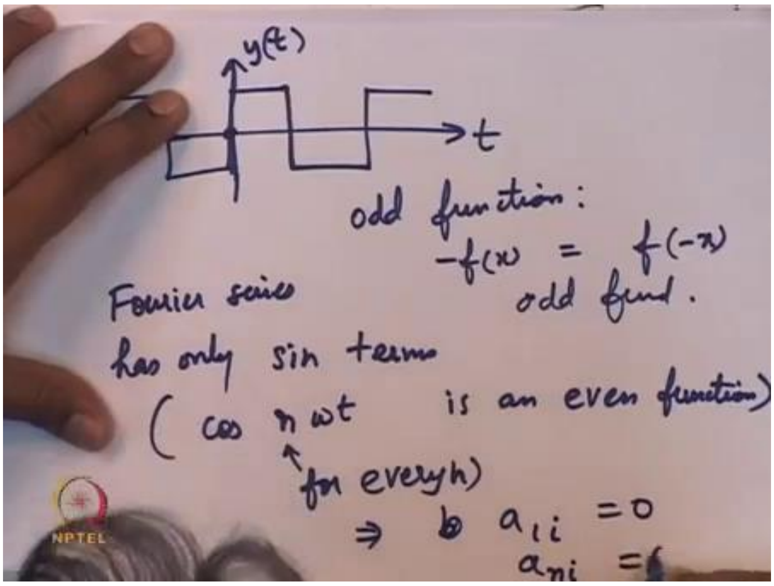

So, let us just draw this input and output, so the output is something like this, so this is y of t and this is t, so this output, notice that this output is a odd function. The output is odd function what else, when you call a function odd f of x equal to f of minus x this is what we would call e 1 in the odd function if some function f of x satisfies this property. Then you say f of x is an odd function of course one should also notice that odd function requires that f of 0 equal to 0 and we have defined sinusoidal non linearity as f of for t for when sin omega t is 0 output is 1.

So for the purpose of this odd one can also put this as 0, another important thing to note is that the Fourier series is un affected. If it changes the value of the function whose Fourier series we are trying to calculate that periodic sinusoidal function value. If you change at just a few points then the Fourier series is totally unaffected why because the

Fourier series depends on this integral formula, you see the integral is more requiring the area under a curve.

If you change the value at just a few points, the value under those the area under a point is 0 as long as the point is finite the area under that point is height multiplied by the width. So, the width is 0 because the width of a point is 0, hence the Fourier series is unaffected by a change in the periodic sinusoidal value at a few points. In other words, on a set of measure 0 that is the technical term, so our odd function this, so our output is a odd function when you give sin omega t as an input. But, output is a odd function for the purpose it making it odd, one should I think define sinusoidal nonlinearity as this.

But, even if you do not if you define the sinusoidal non linearity as taking value one then the input is 0 even. In that case, as I said the Fourier series is unaffected by changing the value of the periodic sinusoidal at a few points. So, odd function, so Fourier series has only sin terms, why because cos is a even function cos n omega t for every n for every n is an even function. But, of course all functions are not either odd or even all integers are either odd any integer somebody comes and gives us is either an odd or an even integer.

But, that is not the case this functions, every function need not be either an odd function or an even function. In general functions we can say has a odd part even part you need both odd and even parts to construct a function if you are trying to resolve it in terms of its odd part and even part. But, coming to this case we have resolved it in terms of its all its odd parts, all the sinusoid terms together comprises odd part all the co sin terms.

So, co sinusoid terms together comprises even part, since the function is already odd we do not require its even function. Now, like cos n omega t to synthesize that function, so using this, what this means is d a, one imaginary part equal to 0, in fact any for any harmonic the co sinusoid term is not required. So, this says that our, now this we are ready to calculate the describing function for our, for this example, so what is the describing function for this.

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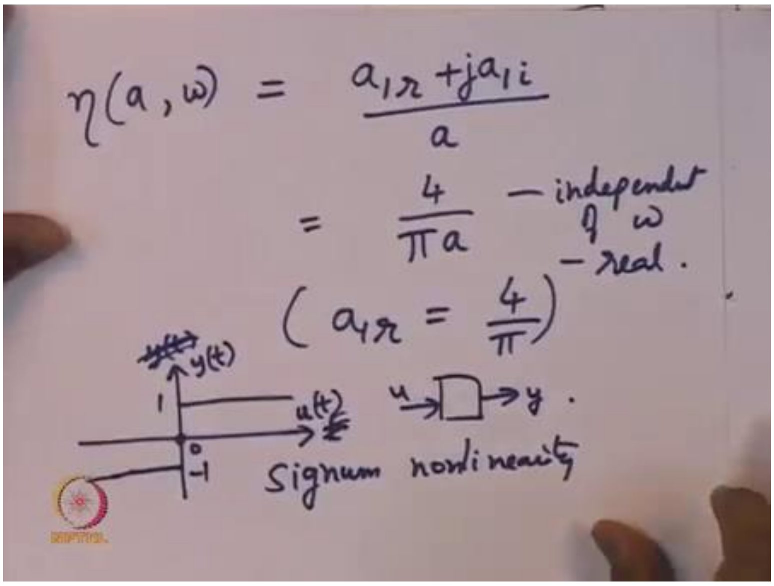

So, eta of a as I said the describing function depend on both amplitude of the input sinusoid signum and the frequency omega of the input sinusoid signum that is equal to a 1 r plus a 1. So, imaginary part j times of that divided by a then this turns out to be equal to that is just look at the formula 4 over pie. But, of course this factor has come, why because we already calculated a 1 r is equal to 4 over pie, so notice that this is first of all independent of omega.

Second this is real, let us just see what is the significance of this we have just now calculated the sinusoidal non linearity, this is 1 minus sinusoidal non linearity. We have found that its describing function eta is equal to 4 pi a this is t, this is the output y of t, sorry this output is going to keep oscillating. Now, this is what I have drawn here is not function of graph of y of t verses t it is input u of t y of t, if the input is positive.

So, for any positive value of the input output is equal to plus 1, for any negative value of the input the output is equal to minus 1. In that sense, this non linearity gives the sign of its input, it does not bother whether the input is increasing decreasing at what frequency it just gives you whether is the input is positive or negative.

If the input is positive it gives plus 1, if the input is negative it gives minus 1, input is u output is y. Now, we have decided we have calculated that we have a notion of gain for this system the gain associated to a sin omega t input and that gain turns out to be equal to 4 over pi a.

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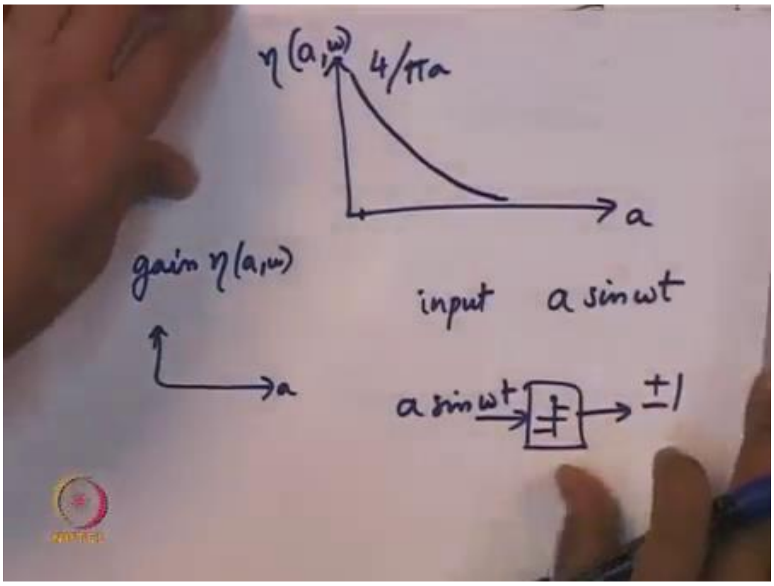

So, again is a function of a, for input a sin omega t we all are used to bode magnitude plots where we speak of low pass filter high pass filter there the independent quantity. So, the x axis the horizontal axis is frequency omega here also we are looking at gain complex gain actually luckily it turns out to be real for the case of sinusoidal non linearity when all it will be real we will see very soon. But, we are not plotting the gain as the function of amplitude and that we have found this is the graph this is what we just now calculated equal to 4 over pi a, polar values of a.

It is very small that is expected you see you give a sin omega t and what you get out is only plus or minus 1 here is this sinusoidal non linearity. So, how much ever high amplitude you give as input output is always plus minus 1, what is the magnification that this system causes magnification is very small. If the input has a very high amplitude why because the output is always plus minus 1 if you are giving a very high amplitude sinusoidal.

Then most of it is sort of getting diminished it is getting diminished to just plus minus one in that sense this graph is reasonable that for large amplitude the gain is very small. But, why the gain is small because the output has to always be plus minus 1 gain is like output divided by input output divided by input measured in some sense. So, suitable sense to get a real number to get a scalar to get a complex number in general, but in our case it is real to get one scalar. We have to have a notion of how we are measuring quantifying the output and the input we have decided to quantify using the first harmonics coefficients.

So, using that method it turns out that this describing function of the sinusoidal non linearity is decreasing like this is decreasing when it would be constant. Let us see various other examples, we will see what happens for linear systems when would it be constant, but for sinusoidal non linearity this decreasing. So, it makes sense because for very high amplitude also the output is just plus minus 1, for a very small amplitude also output is plus minus 1 for very small amplitude. In other words, there is high magnification because we will give you plus minus 1 even if the amplitude is 0.0001.

So, as long as it changes sign the output will jump between plus and minus 1 for very high amplitude also the output is just plus minus 1. Hence, the magnification is small for large amplitude the magnification is very high for small amplitude, so this is the graph of the describing function the gain of the system as a function of a. So, let us see some more examples the example that in every non linear dynamical system of course the first example we should be dealing is linear systems.

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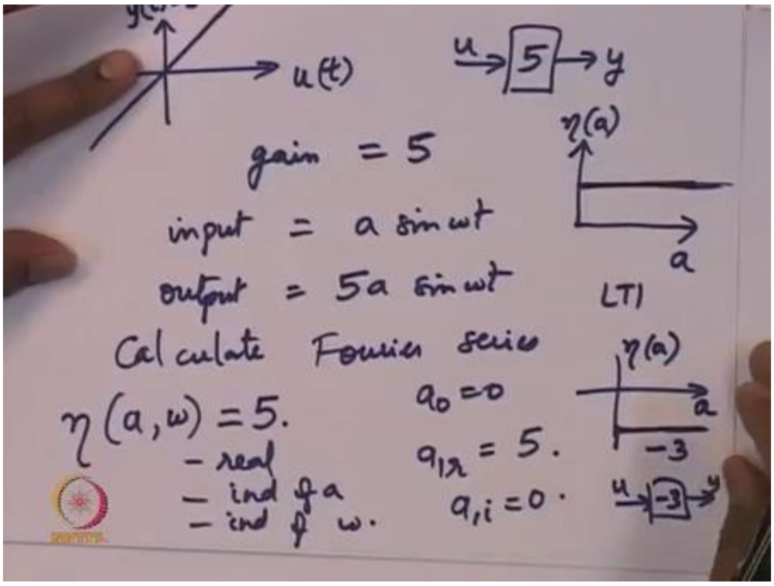

But we took a sinusoidal non linearity because of the ease in which we can calculate Fourier series, so let us take constant gain output y of t equal to 5 here is an input that just multiplies by 5. But, what does it do takes the input sees the current value just multiplies by 5, easy thumb rule to implement, no complicated saturation sinusoidal etcetera. So, what is the gain is equal to 5, how do we see this input is equal to a sin omega t output is equal to 5 a sin omega t. It is a good exercise to calculate the Fourier series Fourier series a 0 is equal to 0 a 1 are will turn out to be equal to 5, a 1 imaginary part will turn out to be 0.

Hence, describing function here turns out to be equal to just 5, it is real important things to note is real independent of a independent of omega. So, also that is expected you see this is just multiplies by 5 how does the describing function graph look as a function of a constant. We prefer plotting only for positive values of a because sin omega t we have amplification by a it is a positive number. So, the describing function the gain of the system is independent of a, that is expected for L T I systems if eta dependent on omega also.

Then this graph would not make sense, because it depends on a and omega, but here it depends only on a and this is the constant why because we have a static linear system that just amplifies by 5. So, amplification by any positive constant would just be equal to that number if it is amplification by a negative number. If it is amplification by a negative number then nobody prevents the describing function from being negative this will be equal to minus 3. Now, for a system that takes input u and gives a output y and multiplies by minus 3 this is how the describing function would look always at minus 3.

So, describing function can be positive and negative it is a convention to plot it for only positive values of amplitude a, input amplitude a. Now, we will see some more examples let us try to already guess how the saturation non linearity describing function is expected to look after that we will consolidate. So, properties of the describing function when it is expected to be independent of a, when it is expected to be independent of omega when the imaginary part is 0, all those we will quickly see.

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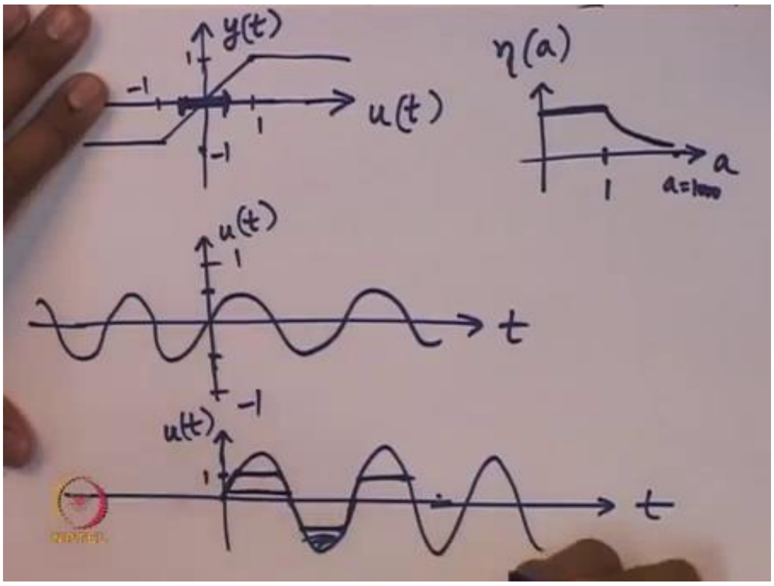

So, the saturation non linearity this is our input, this is our saturation non linearity for at any time u the output is like this. Now, we are going to give some plot something as a function of time input is always a sinusoid as per the describing function definition is concerned u of t.

Now, if this input amplitude is less than 1 it never crosses 1, so we are speaking of a case where the input is always varying between let us say plus and minus 0.6 this is our plus and minus 0.6. So, the input is now varying only inside this range, since the input is varying only inside this range the output is not even getting saturated to any value. If the output is not getting saturated to any value what comes out in the output is again a pure sinusoid exactly the same comes out in the output also. But, because for this range this output is equal to the input for this range because the output is equal to the input it is like as good as system with gain 1.

We are trying to directly plot the saturation non linearity as long as input has being multiplied by amplification a is at most one. The gain of the system is just 1 what is the meaning of this the gain of the system is equal to 1 as long as the input amplitude is at most 1. But, beyond that you see when this t this u of t suppose amplification is more than 1, so that one is somewhere here. So, the output is equal to this then comes back it is always getting saturated to one that is the meaning of the saturation nonlinearity here.

So, if it is getting saturate to plus minus 1 you see there is something that is getting cut, here we can think of that when the amplitude is larger and larger than the output is always getting saturate to 1. For a very small range, for a very small duration of time it is inside this linear range for most of the duration of time it will be in the saturated range. If the amplitude a is very large, for very large amplitude it is equal to 1 for most of the time and the input is actually much larger for most of the time. So, that time we expect that the magnification is actually very small why the magnification is very small for very large amplitude let us say amplitude a equal to 1000.

Now, for amplitude equal to 1000 the output is getting saturated to 1 most of the time and the input is much larger than 1 for most of the time even when you integrate and find this exact Fourier series coefficient. So, one will get that the describing function the magnification is very small because the magnification is very small for large amplitude is what we are talking about for large amplitude because the amplification is very small. For large amplitude we still get plus minus one only most of the time of course there is still a small duration of time over which it is equal. But, that time the input is also very small input is less than 1 that is why, so let us draw another figure.

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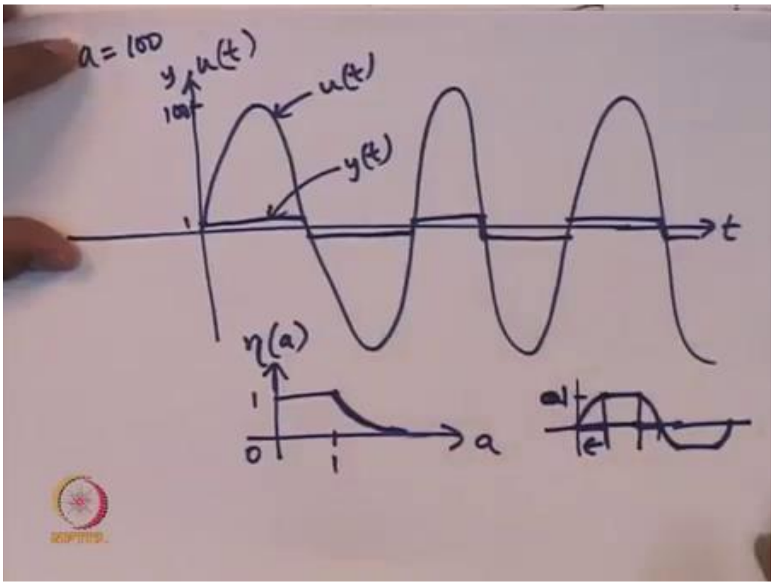

Let us draw a figure for a equal to, let us say 100 u of t is 1 is somewhere here this is 100, this is how the output looks like because the input is much larger than 1. So, most of the time the output is sitting at 1, most of the time you see this is how where y of t is this is, where u of t is. So, this is y and u this graph is part of both y and u you see output is most of the time just plus minus 1 while the input is most of the time much larger than 1. So, what is like a net amplification net amplification is a very small quantity, on the other hand if the input amplitude where itself less than 1 if the input amplitude where equal to 0.6 then the output is reproduced as the input.

So, this justifies our graph of the describing function as a function of a which is equal to plus 1 and then going on decreasing of course it never touches 0. But, comes down to 0 for large amplitude up to amplitude equal to 1 describing function is a constant its equal to 1. Now, like it would happen for any static game linear system for any such system it should have been a constant like this. So, we will check form a formula form the book that it is indeed like this how do we get this one would have to calculate the Fourier series of a function that gets like this the explicitly calculating the Fourier series from here to here.

Then it is a constant form here to here again from here to here for what range it is a constant and when is it increasing like a sinusoid that depends on the value of a of course this is equal to 1 if a is very large then this time itself is much smaller. So, using a careful calculation like that one finds the describing function explicitly as close form formula in terms of a, so let us see in general.

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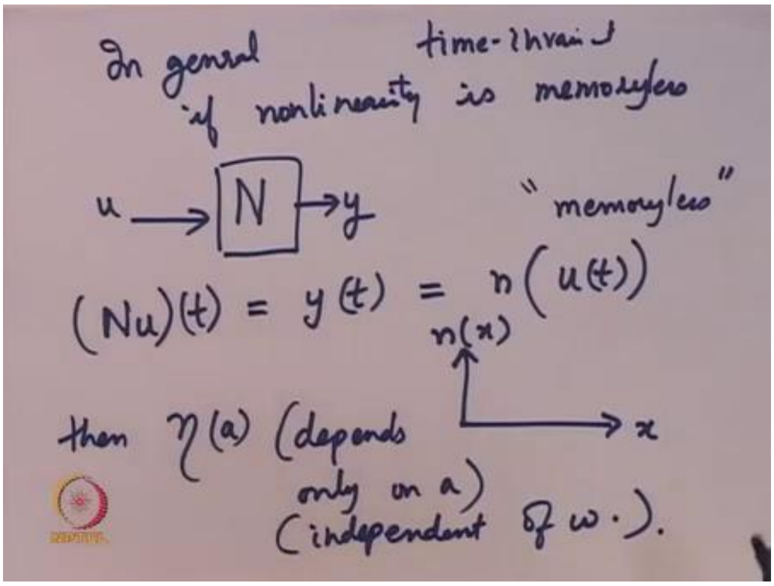

In general, if non linearity is memory less what is the meaning of memory less if this is an operator and n of u is a function of time is equal to y of t. So, n is an operated x on u, but that is luckily given as some other function that does not know that the input is varying. So, we can think of this x n of x if n if the output the way the output dependent on input depends only on the value of u at that time instant.

But, as soon as that value is given to a function n, n of x you will get the value this is the meaning of memory less you will call a non linearity n as memory less. If the way the output n of u is the output the way the output depends on time t can be associated to a fixed function n of x where output y of t is nothing but n of u of t. In that sense, n does not even know at which time instant this is happening it just depends on the value of u at that time instant this is the meaning of memory less.

But, of course we have been doing this for time invariant systems meaning that if the input is delayed by certain amount of time the output gets delayed by same amount of time. This is time invariant systems it is a big class of systems both linear and non linear, amongst the non linear time invariant systems we have described memory less systems. So, what are examples of memory less systems all these saturation nonlinearity the dead zone non linearity these are all examples of memory less time invariant non linear systems.

Now, what are examples of systems non linear systems non linear time invariant systems with memory these are systems with hysteresis we will see two three types of hysteresis is a non linearity. But, of course is time invariant the way the output depends on input does not explicitly depend on time, but it depends on whether the input is increasing or decreasing. In that sense there is a notion of memory associated to it, so hysteresis is an example of time invariant non linearity with memory.

In other words, which is not memory less, so if the time invariant non linearity is memory less than the describing function depends only on depends only on a at most this depends only on a means that independent of independent of omega. So, the frequency at which the input is changing can not affect the gain the magnification of the input to get the output say that there is no omega, omega we have skipped here means that the describing function the magnification. The gain does not depend on the frequency of the sinusoid input, now we will see under what conditions it is real, so when would it be real.

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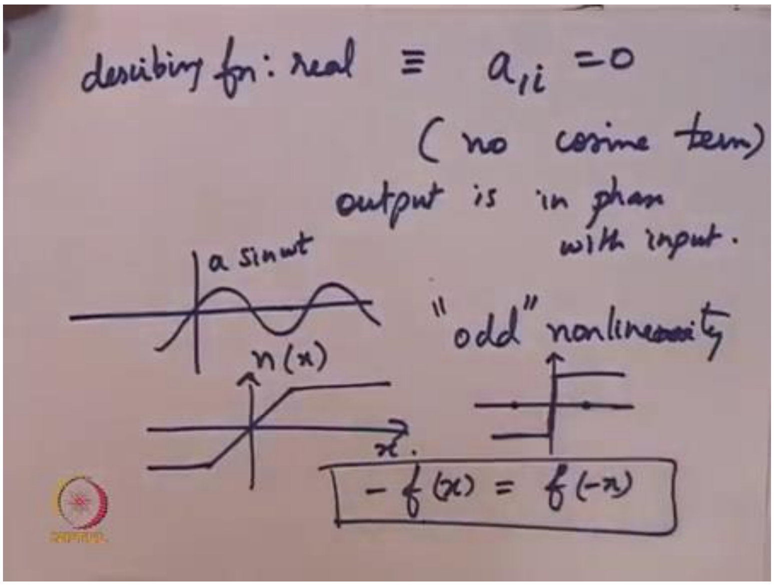

So, describing function real this is same as a 1 i equal to 0 no cosine term, in other words output is in phase with input. Now, notice that your input is a odd function of time a sin omega t, now if your output if your non linearity was also a odd function for example this saturation non linearity. Now, another example was the sinusoidal non linearity, this non linearity when instead of this value when you take this value then the sign. Now, these all satisfy the property that f of minus x is equal to minus of f of x these are all, so called odd nonlinearities with in the class of memory less time invariant nonlinearities we have classified some of them as odd nonlinearity, non linearity.

If this dependence of n as a function of x is an add function odd function meaning it satisfies this property, under those conditions the describing function is going to be a real function. So, there is no imaginary part that is why there is no cosine part why would this happen because when sine, the input a sin omega t is already a even function. So, output will not have a odd function output will not have a even function the input a sin omega t is a odd function and the nonlinearity is also an odd non linearity. So, this ensures that the output will not have a cosine term at all it will not have any even part this is what ensures that a 1 i in fact all the harmonics are also are not required.

Hence, the describing function will be a real function of a only why is it not a function of omega because it is memory less why does it not have a imaginary part because the non linearity is also an odd non linearity. So, when you give a sin omega t as a input u also get a output that is an odd function and hence no cosine terms are required to synthesize such a signal. So, we have found describe some conditions for which under which the describing function is memory less describing function independent of omega. Now, when the describing function is a real function, when the imaginary part is equal to 0, so before we see more examples of describing functions of various nonlinearities.

Now, we will try to see if a describing function has some rigorous development behind it one might ask why study the first harmonic, why not the other harmonics important. So, in what way this describing function carrying some important properties of the non linearity, after all it is an approximation I have only used the word that describing function is an approximation.

It is approximation in what sense is it, is it in some sense the best approximation what property of the non linear system has been captured by taking just the first harmonic. So, this is what we will see in little more detail before we see how the describing function is used for calculating approximate values of periodic orbits.

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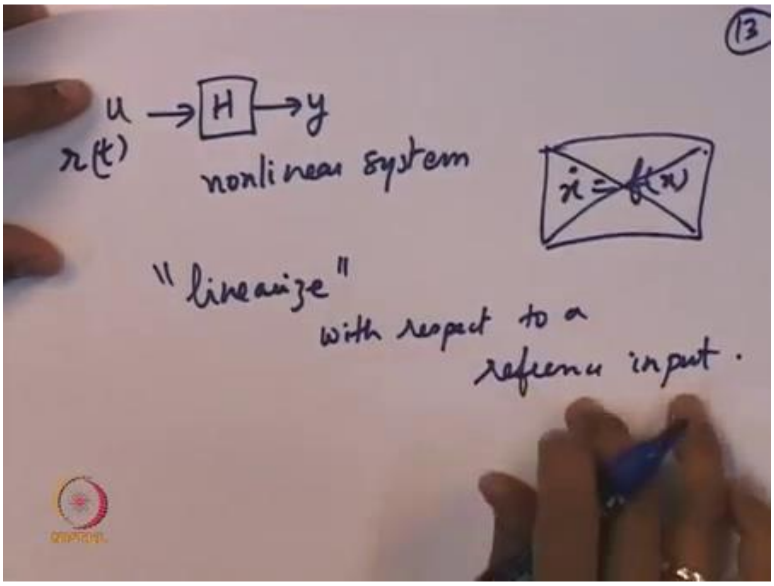

So, consider a system like this, so we have a non linear system like this and we are trying to linearise, so we only saw a notion of linearise for a non linear differential equation. The differential equation is what we took and already linearises once something about where you can conclude about this stability of the equilibrium points post linearization. So, with respect to the linearise system you draw out some conclusion about the stability of the equilibrium point and you can that be utilized for concluding the equilibrium stability properties of the equilibrium point of the original non linear system.

So, that is not the linearization that we are talking about, now here we are now speaking about linearise with respect to an input with respect to a reference input. So, we are speaking about a reference input r of t equal to u of t and we are looking at the output of the non linear system we are trying to find the linear system whose output is as close as possible to output of this non linear system. So, we are speaking about linearise with respect to the with respect to a reference input this is different from linearising. So, with linearising the differential equation in for analyzing the stability properties of the equilibrium point, so these are two different notions of linearization.

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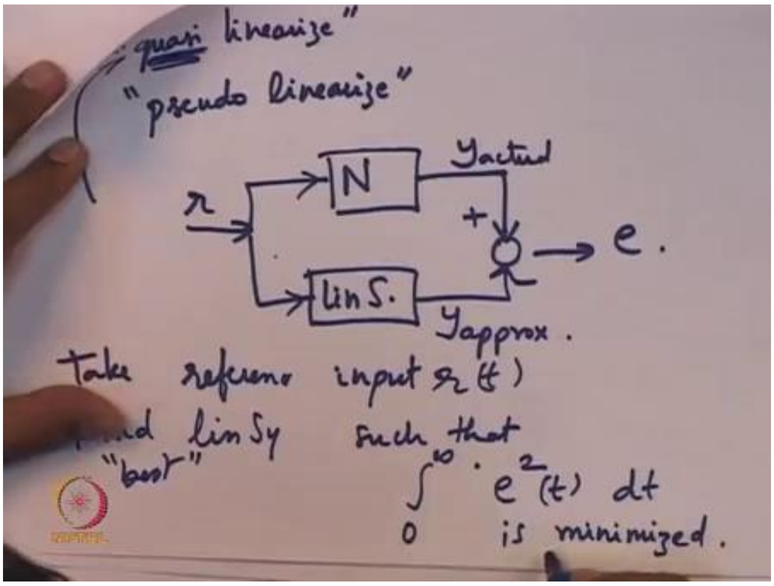

So, this is better called as quasi linearization why because it will turn out that the linearised system optimal linearise system depends on which reference input you use. So, when there is some when there is some notion of dependence it is not linearise in the full sense because of this different linearise systems that you will get depending on which reference input you took. So, whenever this happens people will use a word to say that this is not independent of what you use for linearising. So, that not independent is sometimes conveyed as either quasi or pseudo, so in the context of describing functions one says that the describing function is an optimal quasi linearization.

So, what is optimal about that you will see, now what is quasi about it because the linearise system depends on the reference input that you use for the purpose of linearization. So, let us take this r, let us take this original non linear system n this is the actual output this is linear system this is the error. So, take reference input this is y actual, this is y approx what is approx about it, because we are trying to fit. So, we are trying to find the linear system approximation to the non linear system such that this error e is minimized in what sense at every time instant or some total that is what we are going to see.

Now, take reference input r of t find linear system such that integral from 0 to infinity of e square d t is minimized is that reasonable is minimized over. So, what minimized over all linear systems that you could take, so when we give a reference input we say that this reference input is very important it does not. It is not a problem that this non linear system has different linearization with respect to this reference input we compare y actual the actual output that comes from this non linear system for this reference input. We compare that with y approx what is y approx, it is because we are trying to fit some approximation of this non linear systems approximating it by a linear system.

So, this linear systems output is what we called y approx we will take the difference that is called the error that that difference we will call as a error. We are trying to minimize the energy in the output why this energy because you have taken squares the time domain we are measuring the total energy integrating 0 to plus infinity. So, find best what is best about it this error has got minimized, so of course the first question this integral is even less than infinity. It is possible that this integral is not even bounded it is it does not exist very large e is positive agreed, so this cannot be negative.

But, it could be plus infinity for example if the error is the constant number if it is always equal to plus 5, if it is a non zero constant when you integrate from 0 to infinity. Then you will get something that is unbounded is minimized provided finite, so this is what this is our objective performance objective.

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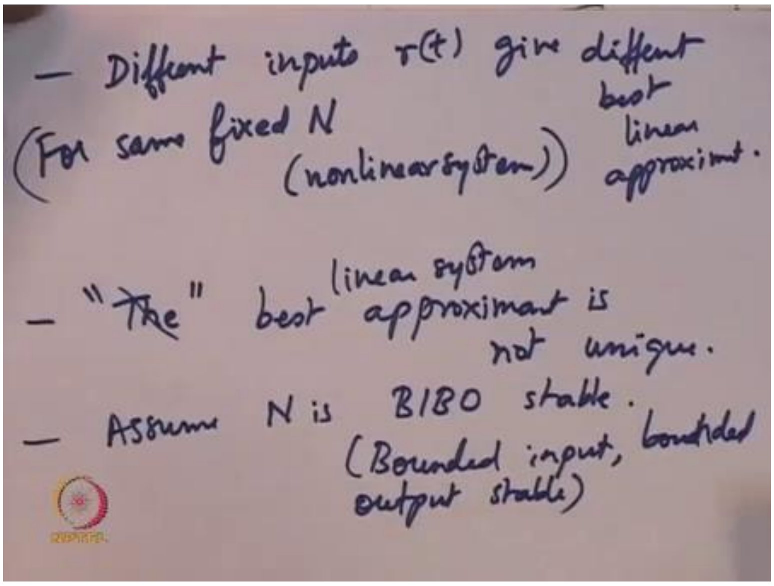

So it turns out one very important result is… Different for the same for same fixed n non linear system different inputs r of t give different best linear approximate. So, first important point even for the same fixed non linear system second it is not even reasonable to call the best approximant. But, best linear system approximant is not unique why what are all those different linear system approximants which are all best and still there is some freedom.

So, what are they assume n is b e b o stable what is b e b o stable when whenever you give abounded input the output is also bounded input bounded output stable what is the meaning of this. So, whenever you give an input that is bounded the output is also guaranteed to be bounded, so assume that this nonlinearity is a bounded input, bounded output stable. So, that you can give a sin omega t and you can be sure that the output has finite power a sin omega t is not a sinusoidal that has a finite energy because you integrate from 0 to plus infinity and the energy is unbounded.

But, at least it has finite power, so we are dealing with finite power signums, now we are trying to deal with finding a linear system approximation with respect to reference inputs. Now, that are all having finite power this all will be made more precise in the following lecture. But, assume that n is bounded for mono stable under these conditions what are all the non linear systems that have the same.

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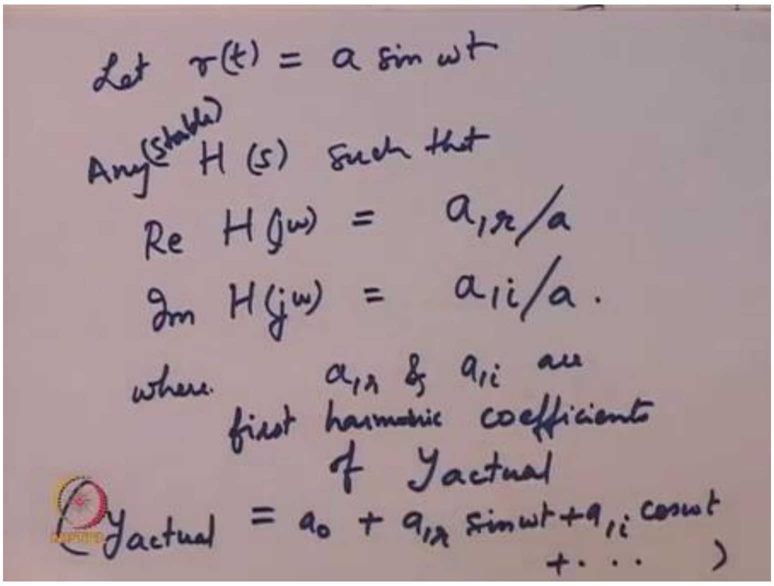

Let r of t equal to a sin omega t any H of s such that H real part of H of j omega is equal to a 1, r imaginary part of H of j omega equal to a 1 i where a 1 r and a 1 i are first harmonics coefficients of y actual. So, what does this mean y actual we are trying to write as some a 0 plus a 1 r sin omega t plus a 1 i cos omega t etcetera, we expand this.

So, we write, we find a Fourier series representation of the actual output where the actual output is for this reference input a sin omega t is the input. So, any transfer function h any stable transfer function h also has to be bounded input bounded output stable. So, any stable transfer function the extremely important theorem that is what makes describing function rigorous. But, of course it has always proved to be very useful derivation of this can be found in Vidhya Sagars book on non linear systems analysis.

So, any stable transfer function H of s whose real part coincides with a 1 r whose imaginary part coincides with a 1 i, what are these a one r a 1 i a 1, 1 refers to first harmonic and r and in refers to the sin omega t part and cos omega t part. So, take any transfer function, which matches only here as long as it matches, here this H is a best approximate what is best about it. So, it minimizes the square error there might be a division by a that is missing, sorry divided by a where this is amplitude of the input a.

If the input is large we might expect that a 1 r and a 1 i are also large, so then H is going to just allow the amplification a to come into the output into its output the y approx that we used. Hence, there is a division by a that is required here, so clearly you if you are only specified a 1 r and a 1 i there can be many transfer functions which have the same value when evaluated at j omega. So, this is what brings in the non uniqueness in H, but then the best part is that all these best approximations, best approximant linear systems all evaluate to the same complex number. So, when evaluated at j omega if they have to be a best linear system approximant to that non linear system. So, once it is a linear system how will the output look like that is what we will use just straight forward impulse response.

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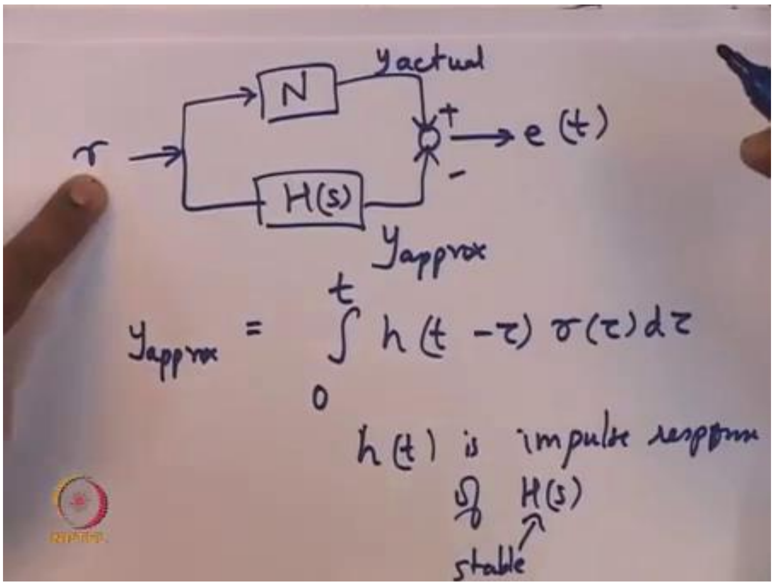

We will call this H of s y approx y actual, so y approx is nothing but equal to integral from 0 to t H of t minus tau r, tau d tau where h of h of t is impulse response impulse response of h of s. So, we have only assumed that this is stable no poles in the right half plane no poles on the imaginary axis also that is when h is bounded input bounded output stable. Now, this is how one will calculate y approximation for any linear system and notice that h can have memory also, this h of s output is calculated by this regular procedure.

The way we do for linear time invariant systems when will the error get minimized it will get minimized in the average sense in the finite. So, in the power sense there is a small mistake in what I told in the previous slide I will correct it right now, so in that power sense with respect to the minimization of the power. Here, you see why we shifted to power instead of energy r is not a sinusoidal with finite energy when r is here. So, that is what I will explain more detail, now these things I have made precise in Vidhya Sagars book.

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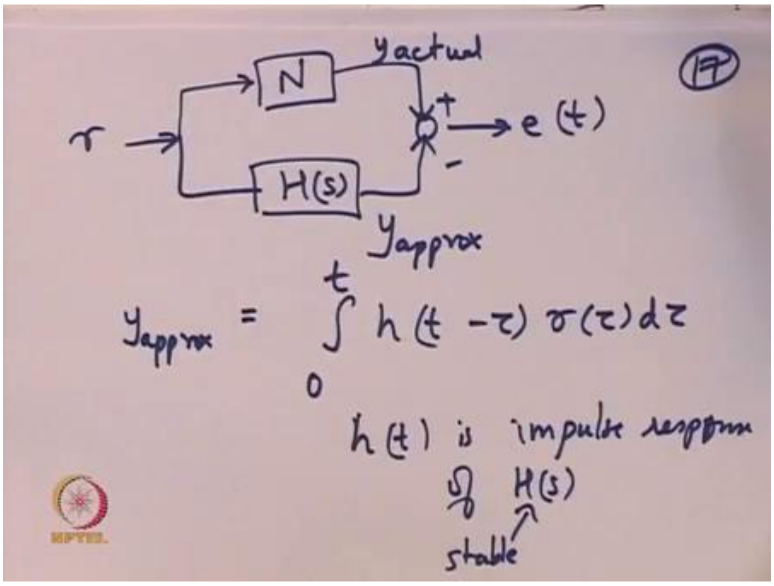

Now, that r of t is equal to a sin omega t 0 to infinity of r of t square is not is not finite, but you divide by t you if you look at the power in the signal limit as t tends to infinity of r of t taking square is same as taking absolute value. Then taking square this is finite we will say finite power we will say this is not finite energy it is not finite energy, but it is finite power. So, our input has finite power and for system that are bounded input bounded output stable the output let us comeback to the this example. The output y actual will also have finite power if you have to minimize a power in the error then this also has to have finite power. So, the transience of h are all dying to 0 because they are dying to 0 they do not have any the power the average power in them is 0.

So, why they have finite energy the transience I am speaking only about the transience have finite energy and, hence the average power is equal to 0. But, only the, only the value of H when j omega decides the average power in y approx and when these two signums are subtracted this average power in e is minimized. So, that is what makes describing function a optimal quasi linearization of a non linearity n this is the theory behind using describing function that is why it gives us the values that we want in a very in a useful sense. Now, even though they are little approximation they gives us the required values, so we will see more in detail in the following lecture.