Nonlinear Dynamical Systems
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Lecture - 24
Popov Criterion Continuous, Frequency-Domain Theorem

So, what we are going to do today is look at one particular technique that is used for non-linear systems and this is probably one of the first techniques that came in non-linear systems. It is amazing it is a frequency domain base technique, but subsequently of course the discovery of circle criterion and so on. This particular criterion is more easier to understand, but how so this criterion is called Popov criterion and it was discovered by Popov in 1962. How the techniques that Popov used of course are one way to view the Popov criterion and in fact the Popov criterion can be looked at from several viewpoints. So, what I would like to do is look at the Popov criterion in terms of loop transformations, so far that we first recall what we did with loop transformations, so you can recall.

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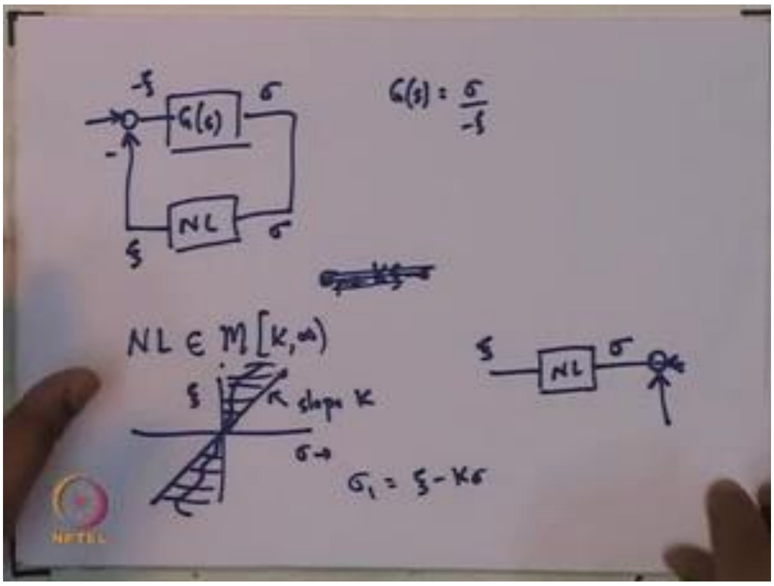

So, we were always interested in a linear plant and the nonlinearity and they are interconnected in this feedback kind of situations. Then, of course without giving any inputs, we are interested in knowing whether this feedback this feedback the circuit is a asymptotically stable. Now, of course this nonlinearity could belong to various kinds of classes and so we had earlier already said like for example, the nonlinearity could belong to belong to the k infinity class.

Now, what we mean by that is that if you draw the nonlinearity characteristics, as think of the input to the nonlinearity as sigma output to the nonlinearity as high. Then, if you plot sigma against high, then here I have this line with slope k and then if the nonlinearity is in this sector, what is means is that the nonlinearity something like that. So, essentially the non linearity is something that lies in the first coordinate in between this line and the and the zeta axis and similarly in, out here.

So, what I am shading that is the area where the non linear characteristics lie. Now, going back to the loop, if you think of this sigma as the input to the nonlinearity and sigh as the output to the nonlinearity, then by the way, this is this is drawn the input to the linear plant is minus sigh. The output of the linear plant is sigma in other words G s can be thought of as sigma divided by minus sigh. Now, in the loop transformation, what is done is this nonlinearity is changed now for nonlinearity in this class.

For example, we can convert this into a new nonlinearity in the 0 infinity loss and the way we do it is the following. So, what we do is in this particular case, we keep the output as it is. So, here we have the nonlinearity noise and let us say the output is sigh, but we modify the input, so we continue to have the input sigh, but what we do is we use a feedback.

For example, we could think of a new input a nonlinearity sigma one and this sigma one is given by let us say k times sigh minus sigma k times sigh minus sigma. So, we think of the new the new input sigma one as being sigh minus k sigma sigh minus k sigma, no I am sorry. So, we in this particular case because it is in the k infinity sector it is not the sign that we change, so let me use a new sheet.

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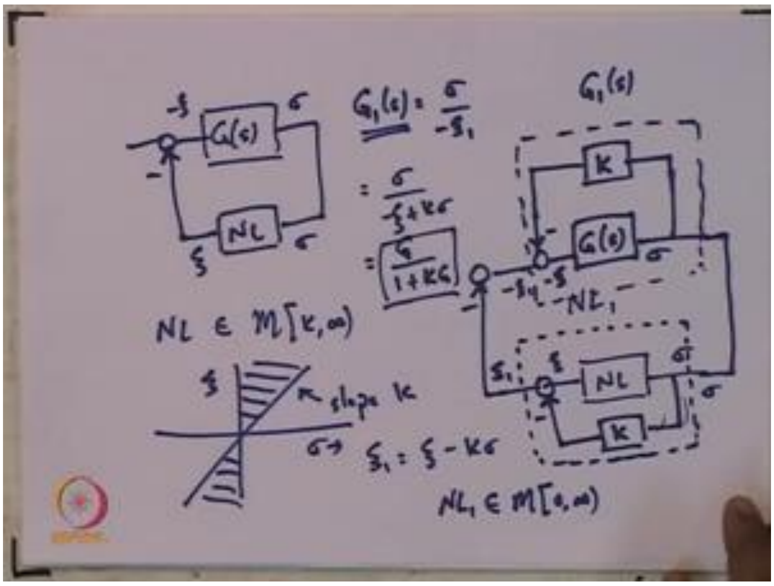

So, we have a the linear plant and the nonlinearity and sigma minus this is sigh, so this is minus sigh this is sigma and we assuming at this nonlinearity is in the k infinity sector. That is like saying you have this line of slope k, so the nonlinearity lies in the hatched portion. So, what we do is we keep the input the same, but we change the output, so how do we do that we take the same linearity by keeping the input the same sigma. Therefore, the output here is sigh, but we construct a new output sigh 1, which is sigh minus k sigma.

So, if we have to construct the sigh 1 into sigh minus k sigma, then we might as well have a loop. So, k time sigma which we add to sigh with a minus sigh and so what we have here is sigh 1, now I am drawing this box here and whatever is inside the box I call it a new nonlinearity. So, let me call this nonlinearity n l 1, now if you think of n l 1 this n l 1 has sigma as its input and sigh 1 as its output. Now, if you now look at the characteristics of n l 1, which has sigma as the input and sigh 1 as the output, now if you look at sigh 1, which is sigh minus k sigma.

Then you see this particular nonlinearity n l 1 will lie in the 0 infinity sector are the reason being that, of course the origins l 1 was laying in the hatched area. So, the sigh could at most given a sigma, the sigh could at most be k given a sigma, the sigh could at most be k sigma. So, if you ate doing sigh minus k sigma it could come right down to 0 and so you have the 0 infinity sector, so this nonlinearity that you have this nonlinearity is in the 0 infinity sector.

Now, the output of this and the input so the input is sigma and the output is minus sigh and so we try and maintain a similar configuration. So, this sigh 1 is the one which is fed back, now if sigh 1 is the one which is fed back and we are interested in this G s. We would of course like the input of G s to be minus sigh rather than minus sigh 1, so what we do is we modify the input and you see and we would like the output of the linear part the output of the linear part is sigma.

We would continue that sigma into the input of this new nonlinearity, but the linear plant we would modify in the following, so you have sigma, but what we do now is we take this sigma and multiply it by k. We would feed that back with the negative sign in here, now if you see and then we would give that input to G s. Now, if you see sigh 1, so out here the signal is minus sign 1 and to that from that k sigma is subtracted, but minus sigh 1 is minus sigh plus k sigma. So, that plus k sigma is cancelled and so what finally you have in input of the original linear plant is minus sigh which is precisely what you had here.

So, the linear plant can be as it was and so by modifying the nonlinearity, we also modify the linear plant and therefore what you have affectively is the same as what you had here and so the way. So, the linear plant that you get is what is contained inside this box and let me call a G 1 of s. If you want to look at the transfer function of G 1 of s G 1 of s has as its output sigma and as its input minus sigh 1 for sigh 1, I substitute and I get sigma upon minus sigh plus k sigma and this.

Then, above and below if I divide by minus sigh 1, I end up with G upon 1, 1 plus k G, so this G 1, this new linear plant is related to the old linear plant in the following way. So, G 1 is equal to G 1 plus k G, so this is the kind of thing that we did, so I mean now we can talk about asymptotic stability of this is the same as the asymptotic stability of this loop, but in this loop because this new nonlinearity is in the 0 infinity sector. Then, we know that the linear plant must be a positive real plant in other words G 1 s if G 1 s is positive real.

Then, this system is asymptotically stable, but G 1 of s being positive real is saying that as far as this thing is concerned it is asymptotically stable if G upon one plus kg is positive real. Now, if you look at it very carefully the original system what we have done in this system is in this loop the in this armed with the nonlinearity.

We have fed back with a minus I mean it is a negative feedback with k, of course considering that the nonlinearity is going this way. This is really a feed forward nut up there, we do the same thing we take the sigma multiply it by k and feed it back. So, upstairs it is a it is a negative feedback and here it is a negative feed forward. In this way we manage to convert nonlinearity in the 0 in the k infinity sector into nonlinearity in the 0 infinity sector, now in the same way as we do these other loop transformations.

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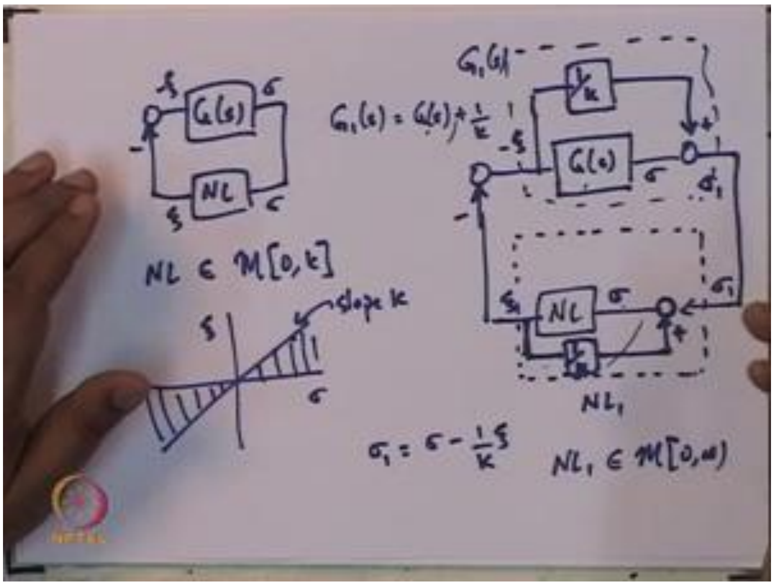

So, for example had the nonlinearity had the nonlinearity been in the 0 sector, so suppose the nonlinearity belong to the 0 k sector. So, that is like saying if you think of this is sigma and this is sigh that is sigma and sigh this is minus sigh this is sigma. Then, this nonlinearity has characteristics, which lie in this hatch portion this is the line which slope k now again our endeavour would be to convert this nonlinearity into a nonlinearity in the 0 infinity sector. So, how does how does one do that, so in the earlier case we saw that we kept he input the same, but changed the output in this case one approach that we could take is we could keep the output the same and change the input.

So, let us do that so you could take this nonlinearity and its output is sigh so the new output will continue to be sigh. What we would do is we will create this nonlinearity, this new nonlinearity and let me call this nonlinearity n l 1. Now, so as that the input is changed, so let me call the input to this nonlinearity sigma 1 and what is sigma 1 well let us take sigma 1 to be sigma minus 1 by k of sigh. Now, if you take sigma 1 to be sigma minus 1 by k of sigh, so we feed back from here with again 1 by k in here.

We add this then sigma 1 added to you know 1 by k times sigh will end up in us getting just sigma here. So, the nonlinearity the original nonlinearity you know the characteristics are sought of preserved out here, but what we have done is that the output, we have fed back. Of course, it is a positive feedback such that it cancels this, now if we do this then this nonlinearity, so y is this nonlinearity now in the in the 0 infinity sector. Well, you can you can see that if sigma was the original input and sigh was the output, but we modify the input to be sigma minus 1 by k sigh.

So, sigma is here and 1 by k sigh, so suppose the point was let say out here that means, so if it was sigh was k times sigma. Then, what we are saying is that the sigma one the new input is going to be sigma minus 1 by k of k sigma that means sigma 1 is going to be 0. So, you are going to get that same output with 0 as the input I mean close to 0 as the input, so one can show that this nonlinearity, this new nonlinearity that you created l n l n 1 is in the 0 infinity sector. Now, going back to the other portion the linear part, so if you have minus feedback, so what you have here is minus sigh.

So, we could put the linear plant there, so if you put the linear plant here from here, you know that with minus i s d input sigma must be the output, but we want sigma one to be the input here. So, we have to modify this, so what we do is we take this minus sigh and multiply it by 1 by k and add it to the sigma. So, then affectively what you have here is sigma 1, so this is the feedback that you will have. Now, this here is the linear part here G 1 of s and so 1 1 of s inter connected to this nonlinearity n l 1 is equivalent to G of s interconnected to this non original nonlinearity n l.

Now, if you look at the arrangement that we have done, what we have done here is of course here of course the sigma is slowing this way. What we have done is we have taken the sigh here and multiplied it by again 1 by k and put it back here with a positive sign. So, here also we will do the same thing, so just like in the last case we do exactly the similar things in the upper and the lower of course, what it means is here it is a positive feedback and here it is a positive feed forward.

Now, in this case of course this new G 1 of s turns out to be nothing but G of s plus 1 by k and then therefore for this system to be asymptotically stable. What we require is that this plant G 1 of s because this nonlinearity is in the 0 infinity sector.

We want this G 1 of s is to be positive, real in other words G of s plus 1 by k this whole thing must be positive real of course, then you could interpret as saying that the Nyquist plot of G of s should lie to the right of minus 1 by k. Now, notice that in the last case and in this case in both these cases, what you have done is some sort of feedback feed forward that you have done whatever you are doing, this arm which contains the nonlinearity. You do a similar thing up there with the linearity and what it affectively does is neutralises the affect of that thing in some way so that the original nonlinearity and the original linearity still maintain the relation that they were suppose to maintain.

Now, the point is that the gains that we used in both the situations were gains that that were constant gains, but perhaps we could introduce gains maybe in such a way or maybe in series with the linear plant. For example, in series with the nonlinearity or in series with in series with the nonlinearity and, therefore in series with the linear plant and so on and modify the loops, so let us just try out one particular example.

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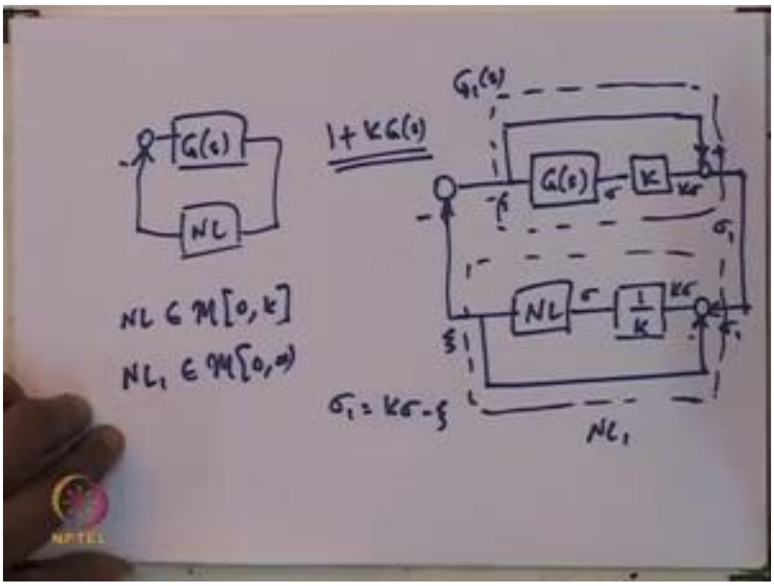

So, for example if the original pant is like that with the nonlinearity like this, now let us not bother about where the nonlinearity lies, but let us do the following. So, suppose the nonlinearity is here and let us assume that the output to the nonlinearity is kept the same.

So, the output is kept aside, but what we do is input, so one would want the input here to be sigma but maybe we put a gain here and maybe the gain the gain that one puts here is let us say 1 by k. So, what that would mean is out here the signal that you have is k sigma and we could think of unity feedback, so just like in the last case, but I have pushed the gain from 1 by k from the feedback loop in here.

So, then we could think of the input as sigma 1 and so in this case you have sigma 1 is k sigma minus sigh and so this here is our nonlinearity. Then, in order to find the equivalent thing to equivalent interconnection to this interconnection, you have a G s here. So, the size of the minus sigh, so the output of this is sigma, but you want to get sigma one sigma one is obtained this way, so what we can do is multiply this by k therefore you get k sigma. Then, you take this minus sigh and feed it forward and therefore what you will get is sigma 1 there. Now, the construction that I have done here is exactly the same as the earlier construction that I did.

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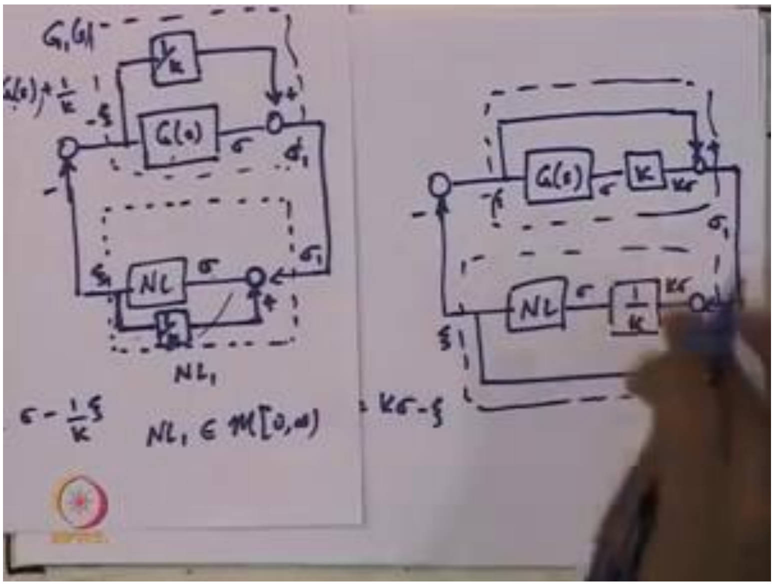

So, this construction and this construction are exactly the same the only thing is this gain I pushed it in here and, therefore this gain has to get pushed in here, but when I do that here it is 1 by k there. When it is get pushed its get pushed to k and that is essentially because here it is in the in the feedback arm and there it is in the feed forward arm. So, when you put it in series, it becomes k whereas, from the feedback arm it remain as it is and so what we have affectively done is we have done, I mean we could just assume that this nonlinearity.

Here, is really a nonlinearity in the 0 k sector and therefore, this new nonlinearity n l 1, so n l one would then be in the 0 infinity sector. This is the original plant G s and the new plant G 1 of s that for G 1 of s is 1 plus k times G s, which of course is roughly the same. As a the result, that we have got earlier only it is the new G 1 of s is going to be k times the old G 1 of s, but you know by just multiplying by s k are k, you do not change anything, now consider having gains maybe we could have trance function, so one kind of Transco function that you could possibly have is...

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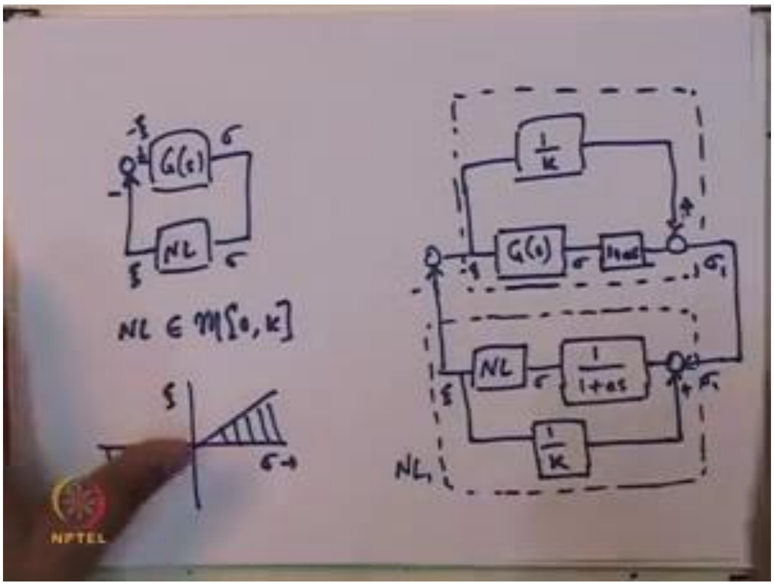

If you think you have G s here and you have nonlinearity and lets again assume that this nonlinearity is in the 0 k sector, what we do is a construction similar to the earlier construction. So, what we do is this nonlinearity be this sigma this is sigh, therefore you have this signal minus sigh here sigma there and a nonlinearity is in the 0 k sector, which essentially means this hatched area is where you have the nonlinearity.

This is sigma this is sigh so what we do is this nonlinearity, you multiply by first order Transco function like this, so if you multiply the nonlinearity by a first order Transco function like this and of course this nonlinearity is the same as this nonlinearity. So, the signal across you wants to maintain to be the signal across there, so you have sigma here and sigh here. So, if you use this sigh in the feedback, so you have minus sigh here, so let us assume you have a the Transco function G s here.

So, what you would get here is sigma and let me let me do the following, let me assume that there is a game here which is 1 by k of the sigh and I feed it back that way. So, if you have something like this you should do a similar thing there, I have this new input sigma 1. So, this is my modified nonlinearity n l one now one would like to do something out there such that this and that sought of matches up.

So, what we do up there is I can again here I have a feedback positive feedback loop, so there is a plus sign here and again a 1 by k. So, I take this with again 1 by k and I am going to add it here with a plus sign and earlier we saw that if you had if you had 1 by k here, then you have k here. So, we will just blindly do that and we will see what happens, so let us blindly do that, so the reciprocal of this is 1 plus a s. So, let us put that k in 1 plus a s out there and let me call this output that comes here sigma one and connect it up here. So, maybe this is looking all to take clamed up, so maybe I will use a new sheet with this particular diagram.

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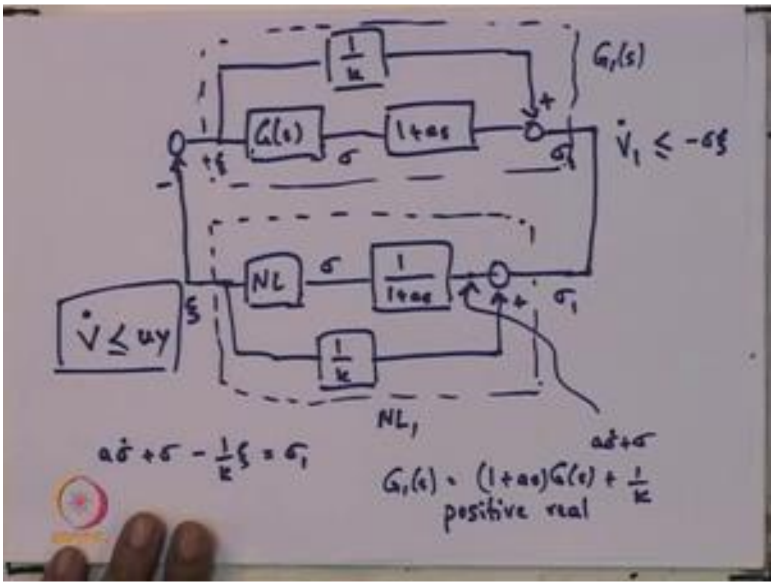

So, we had the nonlinearity and then we used to 1 upon 1 plus a s here and again a 1 plus k which we fed n back in here with a positive sign. And this whole thing is the new non-linear system n l 1, let me call it and so n l output is sigh, input is sigma 1. This is sigma here, we are assuming this is sigma here, we do not yet know, but let us just do the construction and see whether we get something meaningful G s. So, with the input minus sigh, so the output here must be sigma and then we have gain like that and what we do is a similar construction.

So, a gain of one by k through here again with a positive sign, now from the earlier situation here we know G of s is sigma by minus sigh. So, let us just make this assumption, so assuming that the output of the nonlinearity is still which is fed back here. So, you have minus i here, so this must be sigma, now if sigma passes through here what you get here is going to be a times sigma dot plus sigma. Then, when you have this thing, so what you are going to get is minus 1 by k sigh and this whole thing is sigma 1.

Let us assume this whole thing is sigma 1, now if this whole thing is sigma 1, so sigma 1 which is coming in here you have sigh 1 by k times sigh added to it, so what you have here the signal here is going to be a times sigma dot plus sigma. If you have the signal a times sigma dot plus sigma here, then 1 upon 1 plus a s acting on that will in fact give you the signal sigma. So, this whole thing looks consistent, so this whole linear plant that you have let us call that G 1 of s and this whole non new nonlinearity n l 1.

Then, perhaps we could say that this interconnection is asymptotically stable and if we can say that this interconnection is asymptotically stable. That would be equivalent to going back to the original plant and saying this interconnection is asymptotically stable. Now, in here if you look at this linear plant G 1 s this linear plant gone s is given by 1 plus a s times G s plus 1 by k. Now, if this linear plant is positive real, now let us make some assumptions, so if this is passive and this is passive than the interconnection of two passive systems is passive.

So, I mean this being passive and this being passive the interconnections passive, so because each one of them is passive one could find linear part of function or a storage function for each one of them and the storage functions. So, recall that for passivity if you have a passive system, then you have a storage function v such that v dot is less, then input times output, this is how we sought of less than equal to input times output. This is how we characterise passivity, so what we can start doing is the following, so let us assume that this whole plant this whole new linear plant G 1 of s is passive.

Now, this whole thing being passive is like saying that G 1 of s, which is one plus a s times G s plus 1 by k, this is positive real. Then, we have G 1 through several of these theorems earlier like this by which one can fine for this part a storage function b 1 such that b on e dot is less than equal to the input times say output which in this case is minus sigma times sigh.

So, if you can also show that this guy is passive then the sum of these going to be passive so how to show that this particular Transco function is also passive. Now, for that what we will do is we will look at the differential equation, so the differential equation that one could look at is the following this whole nonlinearity is there, so let me write it out in the following way.

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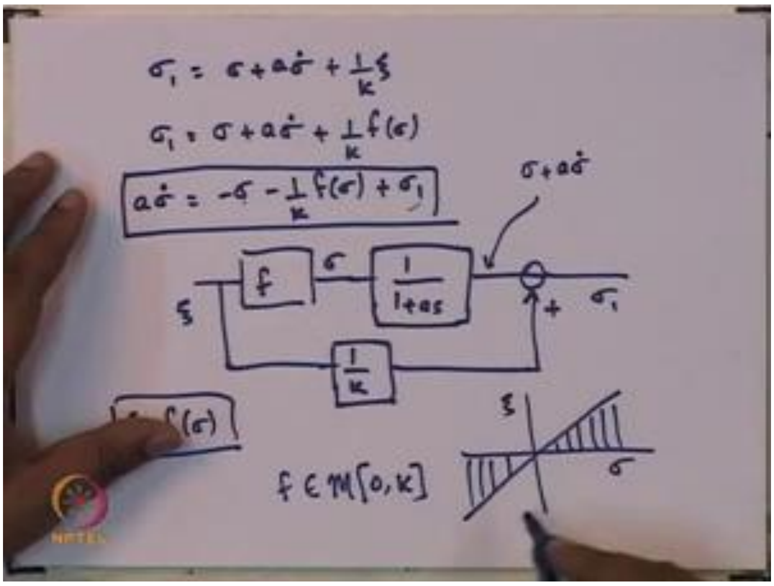

So, let me call the nonlinearity f such that if sigma is the input than sigh is the output and what I am really saying sigh is equal to f of sigma and then earlier we had 1 upon 1 plus a s here. Then, we also had this feedback gain 1 by k, now if this is sigma, then of course from the last line, we saw that this must be this signal here must be sigma plus a times sigma dot. Because of this we know that sigma 1 must be given as sigma plus a times sigma dot plus 1 by k times sigh.

Now, let me right this sigh using nonlinearity thing, let me write this sigh as f of sigma and so I have sigma 1 is equal to sigma plus a times sigma dot plus 1 by k f of sigma. This thing is really like a differential equation, so think of this as a differential equation with sigma as the state variable and sigma one as the input a. So, o 1 could write something like a state space equation saying a sigma dot is equal to minus sigma minus 1 by k f of sigma plus sigma 1.

The sigma one is thought of as the input and sigma is the state and of course one could think of the output equation is sigh is equal to f of sigma. So, take this equation this differential equation the sigma is the state sigma one is the input and this as the output equation. So, these two equations here, now let us make one more assumption, let us make the assumption that this nonlinearity this nonlinearity f is in the 0 k sector, k is related to this k t, where the gain was 1 by k. So, sigma sigh so the f the characteristics of f lie in the hatched area, now if this is the k s then I sought of decide that I will take linear part of function, in the following way you see the characteristics of this f, so you see if the characteristic of the f of the function f.

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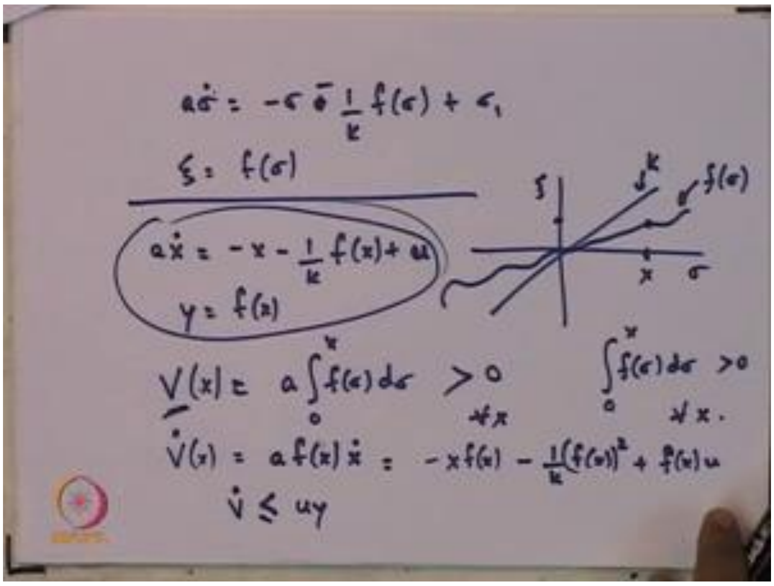

So, we have this differential equation a sigma dot is equal to minus sigma sorry minus 1 by k f of sigma plus sigma 1. You have sigh is equal to f of sigma that is the that is the system and this f is such, so if this is the slope k sigma sigh f is such that f looks something like that. Now, this is the curve f of sigma then of course it would be clear that if you take integral of f of sigma d sigma going from 0 to some point x, then this quantity is always going to be positive.

If that x is somewhere here, then this integral is positive if it is somewhere here, then this integral this integral is negative, but of course the limits are going the other way. So, when you change the limit, it becomes positive, so you get something like this, so this is going to be greater than 0 for all x. So, I have been using sigma as the state variable, so maybe I could just rewrite this as x a x dot equal to minus x 1 by k f of x plus u. Let me call it u the input and, I am saying y is equal to f of x this is the states base equation, it is the same equation as before.

We see that thus function is such that if you take the integral from 0 to x, this is true, so I propose that we use a linear part function V x given is equal to a times integral from 0 to x of f sigma d sigma. Therefore, this function or this linear function candidate has the property that this is greater than 0 for all x greater than greater than 0. Now, if you evaluate v dot of x, so v dot of x from this evaluation is going to be a times f of x times x dot. Now, a times x dot we can substitute in this, so you get minus x times f of x, so first time and then you get minus 1 by k f of x squared, it is that term and then you get plus f x u, now if you take any x, x times f x f x.

So, from positive x f x is positive for negative x f x is negative, so this quantity here x times f x is positive, so minus of x times f x is negative. So, this quantity here is also negative and therefore, one can conclude the v dot x is less than f x times u, but f x is y, u times y. So, this system that you have, this state space system that you have with this particular new polar function, you can see or storage function you can see that this is passive, so if you go back.

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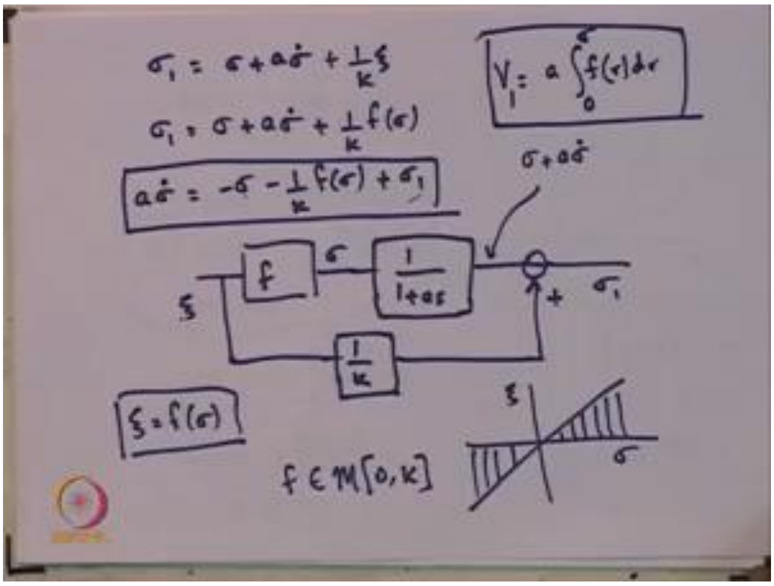

Here, this system that we have drawn here, this systems passive with the storage function given as V is equal to a integral 0 to sigma f of r, b r, let me call this b 1 with this storage function.

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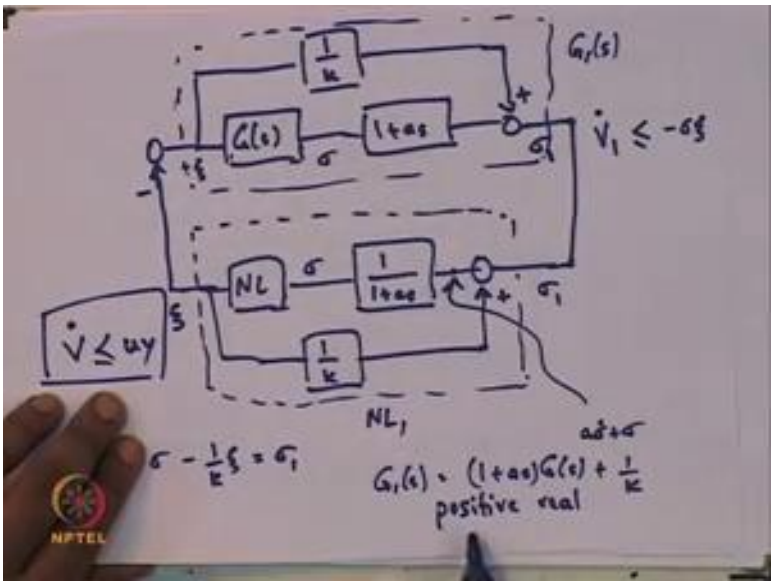

We already saw that in this interconnection that we were looking at this portion because its positive real has a storage function V 1. It has a storage function V 1 such that V 1 dot is less than equal to minus sigma sigh, because the output here is sigma and the input is minus sigh.

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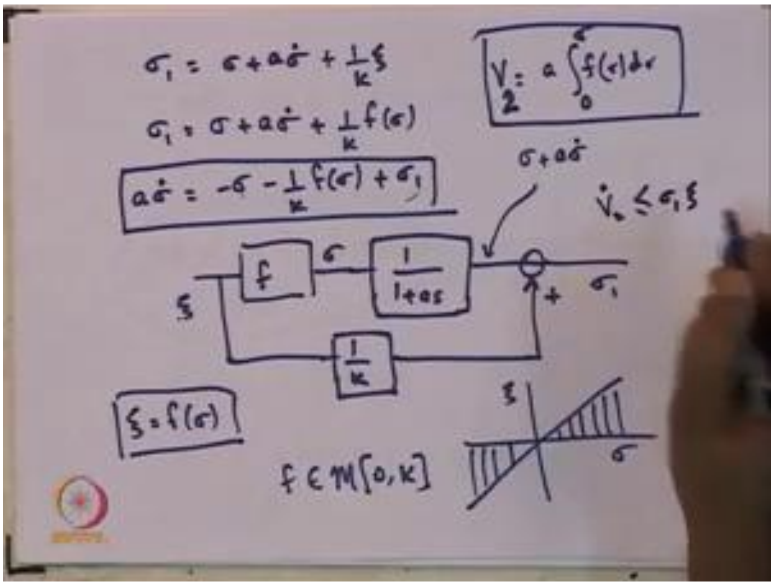

Out here if you take this let me call it V 2, then we know V 2 dot is less than is less than equal to the input sigma 1 times say output sigh, sorry I have to correct one thing here, the output is sigma 1 and not sigma and so it is minus sigma 1 sigh.

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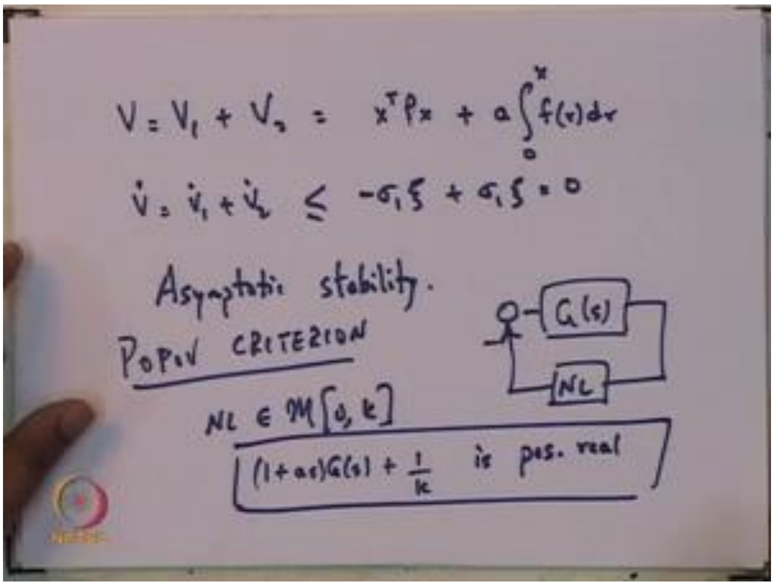

So now if you take the linear plan function of that net system, to be V 1 plus V 2, which is equal, so that we would have been given by because that that linear plant that affective linear plant that you had. So, this is the v 1 this affective linear plant, you could look at the states pace realisation of this and from that you could a write in terms of the states of this thing as some x transpose p x. Here, the p will satisfy all the linear conditions and plus a times integral 0 to x of f of r d r this as the net as the net linear function.

We then have V dot equal to V 1 naught plus V 2 dot, which of course we can calculate, but we know that v one dot this is less than v one dot we know is less than minus sigma 1 sigh V 2 dot is less than sigma 1sigh, sum of these two is 0. So, V this thing that you had this linear function is such that v dot is less than 0 and this quantity here is going to be positive because this quantity is going to be positive. This p has been obtained to be a positive definite matrix, therefore, by linear theorem we have a asymptotic stability.

So, what we are affectively saying, so this is what is called the Popov criterion and what the Popov criterion is saying is the following that if you had a near plant and you had nonlinearity. You had it interconnected in this way and this nonlinearity was in the 0 k sector and the linear plant was such that 1 plus a s times G s by one by k is positive real.

This interconnection is going to be asymptotically stable; of course this nonlinearity must be such that it is memory less nonlinearity in the memory less nonlinearity. This is the affective result of the Popov criteria, so it looks like I am out of time now. So, let me stop for now.