Nonlinear Dynamical Systems
Prof. Madhu. N. Belur and Prof. Harish .K. Pillai
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture - 23
Limit Cycles

In this lecture now, what we are going to look at is a some other characteristics of non linear system, which is equally important, I mean typically when you talk about non linear systems we want to analyze I mean a of course, if you are looking at a non linear system without any inputs, then what you would like to analyze is the are equilibrium points. You want to find out whether the equilibrium points are stable or unstable, what or sort of characteristics the equilibrium point has.

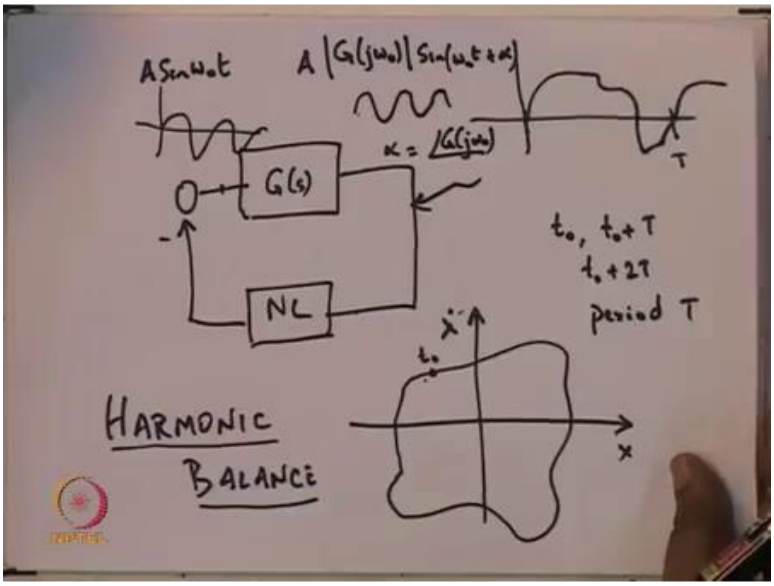
Now, another characteristic, which non linear systems display are limit cycles, so there are these limit cycles that could exist in a non linear system and one would like to know whether these limit cycles are stable, unstable, and so on. Now, of course this has a lot of practical use like for example, oscillator circuits are designed by a by using non linear circuits, such that the non linear circuits actually have a limit cycle, which is a stable limit cycle. So, what happens is when the circuits traverses through the limit cycle, it is a stable limit cycles.

So, if there is any sort of perturbance, you sort of get deviated from the path, but because it is stable, it comes back to the path, and therefore this limit cycle keeps getting described again and again. Now, one important thing about the limit cycle is that suppose you draw phase plane diagram of this non linear system, so you have this limit cycle which is really a closed curve in the phase plane. Now, if this system is evolving along a limit cycle and you look at you know you sense the signal somewhere in your system, because it is going through this limit cycle, the signal that you sense would be periodic.

So, if one wants to analyze or detect limit cycles in a non linear system, then necessarily there must be periodic solutions, I mean if you are sensing the non linear circuit somewhere, you should be getting periodic signals. So one could now utilize this idea of periodicity in signals to try and capture limit cycles in a non linear system. Now, from what we had discussed earlier, if we have a general non linear system. Then you can split it up into a linear part and a non linear part, and think of a linear part as an independent system, and the non linear part as an independent system with some feedback connection.

So, the linear you could think of the linear plant being in the forward loop and the nonlinearity in the in the backward loop and some sort of a negative feedback loop that you have. Analyzing this closed loop system is same as analyzing the original non linear system, so let us look at what it is that I am saying.

(Refer Slide Time: 03:55)



So, suppose you have A, some non linear equation and this we have discussed in some earlier lecture, you can split it up into a linear part and a nonlinearity. Then, think of the original non linear system to be analyzing the original analyzing the original non linear system is like analyzing this closed loop system. Now, in this system suppose you look at this, you have some methodology by which you can see the signal that exists on at this place, now if the system is going through a limit cycle, so you have the phase plane.

So, suppose it was an order 2 system, so you have x and x dot and if it is an order 2 system and it has a limit cycle may be it has a limit cycle which looks like that. So, let us suppose that the given system is traversing through this limit cycle, so if it starts from this point at time t equal to t naught after some time capital T, it will get back to this original point. So, what it means is that it will get back to this point, so it was at this point t naught, the next time it would be at t naught plus capital T.

Then t naught plus 2 t and so the periodicity is so the period is t and if you are sensing the signal here, then what you would get would be periodic signal like that, I mean this being the period and this period this is capital t and this keeps getting repeated. Now, when you have a non linear system, one would like to know if it has limit cycles and if it has limit cycles, then whether the limit cycles are stable or not and things like that. So, one way that one could detect limit cycles is by you know sampling or seeing these signals let us say on a oscilloscope or something.

Then, if the signal is periodic, then you know that the system is right now oscillating in a limit cycle, but analytically if one wants to find out if this given system has a limit cycle, then one of the methodologies used is what is called the method of harmonic balance. Now, the method of harmonic balance is based on some basic principles of periodicity you see for the linear plant. If you give as an input to the linear plant, if you give a periodic signal and the most common periodic signals that we think of are the sin, the sin or the cosine waves.

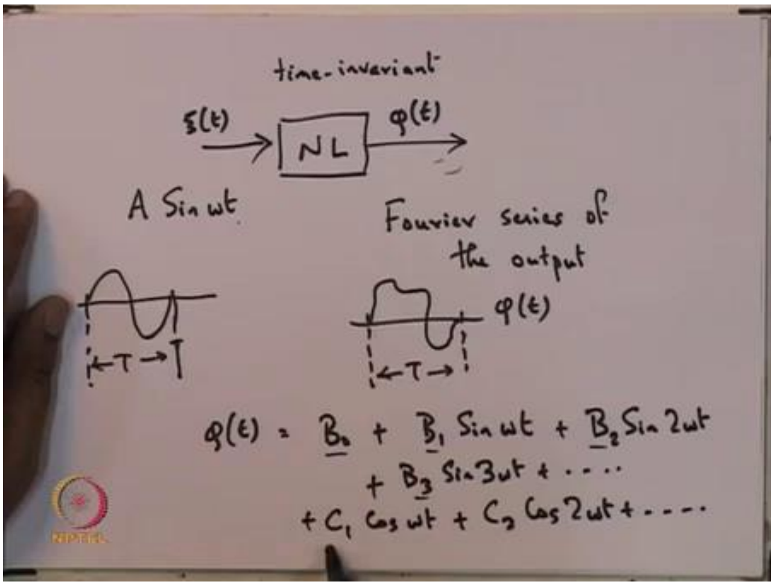
So, if you give a periodic signal as a input to the linear plant, the output of the linear plant will also be sin or a cosine, the only difference being that the amplitude might have got magnified, that is and in fact if you give a signal with frequency omega. Then, you know that the output signal for the linear plant would have magnitude and the magnitude was let us say A. So, you give a signal like a sin omega naught t, then the output would be G j omega naught I mean you evaluate what the transfer function would be at omega naught and take the modulus of that times.

So, the magnitude would be or the amplitude if the amplitude of the input was a the amplitude of the output is a times the magnitude of the modulus of G j omega naught and what you would get is sin omega naught t plus some alpha. This alpha would in fact be that alpha would in fact be the angle of g j omega naught.

So, this is what happens in a linear plant, on the other hand in a non linear plant, I mean and in fact this property that when you give a pure sinusoid as an input to a linear plant you get a pure sinusoid of the same frequency as the output to the linear plant. It is in fact the defining property of linear plants, on the other hand in a nonlinearity when you give a sinusoidal input the output that you get may not.

I mean for example, if we give the input to be a sinusoid the output may not be sinusoid, but of course, depending upon the nonlinearity, when the input is a sinusoid, the output may still be a periodic signal. It will having a period which is the same as the period of the input signal, so what one does is one tries to use this idea.

(Refer Slide Time: 09:30)



So, suppose you have a non linearity, so what you do is as input to the nonlinearity you give sinusoidal input to the non linearity. So, sinusoidal input with a particular frequency omega and the magnitude a now as an output, you get something, but this something of course, need not have. It would be periodic, I mean depending upon the nonlinearity, it may be periodic, now if it is periodic what one could do is so if the output is periodic, what one could do is one could take a Fourier series of the output. Now, if you put certain kinds of constraints on the type of nonlinearity, then if we give a periodic input you can say more things about the output.

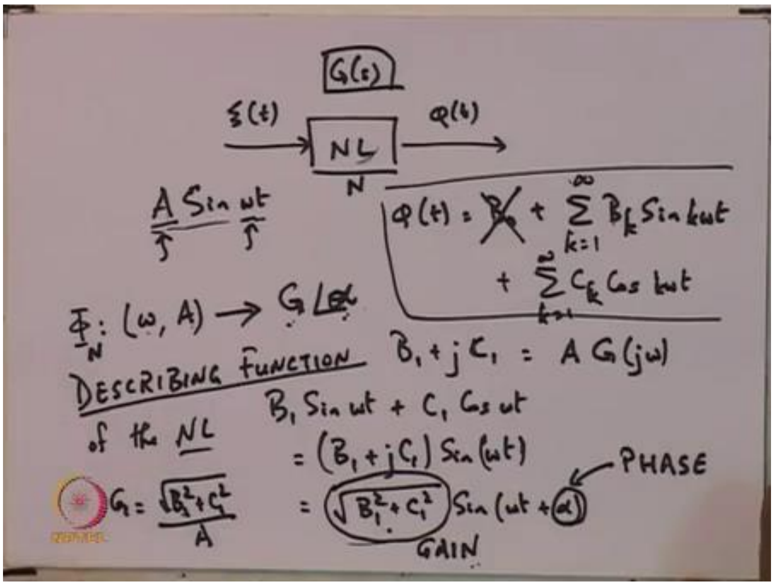
So, for example, if this nonlinearity is very less, non linearity, which is time invariant of course memory less is not that important, but suppose this nonlinearity is time invariant. Now, this nonlinearity being time invariant, essentially means that if you give a certain type of input, now there is some output you get, but if you give the same input after some time capital T, then the output that you will get will again be the same. Now, if the nonlinearity is time invariant, then when you give a sinusoid with you know the period being capital T.

So, suppose you give sinusoidal input and this period here is capital T, then the output of course, you do not expect it to be a sinusoid, but you expect it to be something which is also having period T. Now, if the output signal has period t, then what one can do is you find Fourier series of this output. Now, if you find the Fourier of the signal and so let me call this signal phi of t the input was let us say psi of t and the output of the nonlinearity was phi of t.

So, if you write down the Fourier series of phi of t, what you would get is some constant term. Let me call it B naught plus you will have b 1 sin omega, omega being the same frequency that you give omega t plus B 2 sin 2 omega t because omega t has periodicity t sin 2 omega t will have periodicity t by 2. Therefore, you know the time period t is a multiple of that and so on, so this will also be a be a signal which is having time period t plus B 3 sin 3 omega t and so on.

Then, you might also get c one plus c 1 cos omega t which is also some signal, which has periodicity or period time period equal to capital t plus c 2 cos 2 omega t plus 1. So, this periodic signal phi t in fact can be completely characterized by A, if we know B naught B 1, B 2, B 3 and so on and C 1, C 2 and so on.

(Refer Slide Time: 13:53)



So, now as far as nonlinearity is concerned given an input psi t which is a sinusoid sin, let us say omega t, let us say A sin omega t and as an output, you get this signal phi t and it has the same period capital T as the input. What you do is, you take the Fourier series, so when you take the Fourier series you get phi t to be let us say B naught plus s at i on b i sin i omega t i going from 1 to infinity a very large number plus s at i on i going from 1 to infinity c i cos i omega t.

Now, in this nonlinearity was not a nonlinearity, but was a linearity, so suppose this nonlinearity was I mean this was really not non linear, but it was linear. If it was linear, then this b naught would not be there and the B i's and the C i's for all i not equal to 1 would have been 0 and you would have had a, let me use not i, but k because i can have some other meanings. So, I will just substitute k here, so k equal to 1, B k equal to B 1 and C 1 would be non-zero and all the others would be 0 and then what you have is B 1plus j times C 1.

This in fact is going to be if it was a linear plant, so let us say if it was G of s, then this B 1 plus C 1 is really going to be a times G of j omega, what it does is one assumes that this nonlinearity is not, one assumes the closest linear system to this nonlinearity.

By saying that, if you give a sin omega t as the input, then the output is really B 1 sin omega t plus C 1 cos omega t, of course this can be put together, this can be put together as B 1 plus j C 1 sin omega t, maybe I should not call it. I could do this, I could just call it B 1 plus j c 1 sin omega t. By saying j c 1, what I am saying is this, there is a component with this magnitude, which is lagging the sin omega t by pi by 2. Of course, this could also be written, now as you take the magnitude B 1 squared plus C 1 squared and then you have sin omega t plus alpha, where alpha is this angle, which this complex number would have.

Now, this number that you have, you can think of this as the gain and this as the phase of the nonlinearity when the frequency is omega and the amplitude is A, so for a given nonlinearity what one could do is one could keep varying this various omega's. For each omega, you vary the amplitude a and so for each omega and each amplitude you find out what the gain is and what the phase is. So, what I am saying is you have a function phi, which has omega, the frequency of the input and a the magnitude of the input.

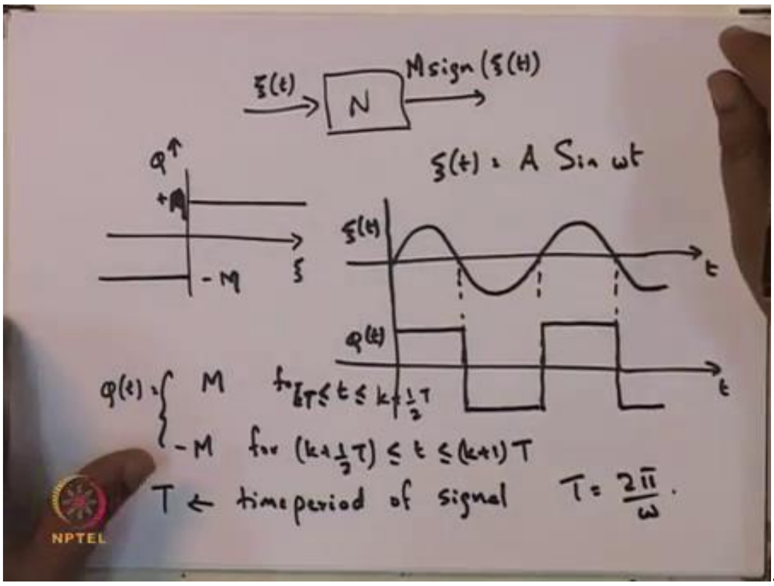
So, it has these two, this function has omega and a as the input and for this nonlinearity, so suppose I call the nonlinearity n. So, I call this phi of n, so phi of this nonlinearity is such that given a omega and given the amplitude, it gives the gain G and the phase phi or let me call alpha. Since, I call this thing alpha, so it gives the gain and the phase obtained in this way, so how do you obtain this well what you do is you take the input to be this sinusoid.

With this particular amplitude, we look at the output and the output, then you take the Fourier series of the output, you neglect all the higher harmonics and look at only at the primary harmonics. So, you look at only B 1 and C 1 and that B 1 and C 1, together you use to calculate what the gain is and what the phase is and so this gain and the phase you put down as g and alpha. Now, such a function, so this function of course, takes as in put omega and the amplitude and gives as an output the gain and the angle. Now, such a function is called a describing function of the nonlinearity, one thing that I would like to clarify here is that of course when you just take the B 1 and the C 1, you get this. So, this will not be the g the gain, I mean if you want to talk about the gain, this would have been the gain had the amplitude been one, but right now the amplitude that I am giving is A.

So, this gain G is really square root of B 1 squared plus C 1 squared divided by A, so you know depending upon the amplitude you see by how much. So, the gain is precisely the number by which you should multiply the amplitude, the input amplitude to get the output amplitude. So, the gain is really you look at the Fourier series, you look at n B 1 and C 1, so you look at the magnitude B 1 squared plus C 1 squared square root of that, but then you divide it, but the amplitude of the input signal.

So, this gain you purely mean whatever is the amplitude of the gain, you multiply to the amplitude of the input signal to get of the amplitude of the output signal. So, this particular function is called the describing function, now the best way to understand the describing function is to probably look at an example of a describing function. So, what we will do now is we will take a particular nonlinearity and we will look at how one calculates this describing function, so the nonlinearity we look at, so let us look at the sin function.

(Refer Slide Time: 22:21)



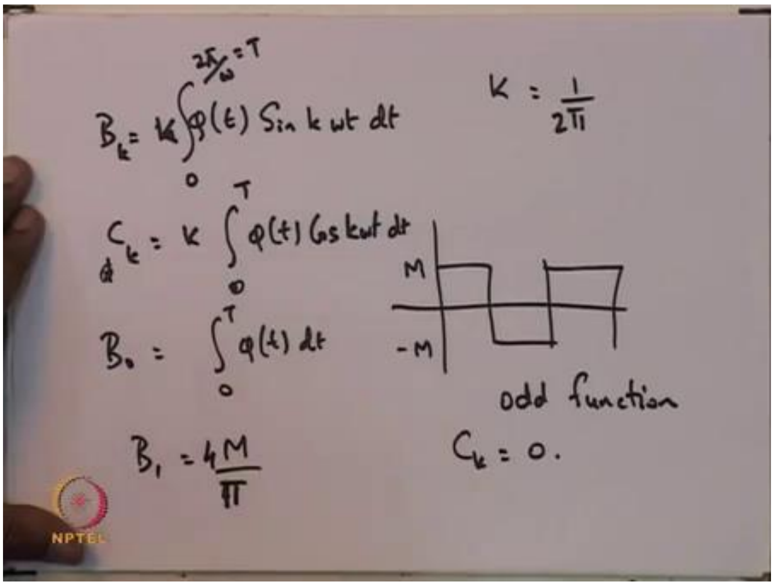
So, we are looking at the Nonlinearity, one linearity and the nonlinearity does the following if the input is psi t the output is the sin of psi t, the best way to draw this nonlinearity, the characteristics of this nonlinearity the input is psi. The output when psi is positive, it gives plus 1 and when psi is negative it gives minus 1.

Of course, instead of just taking just sin, one could also one could also just take, let us say capital M times the sign in which case here you will have m here, you will have minus m. Now, for this given nonlinearity, let us find the describing function, now suppose psi t you give psi t to be a sinusoid. So, assume that psi t is a sinusoid a sin omega t, what do you expect the output to be, so this is phi the output, so in this case the output. So, maybe I will draw it like this, so here is the input, so it is a sinusoid, so this is the input psi t.

So, this axis t and let us see what the output looks like well the output will be plus m till this point. Then, it will be minus m till this point, so the output that you get is the square wave, now if the if the output is a square wave, of course this is also periodic with the same time period as the original input. Now, one can find out the Fourier series which is how does one find the Fourier series. so this particular sequence.

I mean the square wave is essentially if I am taking that this phi of t could be written as plus m for t k t and k plus half t and minus m for k plus half t is an equal to t this and equal to k plus 1 t, where t is the time period of the signal. So, it is the same as t is t is the same as 2 pi by omega, now if you find the Fourier series in the case if how does one find the Fourier series for this square wave well.

(Refer Slide Time: 26:24)

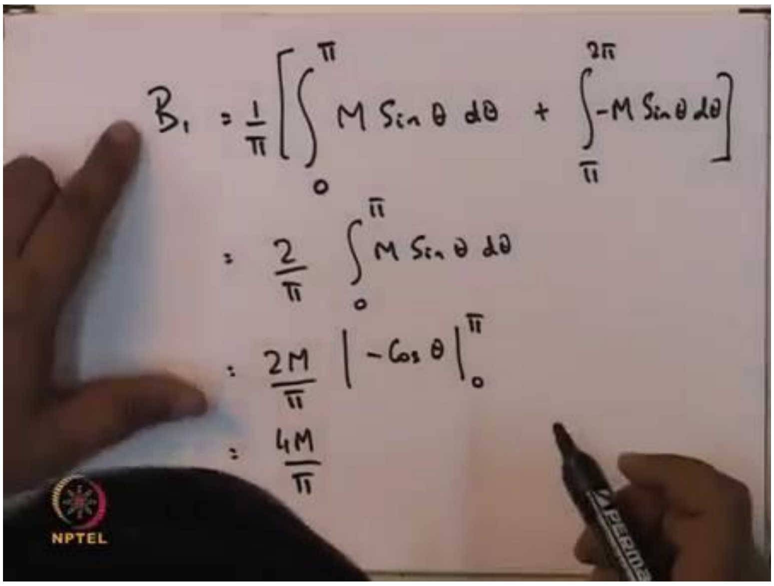


So, the Fourier series for the square wave if found by for example, b K, the way you would find b k is multiply phi t to sin k omega t. Then, take the integral over a period, so here of course omega t i sin case of t if you are taking the period in case of t you could take this integral from 0 to pi by omega that is capital t, which is the same as capital t. Then, this there would also be a multiplication factor in the multiplication factor, let me let me just call it k capital K.

So, this multiplication factor will depend upon how you take this integral, but if you are taking this integral over the whole period, then in that case the k is going to be 1 by pi 1 by 2 pi 1 by 2 pi. So, for this particular for this particular nonlinearity that we were looking at, so we have the square wave and this is m plus and this is minus m. So, suppose we have to, so this is how you find C, B k and how we find C k is again, you have the proportionality constant 0 to capital T of phi t cos k omega t.

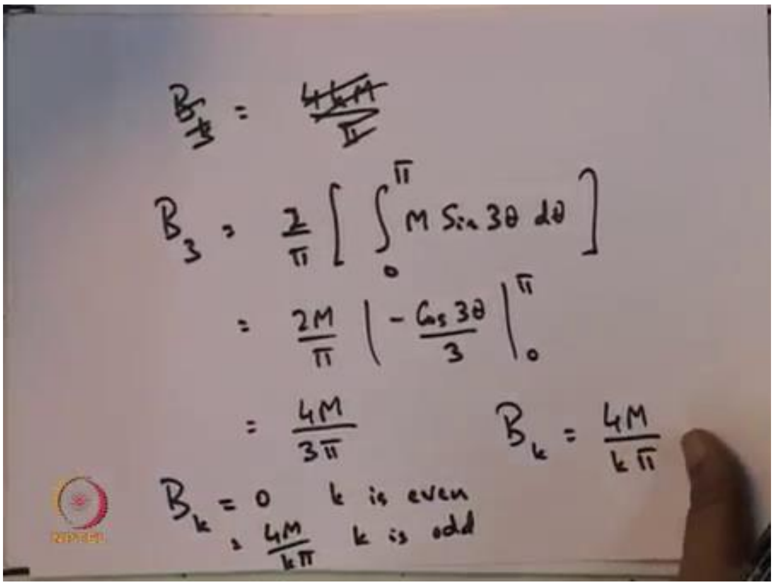
So, this phi t cos omega t multiplied by cos and taking the integral will give us this C k’s and of course, the B naught you get by just integrating 0 to t of phi t. Now, in this case when you calculate, it should be clear that if you take any, if you evaluate any of the c k’s in this particular case. You see what you have G here is an odd function. Any odd function if you have, a periodic odd function and you are going to take the Fourier series all the C k’s will turn out to be 0. On the other hand, if you calculate the B k’s in this particular case you will get B 1 to be m by you get 4 m by pi, if I am not mistaken, may be perhaps I should just calculate it out.

(Refer Slide Time: 29:36)



So, let me calculate B 1, so this B 1 is like integrating from 0 to pi of m times sin theta d theta plus integral from pi to 2 pi of minus m sin theta d theta, but this integral is really the same as this integral. Of course, there is some constant, so I will fix the constant later, so this is the same as two times integral from 0 to pi m sin theta d theta. Now, I can pull out the answer, I have two m and I have integral of sin theta is minus cos theta evaluated at pi and 0 and so this should give me cos theta evaluated at pi is 1 because theta at 0 is also 1. So, I get 4 m, so as it turns out that this whole thing is 1 by pi of this whole thing, so it is 2 by pi here, 4 m by pi 4 m by pi. So, I was right here in the first place, this constant k should have been 1 by pi, so this B 1 turns out to be 4 m by pi.

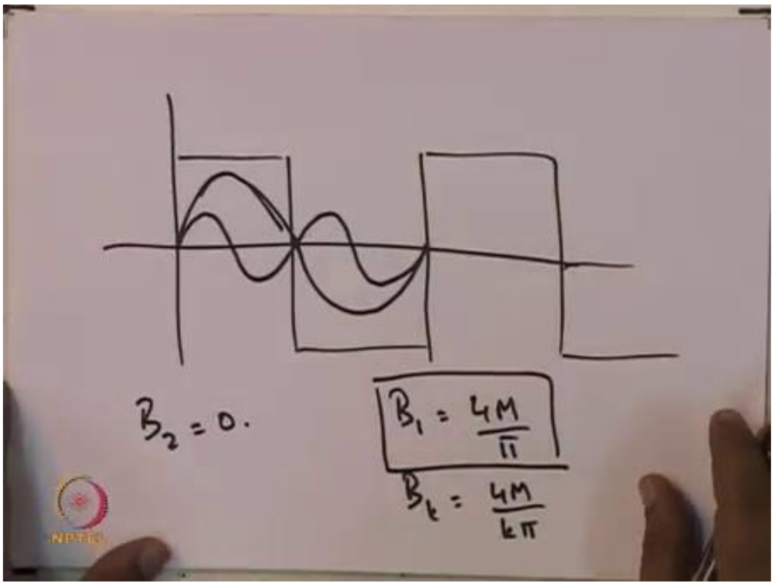
(Refer Slide Time: 31:29)



In general, we can see that B k will turn out to be this, if you have to calculate B k, this is like if you evaluate all the B k's, these B k's will turn out to be 4 times k m by pi, no sorry may be it may be we just calculate B 3. So, B 3 turns out to be 1 upon pi and now you have to take the integral from 0 to pi of m, 2 times of m sin 3 theta d theta, which is 2 m by pi and this integral is minus cos 3 theta by 3 evaluated at 0 and pi.

So, this turns out to be a k, so that is one-third plus one-third that is two third's, so you get 4 m by 3 pi. So, in general when you calculate b k what you are going to get is 4 m by k pi, I have to be a bit careful here, because I am going to get b k equal to 0 when k is even and equal to 4 m by k pi when k is odd.

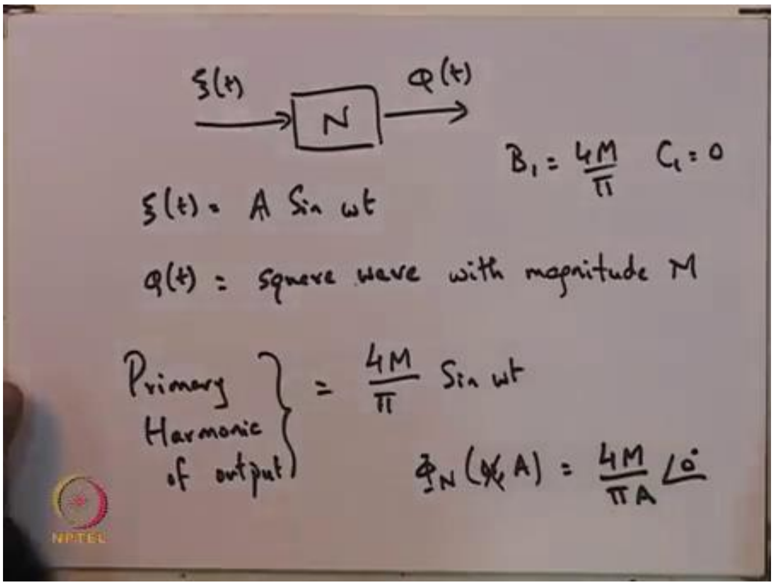
(Refer Slide Time: 33:36)



So, why I am saying this is because we are looking at a square signal, I mean a square wave like this and so suppose I take a sinusoid which is an even multiple. So, suppose I am taking an even harmonic, then if you look at the first half, the positive thing is multiplying sin and in the second half, it is negative thing which multiplies the same signal. As a result, when you do the integral, which is what you are doing to calculate, so in order to calculate B 2, then B 2 will turn out to be 0's because there is as much positive as there is negative.

On the other hand if i am looking at the primary harmonic, then that is the positive half and that is the negative half. So, when it was positive, this was also positive when that is negative, this is also negative, so the B 1 turns out to be as we saw 4 m by pi. So, whenever you take a odd harmonics, you will so odd harmonics B k will turn out to be 4 m by k pi, whereas B for the even harmonics, it will turn out to be 0. Anyway, we are not concerned with this entire all the full Fourier series, what we are only concerned with is this particular thing B 1.

(Refer Slide Time: 35:14)



Now, coming back to the nonlinearity, so you have this nonlinearity and when the input is psi t the output is phi t and what we have just seen is when the input psi t is a sin omega t. Then, the output phi t is square the square wave with magnitude capital m and so if you want to talk about the gain the gain is going to be 4 m by pi 4 m. As we look at the Fourier series of the output of the square wave, we saw that B 1 was 4 m by pi and c 1 was 0. So, the output, the primary harmonic of the output, the primary harmonic of the output is going to be 4 m by pi sin omega t.

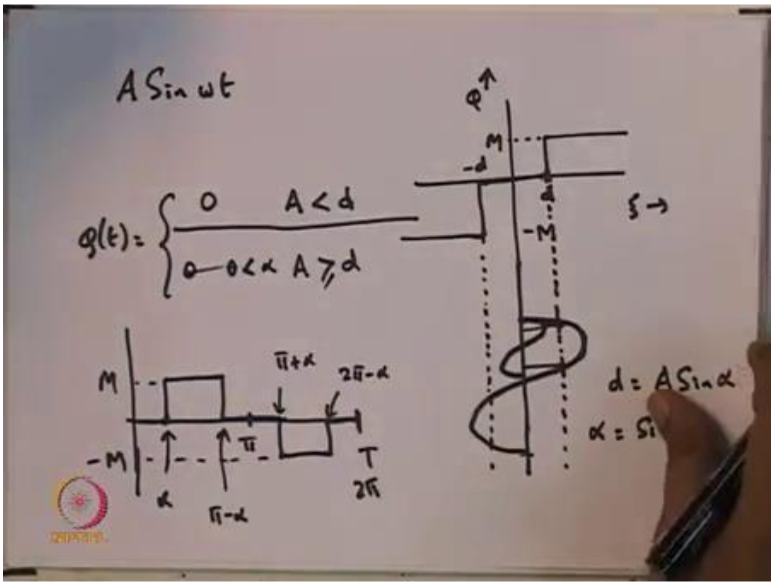
Therefore, if you want to now talk about the describing function of this small linearity phi of n, then phi of n of omega a is going to be 4 m by pi a with angle 0, why is the angle 0. The angle is 0, because when you give a sinusoid, the output of the primary is in phase with the input. So, there is no angle alpha here that is why the angle is 0 and when the primary and the input is A, we saw that the output magnitude is 4 m by pi.

So, the gain is the output magnitude divided by the input magnitude, which is why it is 4 m by pi A. So, the describing function for this particular nonlinearity is four m by pi a note that this nonlinearity the describing function is independent of omega.

So, it really does not matter what this omega is as you change the omega A, you get periodic signal of a different time period, but then the analysis of B one and C one is exactly the same as what we have done before. So, again you will get the magnitude to be 4 m by pi, it does not matter what this omega A is, you will get 4 m by sin omega t as the output if a sin omega t is the input for whichever omega you think of. That is why the describing function, though I am writing here that it is function of both omega and A, it really is only a function of A.

This omega does not play a role, so in this particular nonlinearity, we could suppress omega, if I like it and we could just think of the describing function as only depending on the input magnitude A. Now, one can also look at describing functions of more I mean that that was probably a very simple nonlinearity. One could look at other nonlinearities like for example; one can look at a nonlinearity, which is let us say really with a relay with dead time.

(Refer Slide Time: 39:00)



So, if you look at the input output characteristics of such a relay with delta in, so this is the input, this is the output and it could be that for some time for some input, there is no output. Then, there is an output and here similarly, for some negative inputs there is no output. When there is an output, let us say this magnitude is again m and this minus m, so up to some value, let us say d up to some value d, there is no output b on d, you get plus m. Similarly, up to minus d, there is no output and B on that you get b on that you get minus m. So, you have a non linearity with these characteristics; now suppose to this nonlinearity you give an input which is a sin omega t. Now, if we give an input a sin omega t, what would be the output, so let me write down what the output, so phi t would be equal.

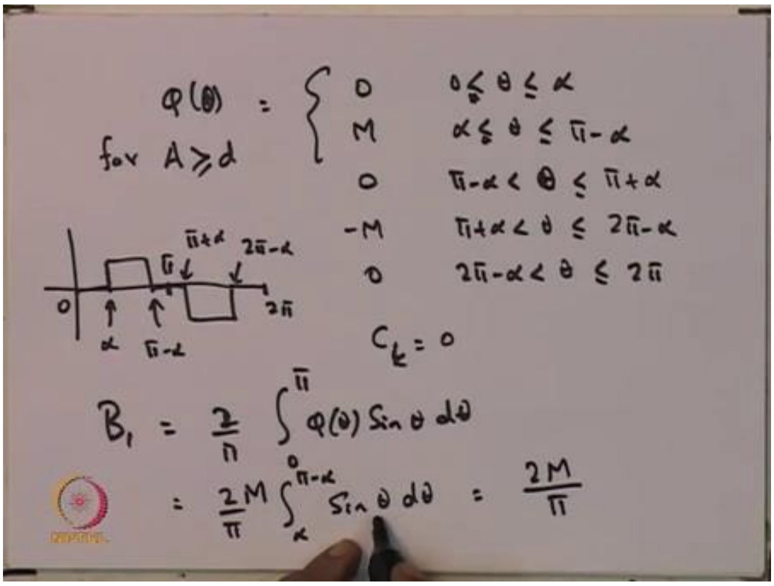
So, let me mark what you what one would get as an output by doing something here, so let me draw the sinusoid here and then I can just directly check what the output would be, because on this axis if I draw the input like this, then I can check what the output is. So, suppose the input is like this, so some sinusoid like that where the magnitude A is less than this small d. Now, you see if the magnitude A is less than small d, then when this is the input the output as you can read out from this map is 0.

So, the phi t will be 0 if the magnitude a is less than d, on the other hand if you consider input, where a is greater than d, then what is going to happen is so that is d and that is minus d for you. So, what will the output look like, so let me draw output here, so initially this is how the sin wave took off. So, initially for this much time, so up to this point for this much time the input is 0, then for this portion of the input it is plus m. After that, it becomes 0 and it is 0 for all this time, so it is 0 for all this time and then it goes negative and then it again becomes 0 and this could be the time period t.

I could also write this, let us say 2 pi, it is simpler just use the angles, so you have this signal with this being plus m and this being minus m, you have this signal. Let us think of this angle as pi and this is 2 pi, you will get this signal and periodic of this signal, so the point b for a less than d is going to be 0 and for a greater than d greater than equal d. Well, it is going to be 0 up to this point and what is this angle this angle is precisely when do you get this angle, when the input has precisely become d, which means d is equal to A sin alpha.

In other words, alpha is sin inverse of d by A, so this angle here is alpha, then of course you can calculate what this angle is this and it should be clear that this angle is pi minus alpha. This angle here is going to be pi plus alpha and this angle here is going to be 2 pi minus alpha. So, the output signal, if you want to describe the output signal for magnitude A, I mean the input magnitude greater than d, then this signal is going to be 0 for theta less than alpha, perhaps I will write in the next page.

(Refer Slide Time: 44:18)

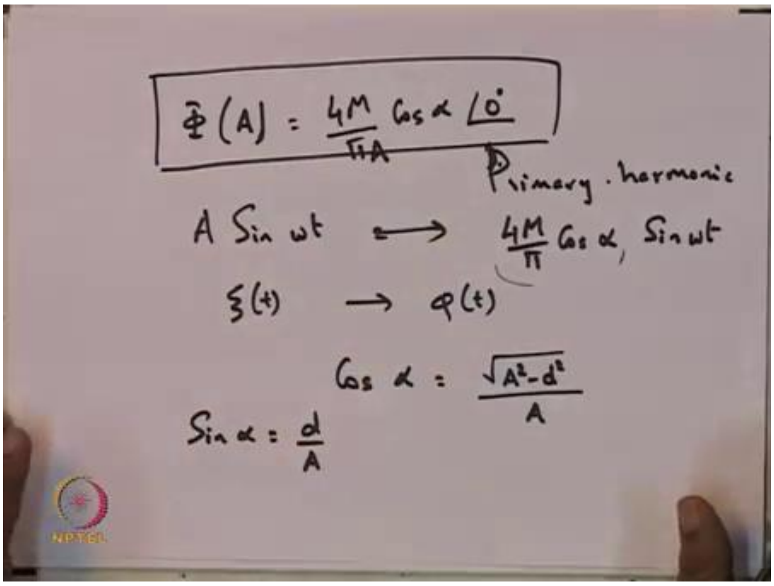


So, the output phi of t for a greater than equal to d is going to be 0 for angle 0 is less than theta is less than alpha and then it is going to be plus m for alpha less than equal to theta less than equal to 0. So, I should not be using less than equal to, but I guess you understand what I mean by that pi minus alpha, it is again 0 pi minus alpha less than theta. May be I will just consistently not use less than only for the lower limit and for the upper limit less than equal to less than equal to pi plus alpha. It is going to be minus m for pi plus alpha less than theta less than equal to less than equal to phi of theta less than equal to 2 pi minus alpha.

Then, it is again going to be 0 to pi, so the full cycle is given in that way as we had drawn a near, this being alpha this being pi minus alpha pi plus alpha 2 pi minus alpha this is 2 pi. This here is pi 0, now if we have to now calculate the primary harmonic, again this again a odd function. So, we can straight away say that all the C i's are going to be 0, all the C k's are going to be 0, but we are of course only interested in finding B 1.

So, if you have to calculate B 1, B 1 turns out to be 1 by pi integral 0 or rather 2 by pi integral 0 to pi of phi of theta sin theta d theta. So, how much is this well this integral from 0 to pi can further be broken down to into 0 to alpha, which will be 0 and pi minus alpha to pi which is also 0. So, what is just left is alpha to pi minus alpha of i theta, there is m, so I pulled out the m and I just have sin theta d theta and this turns out to be 2 m by pi and I will have minus cos theta, so, I have two times cos of alpha.

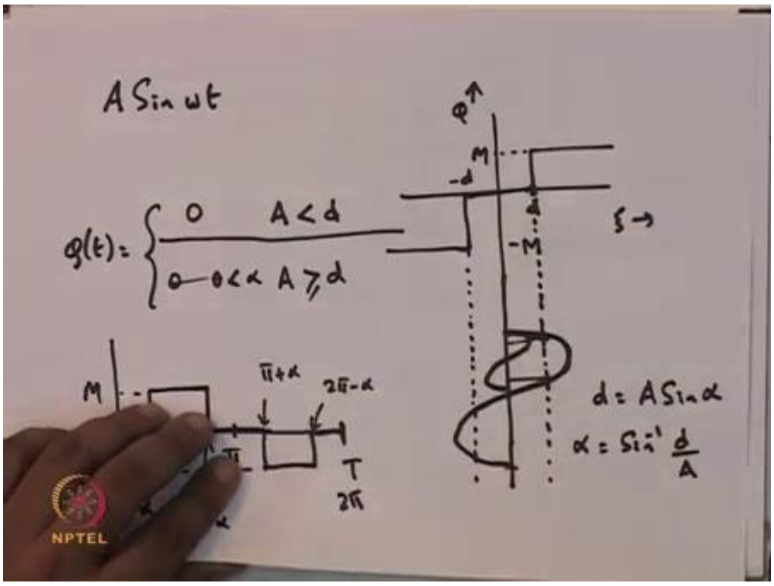
(Refer Slide Time: 47:33)



So, what this means is that for this particular nonlinearity that we are looking at phi of a, I mean I am suppressing the omega because when you give the input to be sin omega t. Then, for the output, so the input is psi t and the output is phi t and then the primary harmonic of the output is going to be 4 m by pi cos alpha times sin omega t. Of course, this cos alpha you can write this down, because we already know that alpha or sin alpha we know is d by a and therefore, the cos alpha is going to be square root of this. So, we can write this cos alpha in terms of a and d which are parameters that came from the nonlinearity.

Therefore, in this particular case the describing function and again here the describing function is independent of the omega, because on every time we give a sin omega t, you will get a square wave of that kind. When you calculate the B 1 for the square wave, it would be like this and so phi of A is going to be 4 m by pi cos alpha, but this is not the gain the gain is pi a. So, this is the gain and the angle is going to be 0, so this again is the describing function for the nonlinearity which we showed.

(Refer Slide Time: 49:33)



In this way, so we have we have sort of calculated two different describing functions for two different nonlinearities. So, what we would do in the in the next lecture is I will talk about how we are going to use these describing functions, in order to detect whether a given system has a limit cycle or not.