Nonlinear Dynamical Systems
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Lecture - 21
Loop Transformations and Circle Criterion

So, in the last lecture what we have started doing was what are called loop transformations. So, I have started initially with on non linearity in the k 1 infinity sector and I showed how we can convert that non linearity or we can think of that non linearity as non linearity in the 0 infinity sector. So, in some way the k 1 infinity sector you expand it out, so that it becomes the 0 infinity sector. So today I will continue with that, but probably I will also revisit the k 1 infinity and sort of a try and wrap it round nicely. So, may be what we will do is we will continue with what we are doing in the last lecture that is after we had finished the k 1 infinity sector, we had started looking at the 0 k 2 sector.

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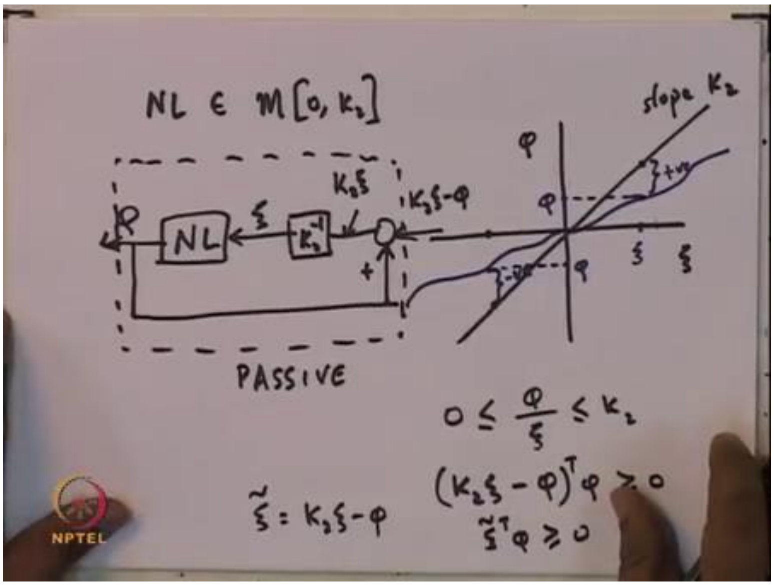

So, let us think of a non linearity in the 0 k 2 sector. So, we are thinking of a of a non linearity which belongs to the 0 k 2 sector, so what I mean by that is so here psi is the input to the non linearity phi is the output to the non linearity. We have this line, which has slope equal to k 2 and what we assign is that we have a non linearity which lies in the 0 k 2 sector. So, it lies something like that that is a non linearity, now that non linearity of course one could think of the non linearity has phi by psi that means at any point of you take the phi and divide by psi, you get this this particular sort of triangle. Of course, the slope, the tangent of the angle that it obtains is smaller than the angle by the slope k.

So, this is less than equal to k 2, but it's greater than equal to 0 and then we could rewrite any non linearity in the sector by means of a quadratic form. So, the quadratic form that we could write this as is the following, so you take k 2 psi k 2 psi minus phi. So, when you do k 2 psi minus phi, so if I am thinking about this particular psi here and k 2 psi is here and phi is here. So, this quantity here is positive if I on the other hand take a psi which is negative k 2 psi is here phi is here and so this quantity is here is negative. So, if I multiply k 2 psi minus phi to psi I would get something positive, but that is not the quadratic form that I am going to take.

What is the quadratic form I am going to take is you see phi here corresponding to negative quantity the corresponding phi is also negative and here the phi is positive. So, what I am going to do is I am going to take the quadratic from given by k 2 psi minus phi transpose phi and this is going to be greater than equal to 0 for any non linearity lies in this this particular sector 0 k 2. Now, suppose now you have this original non linearity and you have the input psi and output phi. We want to convert it into a non linearity which has its output it has exactly the same output as the original non linearity, but its input is modified and this new input psi tilde transpose phi greater than equal to 0 is psi tilde is given by k 2 psi minus phi.

So, how to modify this so that the psi tilde becomes k 2 psi minus phi, so the way we are going to do it is the following, so let me put a gain here which is k 2 inverse. So, if I put a gain here k 2 inverse then what should have been here is k 2 times psi, so k 2 psi multiplying k 2 inverse will give me psi. Now, I have to get this k 2 psi and so what I do is I take this phi and I feed it back here with the positive sign, so I am adding phi to something so that I get k 2 psi.

So, the something to which i have to add phi to get k 2 psi is k 2 psi minus phi and now if I think of this non linearity in the box which is this non linearity with this gain and this feedback. Then this non linearity has its input k 2 psi minus phi and it has its output phi and therefore this non linearity by this equation or this equation is therefore, a passive non linearity. So, the transformation that I do for something in the 0 k 2 sector is in this way and when I do the transformation in this way I get this new non linearity which has the original non linearity psi has input in phi has the output. The new non linearity continues to have this phi as the output, but it has k 2 psi minus phi has the input.

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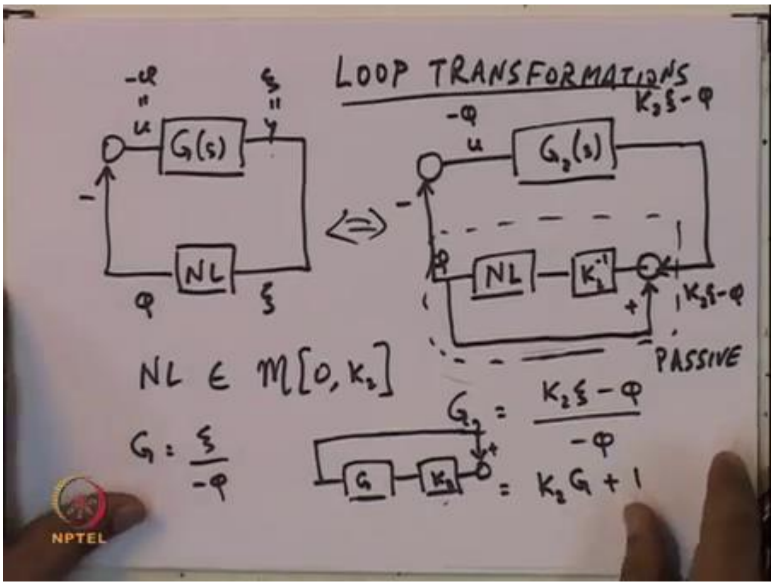

Now, if we are one looking at Feedback structure where you have a plant G s and you have this non linearity and this non linearity is in the 0 k 2 sector. We are looking at this feedback connection between the linear plant and the non linearity, we modify this non linearity to that the new thing. So, the new the new non linearity is obtained in this way you have gain here k 2 inverse and whatever is the output you feedback with a unity feedback with a positive sign. So, that is what you have here is really k 2 psi minus phi and the output remains same phi, so here you have psi phi and this is u input of the linear plant and the output y.

So, let me call this new plant that I have here which would be a modified version of this plant let me call that G 2 s so that G 2 s has its input the same as the input of G s, but it has its output this k 2 psi minus phi. So, let me instead calling this u let me call this minus phi because this u is equal to minus phi and here y is equal to psi and here what I should have is k 2 psi minus phi. Therefore, the new plant that you get there G 2 is output by input, so it is k 2 psi minus phi divided by minus phi. From here, you know that G is psi divided by minus phi, so if you now evaluate this this turns out to be k 2 times G this transfer function plus 1.

Now, if you had a linear system with this non linearity in the 0 k 2 sector and you look at the closed loop system this is the exactly the same as looking at this modified non linearity along with this plant G 2 s. This G 2 s is really k 2 times G plus one of course, how do we realize k 2 times G plus one if you have the original G you can put a gain of k 2 in serious with it and the plus one you can get by having a unity feed forward. So, G 2 s is really this net transfer function, so if you see just like in the last case you have some sort of a symmetry because in the non linearity, you are putting this k 2 inverse.

Then, you are having this feedback which is positive feedback and therefore G s will also get modified, but this time gain that you have in series with G s is going to be positive k 2 I mean it is going to be k 2, whereas with the non linearity with k 2 inverse. Here, in this loop in this portion in the non linear portion this was the feedback, so out here it is the feed forward if you recall in the k 1 infinity sector, we had used a feed forward in the non linearity. As a result of the which you had a feed back in the in the linear plan here in the non linearity using a feedback. Therefore, in the linear part you will have a feed forward, now this kind of transformations go under the name of loop transformations.

Now, by doing this loop transformation, you have got a new non linearity here and this non linearity is passive this whole thing that am marking this whole non linearity is passive therefore, this G 2 s. This modified linear plant if this modified linear plant is strictly positive real and it is stable, then the original plant along with this non linearity of course if this is strictly positive real and is stable. Then, this resulting feedback system is going to be asymptotically stable and that is the same as saying that this particular system is asymptotically stable.

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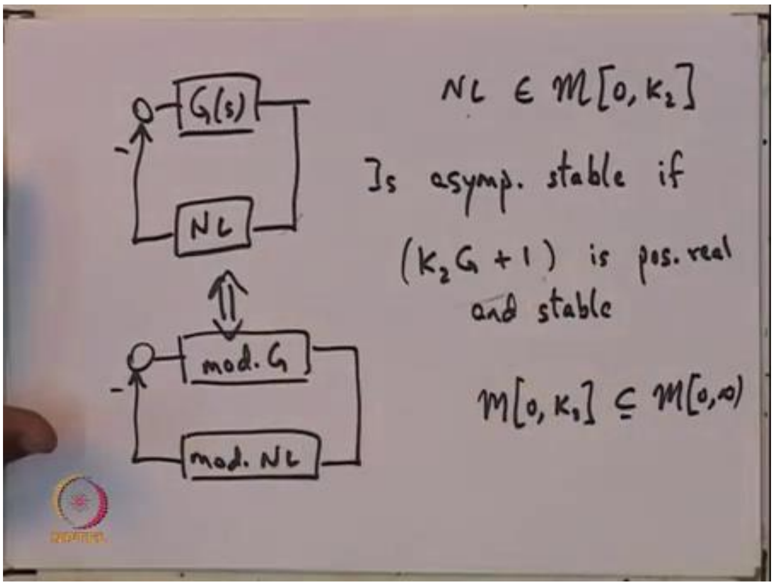

Therefore, from this what we can conclude is if you have a non linearity and you have a linear plant G sand you are looking at this feedback structure and if the non linearity lies in the 0 k 2 sector then this resulting system is asymptotically stable. If k 2 times G plus one this transfer function which is a modified transfer function that G s becomes because you have modified non linearity, these two are equivalent and the modify G is k 2 G plus one this is positive real and stable. So, what we are really doing is when we are looking at non linearities which are in sectors which are really in some sense subsets of the earlier situation means initially the passive theorem passive lemma and so on.

We have proved for non linearities in the 0 infinity sector, now we are looking at something in the 0 k 2 sector where k 2 is strictly smaller, then infinity. Then, one would expect more transfer functions to be inter connected to non linearity resulting in something which is asymptotically stable. Of course, if you have nonlinearity in the 0 k 2 sector you see this this is true that all the non linearities is 0 k 2 sector. This is subset of the nonlinearities in the 0 infinity sector, so of course if you take a plant here which is positive real and stable, then the resulting system is anyway going to be asymptotically stable, but what this result tells you is that you need not necessarily take G s which is strictly positive real and stable.

You could take a G s such that k 2 times G s plus one this resulting transfer function is positive real and stable. So, we could have a G which is not positive real or stable and you could have k 2 G plus one resulting in something which is positive real and stable. If that is true, then that G along with the original non linearity that will again result in a system which is asymptotically stable. Now, I had used some sort of quadratic form, so let me just revisit this quadratic forms and there, I mean depending on taste of people, there are new additional definitions given to many of these systems.

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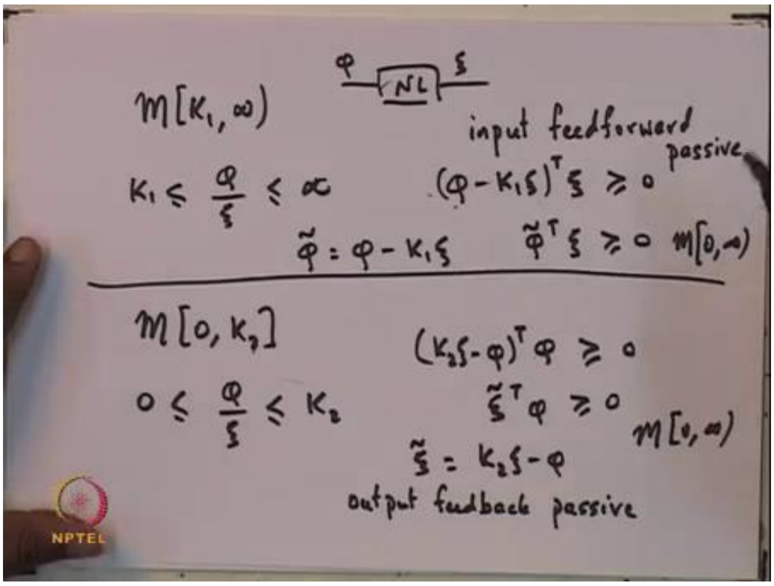

So, when we consider the non linearity in the k 1 infinity sector, when we had looked at the non linearity in the k 1 infinity sector what this means is the output of the non linearity divided by the input lies between infinity and k 1. Then, this I could rewrite as phi minus k 1 psi transpose psi being greater than equal to 0, so this this inequality that I have here of the of a non linearity in this class I could write in this way and this then I could rewrite in this particular way. Then, when I could rewrite in this way I could think, so the original the original non linearity was there with input psi and output phi. So, I retain the input as it is, but the output and modify to phi tilde and so I look at it this way where phi tilde is given as the original phi minus k 1 psi.

Then, this new modified non linearity is passive, similarly when we take a non linearity in the 0 k 2 sector then the inequalities similar to this inequalities that you would you kept output by the input is less than k 2 and is greater than 0. Then, this particular case what I did was I kept the output the same, so I wrote this inequality in quadratic form and so the quadratic form that I wrote was phi and multiplying phi multiplied k 2 psi minus phi. This is greater than equal to 0 and this k 2 psi minus phi I define that as the new input, so I kept the output the same, but I modified the input and so psi transpose phi greater than equal to 0, where psi tilde was given by k 2 psi minus phi.

So, what I am really doing is I take a non linearity in this sector there are this inequalities which are satisfied, but that is equivalent to saying that it is this particular quadratic form that is satisfied. If this quadratic form is satisfied and I think of a new non linearity which has phi minus k 1 psi as the as the new output I keep the input same I change the output. Then, this new non linearity will actually be the is new non linearity actually the in the 0 infinity sector, similarly, if I take something 0 k 2 sector, this is one way define it, but I am re defining this inform in the quadratic form. If I redefine this in quadratic form, I think of this here as the new input, so I keep the output the same I change output into this new input and resulting system is again in the 0 infinity sector.

Now, if you look at this case non linearity this case, then what we doing is we kept the input the same, but we have modified the output. On the other hand, if you look at the non linearity in this sector, we have kept the output same and we modified the input. Now, the way we modify the output in this particular case is we give a feed forward with the original plant, we give a feed forward. So, such systems are also sometimes called an input feed forward passive and similarly these systems what we did was we kept the output was same, but we gave a feedback from the output.

Therefore, these are called output feedback passive, now these are all definitions, but the important thing to realize is that whenever you have the some nonlinearity in some sector like k 1 infinity. You can convert it using this this quadratic form into something which is passive and similarly, if you have some nonlinearity in 0 k 2 sector, then by modifying the input. I mean this is the quadratic form that satisfy and so if we modify the input then the new non linearity that you create is in the 0 infinity sector and once things on the 0 infinity sector. Then, you know that if you if you put a positive real stable plant in a feedback loop with this such a non linearity, you get asymptotic stability.

So, whatever was a linear plant that you connected that will undergo transformation to be a new linear plant, which you associate with this particular non linearity and that new linear plant should now be the positive real and should be positive real and stable.

Similarly, in this case now of course we could also we could also look at the nonlinearity which is in the about you know k 1 k 2 sector.

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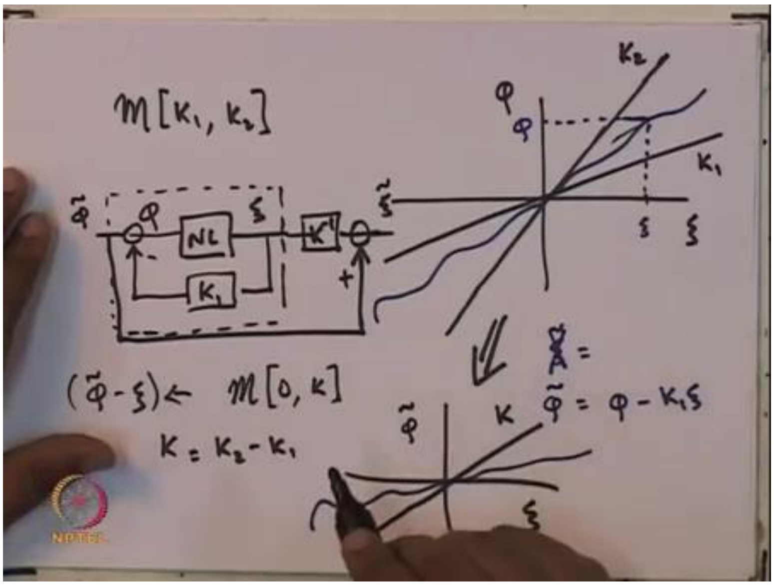

So, one could look at m some nonlinearity in the k 1 k 2 sector, so what do we mean by that this is the input this is the output let us assume this as slope k 1 and you have another line which as slope k 2. What you saying is that we have a non linearity, which lies in the k 1 k 2 sector, now for nonlinearity that lies in the k 1 and k 2 sector. We can again write some sort of quadratic form, but and then use that quadratic form that convert this k 1 k 2 sector into 0 infinity sector that means passive. Now, one way go about in this is what one could do is we could convert this k 1 k 2 into a 0 k sector and how does it one convert to 0 k sector.

Well, you have this non linearity with input psi and output phi and now if you think of input as this psi the output of this particular phi, what one does if we modify the input to the new input psi tilde which is given by. What we do is for the same for the same psi you have a new output phi tilde which is given by whole psi minus k 1 by psi.

How to do it here, but what we doing is the new, so this is the phi tilde now what can we say about this new what can we say about this new non linearity. That means the nonlinearity, which is there in the box, well what we can say that this new non linearity whose output is phi tilde and input is psi this new non linearity belongs to the 0 k sector where k is k 2 minus k 1. Now, we should be clear why k 2 minus k 1 because you see we are talking about non linearity in the sector. So, one of the worst cases would be I mean we could really think of the nonlinearity as a linearity which is output is k 2 times the input. Now, when we put this transformation then the output becomes k 2 times psi minus k 1 times psi which is k times psi.

So, what we have effectively done is we have rotated this round and so the new nonlinearity that you get psi is input phi tilde is the output and you will have slope k which is k 2 minus k 1 and the non linearity. Now, will lie like that, now once you something in this 0 k sector we can use this particular things to convert it now into something into the 0 infinity sector. So, what do we do well to the nonlinearity you put k inverse here and then what ever is the output you feed it back and you feed it back with a with a positive sign. We show here now the phi tilde, but now we have modified with input now, it becomes psi tilde, now this new non linearity with psi tilde and phi tilde is in fact in the 0 infinity sector.

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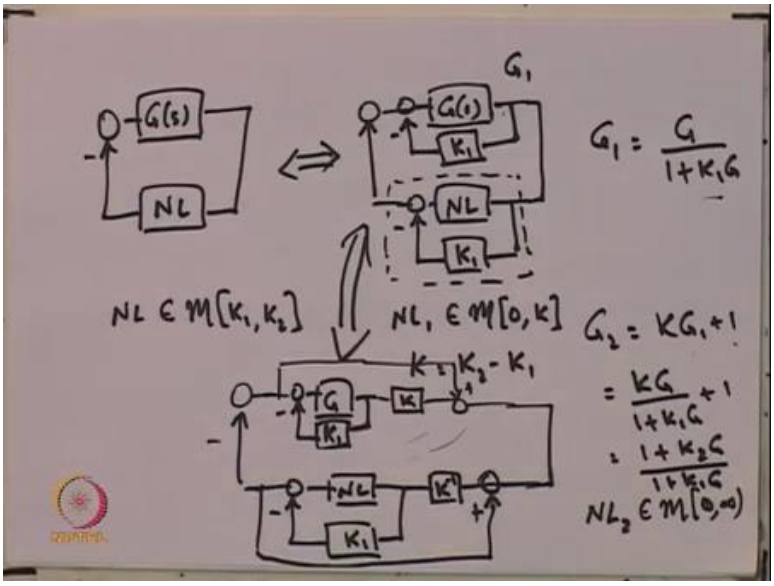

So, here have some transformations that are going wrong you have G s and you have this non linearity and this non linearity is in the k 1 k 2 sector. So, you do the first round of transformation and so you get this new non linearity, so the new non linearity you get is the old non linearity with a gain of k 1 put here. So, this is the new non linearity instead of this we put additional thing and as a result the G s gets modified and how does G s gets modified and G s gets modified by a feedback. Now, this is equivalent to this where you have modified the non linearity and you have modified the plant, this non linearity which is there in this dotted box that I am talking about if I call that non linearity n l one this n l one belongs to the 0 k sector.

The k is equal to k 2 minus k 1, now this is further modified now in the following way so you have non linearity you have this k 1. So, this is a non linearity and now you put in k inverse here and you take a positive feedback and this new non linearity that one is having, let me call this a new non linearity n l; two this belongs to the 0 infinity sector. Of course, when you do this you have to do the corresponding change in the transfer function and the change would be. So, you have the linear plant and you had the feedback and then you do exactly this, but here you multiply by k and then you feed forward you feed forward.

This is the resulting linear plant that you have, so this is equivalent to this and this is equivalent to into this. So, if G s with this non linearity in the k 1 k 2 sector is to be stable and this linear plant with this non linearity this must be asymptotically stable. Now, what is this this one could get from this by the transformations that it has to go through, so if I call this linear plant G 1 and then I know G 1 is equal to G upon 1 plus k 1 G and then the G 1 is converted to this one. So, if I call this linear plant G 2, so G 2 I know is k times G 1 plus 1, so substituting G 1 this is the same as k time G upon one plus k 1 G plus 1.

This is same as one plus k 2 G upon 1 plus k 1 G, so if one is given non linearity in the k 1 k 2 sector and we are asked to find out all the linear plant which is an inter connecting with this non linearity gives rise to a system which is asymptotically stable. Then, we could do a loop transformation on this k 1 k 2 convert into this non linearity in the 0 k sector and that is this thing, but what would mean is that this G you will get modified to this G 1 G upon 1 plus k 1 G. Then, something in this 0 k sector we can convert into something in 0 infinity sector that means you can make this non linearity covert this non linearity into passive non linearity by this additional thing that you do here some output feedback.

Now, once you do this the linear plant will also will have will also have to be modified accordingly and then linear plant is also modified accordingly. Then, this linear plant is modify to this linear plant, but that is saying that this linear plant G 2 k times G 1 plus 1, but then G 1 itself was dependent on G when you put all of them together you get one plus k 2 G upon 1 plus k 1 G.

Now, theoretically if you given this non linearity in k 1 k 2 sector for any given plant G you could calculate 1 plus k 2 G upon on plus k 1 G and check whether this plant is positive real, if it is positive real. Then, the original system is asymptotically stable, so this transformation that we have this goes under I mean this is a theorem on its own and it goes under the name of the circle criterion, so let us find out what is circle criterion is all about.

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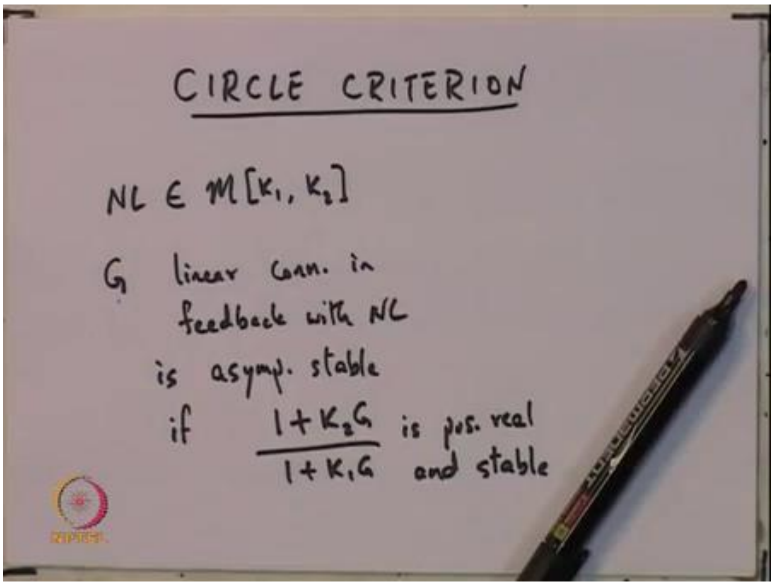

So, what we have said is we have a non linearity in the k 1 k 2 sector and you have a linear plant connected to a non linearity, so linear plant G linear connected in feedback with non linearity n l is asymptotically stable. So, this is what we have shown the last time one plus k 2 G upon 1 plus k 1 G is positive real and stable given G what one could do is one could calculate this one plus k to G upon 1 plus k g. Then, check whether this transfer function is positive real, but you know checking for positive realness one way to check for positive realness is by using the Nyquist plot.

Now, is there a way to check whether 1 plus k 2 G upon one plus k 1 G is positive real, but they still want to use the Nyquist plot of the original G the terms of that is possible. This way of predicting whether 1 plus k 2 G upon one plus k 1 G is positive real and stable by using the Nyquist plot of G this is what circle criterion is. So, we will look at this transfer function and we will use the Nyquist plot of G and try to say whether this transfer function is positive real and stable, so this is what we do?

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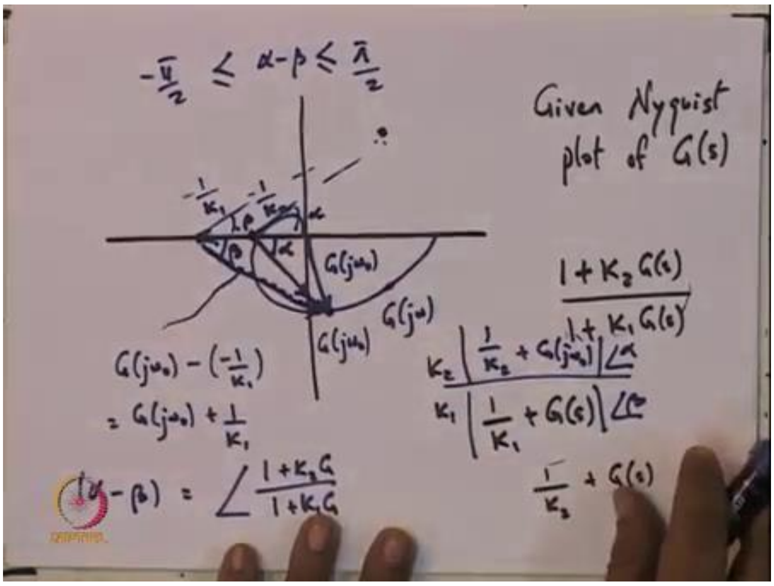

So, for that let us look at the complex plane, they have given the Nyquist plot of G of s, so Nyquist plot of G of s essentially means so maybe we have something like that. So, this is G j omega, so I have various omega we have evaluated G s and we have plotted that that is Nyquist plot. So, let us take some particular point here, so this is lets a G at j omega naught, now we are interested in finding something out about this transfer function which is one upon k 2 G s upon one plus k 1 G s. So, let us look at the denominator, so this denominator I could pull k 1 out and this is a same as 1 upon k 1 plus G s. Now, if I am going to evaluate this at omega naught, well this vector here is G j omega naught and if this point here is minus 1 by k 1, then this vector is minus 1 by k 1.

Therefore, this vector here is this vector here is G j omega naught is minus 1 upon k 1 which means this is really G of j omega naught plus 1 by k 1. So, whatever is in the denominator is obtained by looking at this particular vector now in the same way one could also look at the numerator and further numerator if one pulls out k 2. You have one by k 2 plus G s and G j omega naught is the same a 1 by k 2 will be again a point here minus 1 by k 2 point here in the negative axis because we assuming the k 1 k 2 both positive.

Of course, k 2 was larger number than k 1 an so 1 by k 2 would be a smaller thing and so let say sorry k 2 is larger. So, 1 by k 2 is going to be smaller so this is minus one by k 2 and so you will have similar vector here. Now, if you want to evaluate this at G omega naught, what we are really evaluating is this vector the magnitude of this vector in the denominator and the magnitude of this other vector 1 upon k 2 with plus G j omega naught in the numerator. Of course, because I have pulled out this k 1 and k 2 this will be k 2 by k 1, so the magnitude of this magnitude of this, but what we wanted to know was this transfer function was positive real.

What would that mean is that the resulting Nyquist plot should have an angle which lies between plus 90 degrees and minus 90 degrees, but this angle of this transfer function is essentially the angle of this which let we call it alpha and the angle of this. Let me call it beta so this angle is alpha, so this is alpha here and this angle is beta and so for this transfer function one upon k 2 G upon 1 upon k 1 G the angle of this particular transfer function is really equal to alpha minus d. Now, if this transfer functions where to stand for something which is positive real then this angle alpha minus beta must be less than equal to this. So, the magnitude will give us a magnitude and this angle must lie between plus 90 and minus 90 or plus phi by 2 and minus phi by 2.

If you thinking in this radiance, so this is must be less than equal to phi by 2 and greater then equal to minus phi by 2, now what that means is if you take any point. Then, if that point is to be point on the on the on the Nyqusit plant of G s, then to know the angle corresponding to that point we draw this lines from minus 1 by k 2 and minus one by k 1 and now look at the angles. So, you have alpha and beta and if the difference between these two angle alpha minus beta lies in this range. Then, we do the transformation then the resulting point is going to lie in the right half plane, so everything essentially depends on these two points minus 1 by k 2 and minus 1 by k 1, and so let us now look at how those two points get related.

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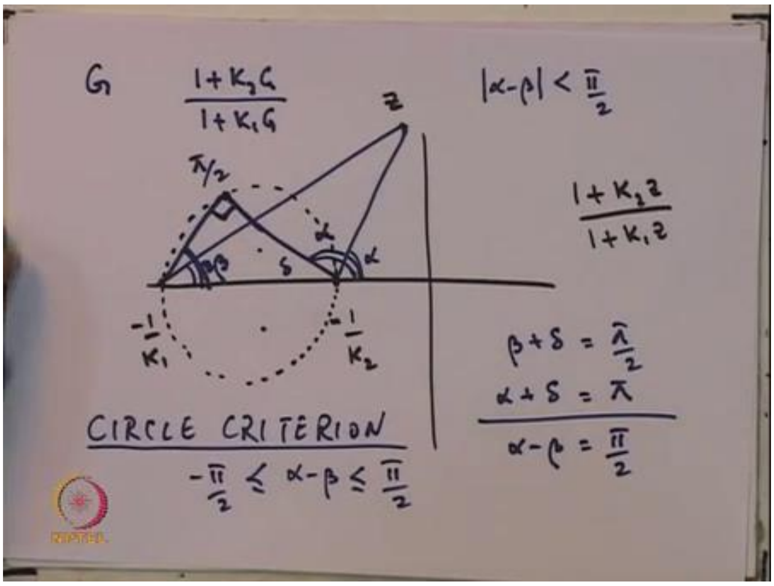

So, let this point be minus one by k 2 and this point be minus one by k 1 if we are interested in let say some point here then what one does is we look at this vector and look at this vector look at this angle alpha, look at this angle beta. We are saying that alpha minus beta should be less than equal to phi by 2 and should be more then equal minus phi by 2 and this guarantes if alpha minus beta is this in range. Then, this guarantees that this point z, so suppose I call this points z then guarantee that 1 plus k 2 z upon 1 plus k 1 z the point to which z will this by linear transformation lies in the right half plane. So, all the points z such that alpha minus beta is in this range or permissible points, where the Nyquist plot of the original plant G s could exist.

Now, how do we find all those points there alpha minus beta satisfies this inequality, now if you recall high school geometry then you might remember that if you have a circle, this might not look like a circle. Assume this is circle this is a circle whose diameter this distance between minus 1 by k 1 in minus one by k 2, and if you take any point if you take any point on the circle and you look at these two lines.

Then, in high school you would have learnt that the angle subtended by these two this angle here is phi by two if this angle is phi by two then what can we say about this particular angle alpha minus. This particular angle beta well we know beta plus this angle if I call this angle delta we know beta plus delta is equal to phi by 2, but we also know alpha plus delta is equal to phi. So, if you subtract the second one from the first one you get alpha minus beta is precisely equal to pi by 2, so this is something that we would have learned a high school geometry that if you draw this circle. Then any point on the circle if you subtend it statins an in the angle 90 degrees as a results the quantity alpha minus beta for any point on this circle is going to be precisely phi by 2.

So, then it turns out if you take any point outside the circle then the angle will be less then or the modules of the angle the modules of the angle will be less then phi by 2 and if you take any point inside the circle then angle that going to get subtended. Its modules is going to be greater than phi by 2, so this is really a circle criterion, so what it says is that forso given G if a Nyquist plot of g. So, given G, so suppose we want to find out about this transfer function 1 plus k 2 G upon one plus k 1 G given G.

Then, from the information about the k 1 k 2 you can plot these two points minus k 1 by k 1 minus one by k 2 and you can look at the circle and if the Nyquist plot of G does not enter this circle. Then the Nyquist plot of the transformed transfer function is going to lie completely in right half plane and that is the circle criterion. Now, this sort of throws up a lot of very interesting things which one would like to talk about, so I would talk about what are the various kind of to interpretation that you can get with the circle criterion in my next lecture.