Nonlinear Dynamical Systems

Prof. Madhu N. Belur and Harish K. Pillai
Department of Electrical Engineering
Indian Institution of Technology, Bombay

Lecture - 20
Kalman- Yakubovitch – Popov Lemma/ Theorem and Memoryless Nonlinearities

So, in the last lecture what we saw was that if you took linear plant which is positive real and the definition of positive and as these ambiguities, but what we effectively would mean is a Nyquist plot lies in the right half plain, the given system is linear. Now if you take such a plant, and then such a plant is of course passive. Now, if you take a non linearity such that the characteristic of the non linearity lies in the 0 to infinity sector, then one can think of that non linearity also be a passive system. As a result what happens is when you inter connect the linear system given by a transfer function which is positive real with non linearity which is in the 0 infinity sector.

Then you end up with resulting system which when you do not given input is as I mean of course the origin is an equilibrium point of this system without inputs and this system is asymptotically stable. In fact when the transfer function is taken to be strictly positive real, then we can see more in fact we could say that the resulting system is exponentially stable. So, let me just rewrite what I have just said in terms of some diagrams, so that it would be clear to be you what I am trying to say, so what we are doing is the following.

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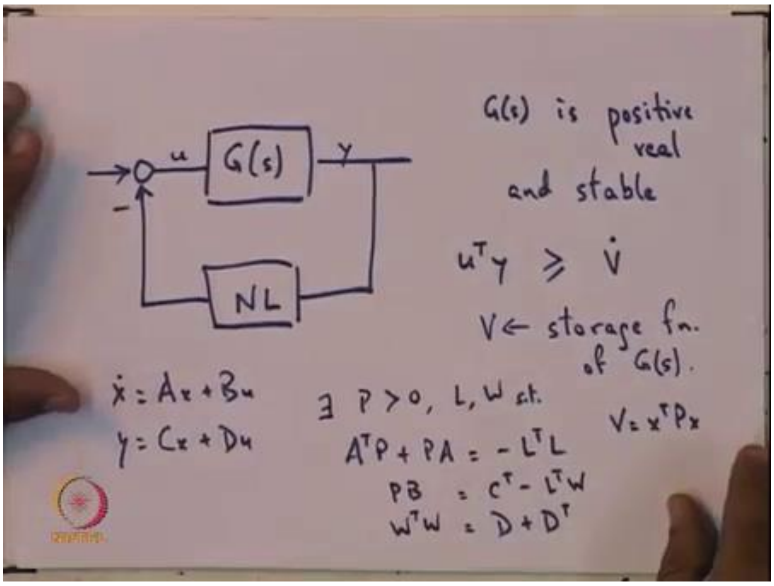

So, you have a linear plan G s and you have the feedback connection and you have this non linearity. So, this negative feedback, now this linear plant that you have we are saying that this linear plant is positive real. Of course, in the last class, I had given various interpretations, what you mean by positive real, but what I would mean by positive real here is a Nyquist plot of this plant is in the right half plain. In addition G s this is stable. Of course one could also use definition to say that G s plus G transpose s star is greater than equal to 0 for all real, for all s whose real part is positive that is another equivalent definition.

Now, by making this assumption, so let me write down G s is positive real and stable. Now if I call the input of G s as u and the output of G s is y, then what it means when you said G s is positive real is stable is that u transpose y is greater than equal to v dot. Here, v is the storage function of G s, we have already talked about what the storage function could be and in order to find out what the storage function could be in work the positive real lemma or Kalman Yakubovich Lemma from where we can get this matrix p. You know if we recall a when you have G s in positive real and stable, then that is equivalent to is set of equations. That means if you if you take the minimal representation for G of s to be a x plus b u y equal to c x plus d u.

Then, using these matrices you can write down there exists p which is positive definite and to other matrices l and w such that first of all a transpose p plus p a is equal to minus l transpose l. Then, you have p b is equal to c transpose minus l transpose, then w and lastly you have w transpose w is equal to d plus d transpose. So, this p that get you is which that positive definite you take the storage function v to be x transpose p x like p yes that this p that you get from positive real lemma, now what about the non linearity now the non linearity that one picks is of the following type.

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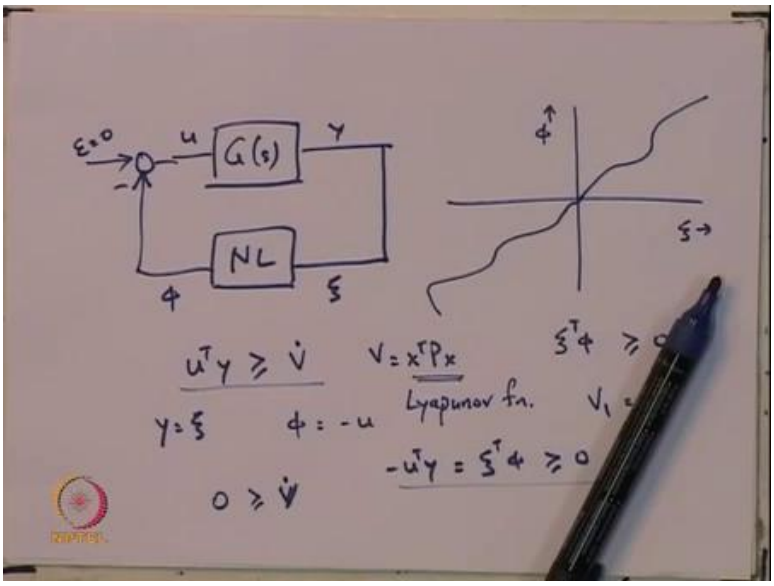

So, the nonlinearity, so let me draw that diagram once more, so here you have the non linearity we are calling this output of linear thing us or r y the output y and the input u. Let me call the input of the non linearity psi and let me call the output of the non linearity phi. Now, the non linearity is such that if you draw the characteristics of then non linearity, so you have psi on the access and you have phi on this axis then the non linearity is something that lies in the first and the third quadrant. Now, if it is lies in the first and the third quadrant, then it is clear that if you multiply the input of the non linearity psi to the output of the non linearity just phi. Then, psi or psi transpose phi one is considering the vectorial the vector situation is multi input and multi out equation is clearly greater than 0.

Now, since this is greater than equal 0 1 can say that this non linearity as a storage function v one which is 0 storage function. Now, earlier in the last slide I have said that you have u transpose y is greater than equal to v dot, where this v was given has x transpose p x that is p coming from the positive real lemma. Now, if you look here there are these inter connection equations which is y is equal to psi and phi is equal to minus phi is equal to minus psi. Therefore, this psi transpose phi, in fact psi transpose phi is really equal to minus u transpose y and its greater than equal to 0.

So, if you add this equation and this equation you end up with 0 is greater then and equal to v dot. Now, what this means that if you use this v that is x transpose p x as lyopuna function for this closed loop system assuming that this input that external input is 0. So, if have that external input to be 0, this is like a system with no inputs and this system with no inputs will have these conditions satisfied and if you have this conditions satisfied then 0 is greater than could v dot if v to the x transpose p x. That can act like a Lyapuna function of this close to system and then what we have here is its derivative which is less than equal to 0. So, in fact from that you can conclude that this resulting system is stable, now if you remember, we had also talked the Kalman Yakubovich Popov lemma.

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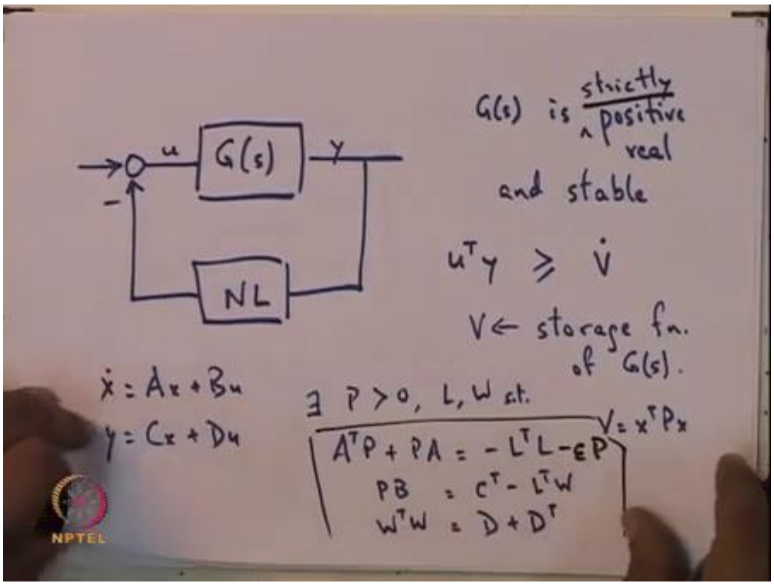

What we had said was that if you take this thing to be strictly positive real, so maybe I will mark black then the change in the equations. Instead of this, you will get minus epsilon p also into this equation into this first equation all the equations remain as they are…

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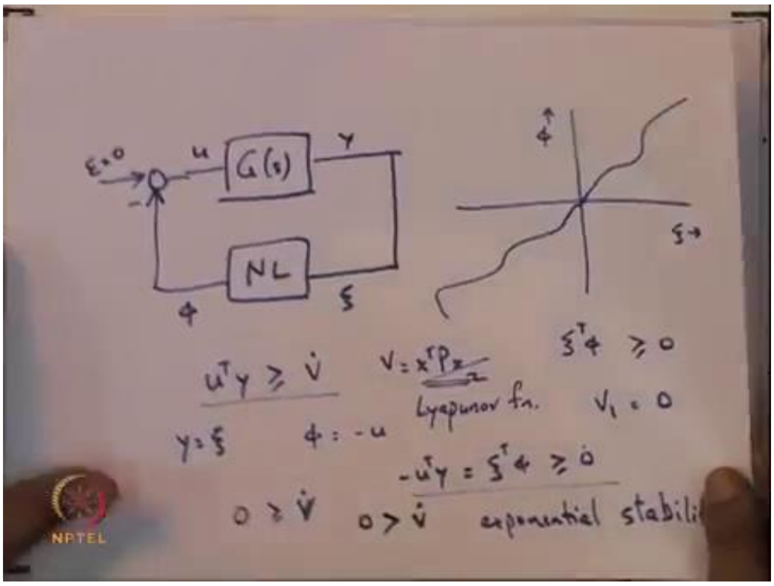

What this means is when you substitute here instead of v dot being less than equal to 0, you will get 0 strictly. Then, v dot then as a result because of that epsilon p which is their using the Kalman Yakubvich Popov lemma, which is G s was taken to be strictly positive real. Then, we would we can conclude exponential stability, so what we are going to do in this class is we will now explore what new conclusions we can draw using this rather password result. Before we do that, I would like to use this a this v as which we have got this from positive definite matrix pas lyapuna function and show that this resulting system is a asymptotically stable.

Well, in order to show that I essentially only have to show that u transpose y is greater than equal to v dot and then the rest of it we have already seen it. If you take u transpose y greater than equal to v dot and you already know when you take a nonlinearity like this psi transpose phi is greater than equal to 0 is some both of this we can get 0 is greater than v dot. So, all we have show that is u transpose y is greater than equal to v dot one small adjustment of a constant, I have to do, I should not be taking v as x transpose p x, but I should be taking this as x transpose p x by two half.

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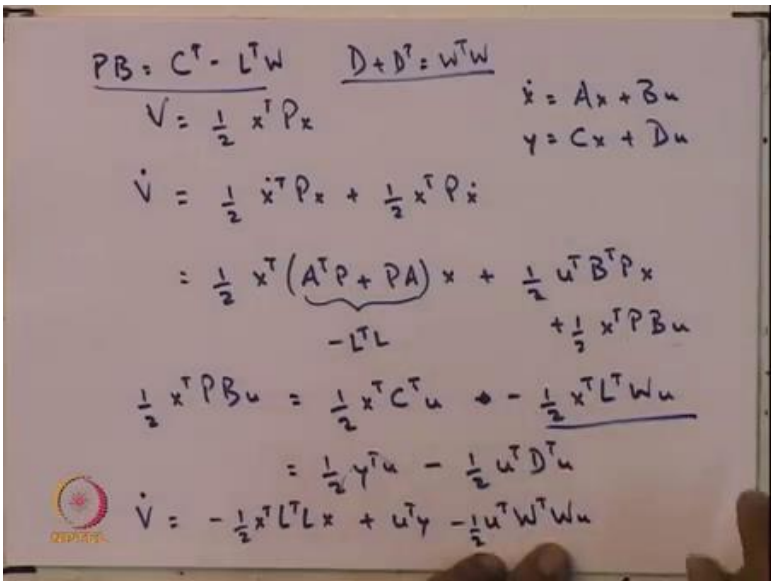

Now, if you take that then so we are taking to the half x transpose p x, so well in that case v dot is half x dot transpose p x plus half x transpose p x dot, but we know that x dot is a x plus b u from the equation of the system. So, substituting that in here we will get half of x transpose a transpose p plus from here similarly, so p px plus I will get half u transpose b transpose p x plus half x transpose p b u. Now, this from the Kalman Yakubvich Popov lemma or rather the positive real lemma, this is the same as minus l transpose l. So, I can put that in there, but if I look at these two terms p b, we know that the one of the equations that we have from the positive real lemma is p v is equal to c transpose minus l transpose w.

So, for p b, I can substitute c transpose minus l transpose w for this, similarly for this so I will just expand this out and show that comes to, so half x transpose p b u becomes half x transpose c transpose u plus or rather minus half x transpose l transpose w u. Now, if you look at the output equation of the linear plant i is equal to c x plus d u and so the c x, I can substitute as y minus d u.

Once I do that, I get a half y transpose u minus a half u transpose d transpose u, so this is from the x transpose p b u from this thing you would get exactly the transposes of this. So, when you put them all together then you end up with v dot is equal to minus a half x transpose l transpose l x that accounts for this and then from both of this u will end up with u transpose y because this half. Then, another half coming from there and then you have this term and there will be a similar term coming from the other portion.

So, putting both of them together, you will have u transpose d transpose u and you will have u transpose d u and d plus d transpose, one of those equations you had was plus d transpose is equal to w transpose w the positive real lemma gave us this equation. So, using that you will get u transpose w transpose w u a half of that and then these last two terms which is there will be this term and the transpose of this both with the minus sign.

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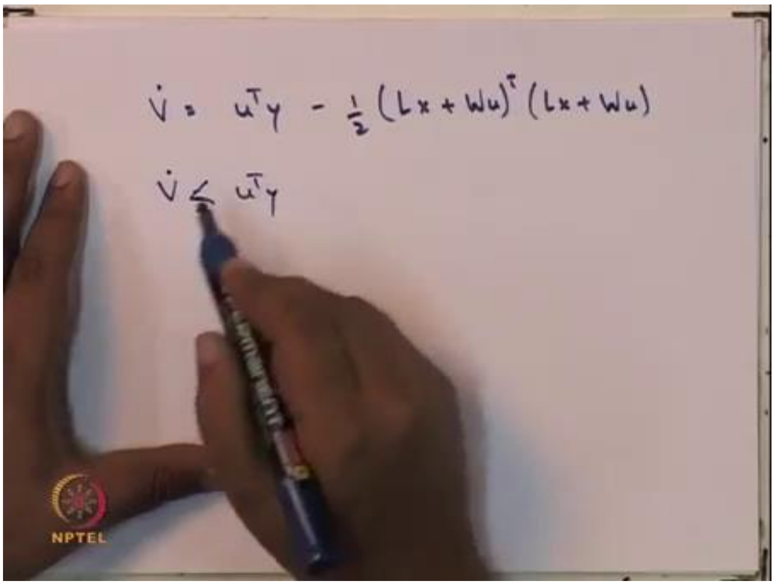

So, if i put all of them together then what I end up with is v dot is equal to first of all u transpose y which we had here and these two terms and the other two terms this and the transpose. Putting all together, you will get minus half l x plus w u the whole thing transpose w l x plus w u. Now, you see this quantity here is positive quantity and therefore, you can conclude v dot is less than equal to v transpose y and this is essentially what we wanted.

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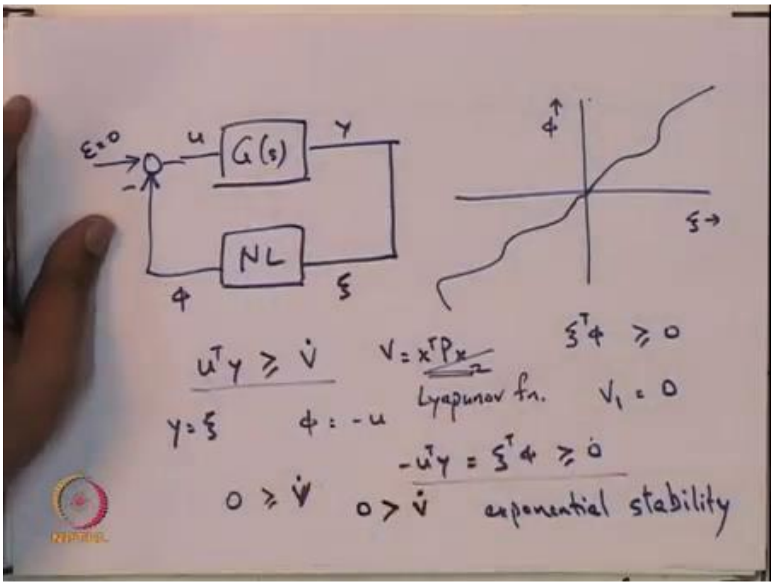

In order to conclude from the sheet that u transpose y is greater than equal to v dot 0 is greater than you already have this about the non linearity. When you put both of them together, you get 0 is than greater than equal to p dot, and therefore you can show you can show stability.

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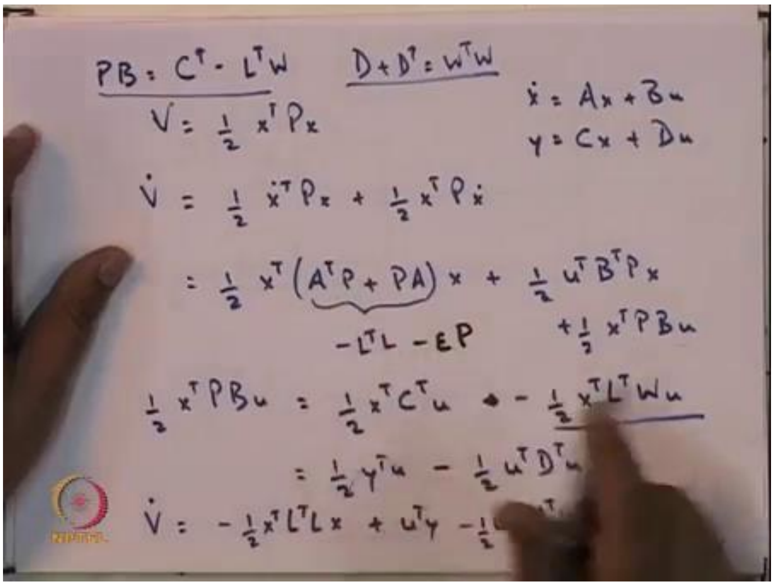

If one assumes that G of s is strictly positive real, and then in that case in this equation here, a transpose p plus p a there will be one additional term here and this additional term if i call it epsilon p, this additional term.

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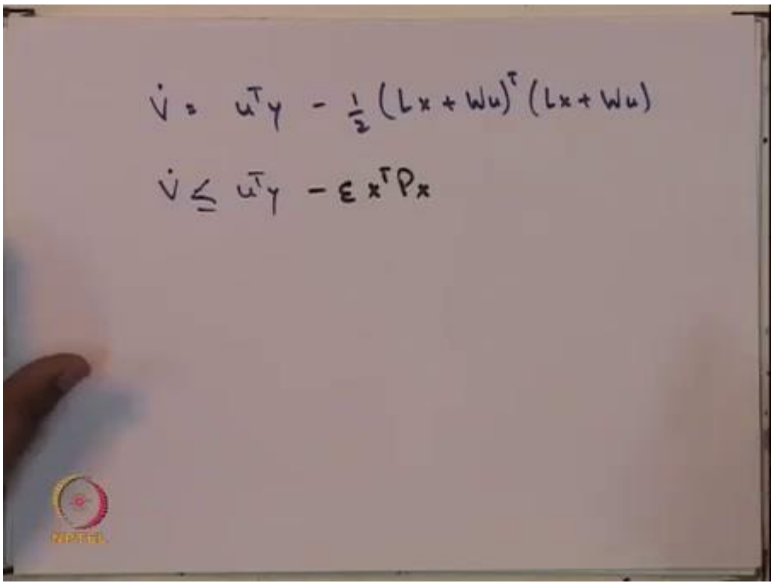

This additional term will finally end up in the last equation and here you would have minus epsilon x transpose p x and this will now let us prove that the resulting system is in fact asymptotically stable. Now, what I am going to do is I am going to make use of this rather powerful theorem and I am going to look at all kinds of non linearities. Then, we divide new results about which non linearity's with the when put in a feedback connection with certain kind of linear plant would result in an asymptotically stable system, so let me conclude about the this first.

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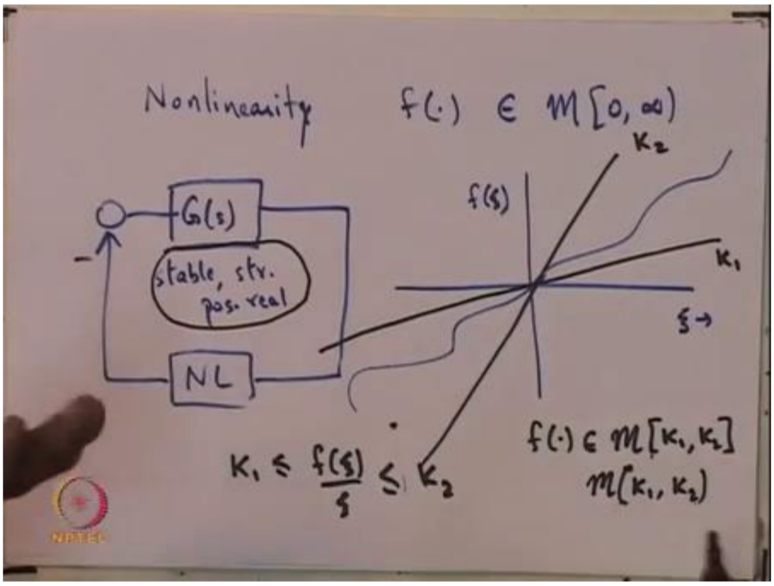

So, what am saying now is suppose you take a non linearity, now let me call it f that belongs to the 0 infinity sector. Now, when I say that non linearity belongs to 0 infinity sector, what am I mean by that is that if psi is the input of the non linearity and this is f of psi is the output of the non linearity then this non linearity lies in the 0 to infinity sector. That means it lies either in the first quadrant or in the third quadrant if you have any such nonlinearity any nonlinearity, which belongs to this class and you take a G of which is stable and strictly positive real. Then, this feedback connection of this strictly positive connection with the stable plant to the non linearity results in something, which is asymptotically stable.

Suppose, now the non linearity that we consider e is this particular non linearity, now if you consider this particular non linearity. Then, you can see that of course it is true that this non linearity lies in the 0 infinity sector, but in fact you can say more things about this non linearity. Suppose, I draw a slope like this call it k 1 and suppose I draw another slope, let us say like that call it k 2, then in fact this non linearity f lies in the k 1 k 2 sector. So, what i am trying to say is that the non linearity is such that k 1 is less than equal to f psi by psi which is less than and equal to k 2.

So, in some sense by declaring f to be from sector from 0 to infinity, we are doing a over kill because we can get a much tighter bound for non linearity. In fact you can say that non linearity lies in the sector k 1 k 2 in fact out here as you can see, so here the way I have drawn it because the k 2, there is some portion here of the characteristic which as the slop k 2. Instead of writing this sometimes people would write something like this k 1 k 2 and open interval k 1 k 2. Now, when you open a open interval k 1 k 2 are a closed interval k 1 k 2 that difference between them is essentially to do with this in equalities whether they are strictly in equalities or just you know non strict in equalities.

So, when you have non strict inequalities, then you would put the closed bracket when you have strict inequality you would put the open bracket and of course there semi open semi open closed kind of intervals for the non linearity. Now, because this non linearity lies in the 0 infinity sector, therefore we know that if you take any stable strictly positive strictly positive real transfer function and connect it in this feedback loop with this particular non linearity. You will get an asymptotically stable system, but since we can put a tighter bound on this non linearity that means this non linearity actually lies in k one k two sector.

Therefore, one would expect that apart from these plans there are other plans also which you could you could connect with this non linearity and these plants may not be belonging to this class that means they may not be strictly positive real or stable. Despite that the resulting system is asymptotically stable and the reason one why one would believe that is because it is true that this particular non linearity is in the 0 infinity sector, but in fact we can say that it is in the k 1 and k 2 sector. So, if it is in the k 1 k 2 sector it would be very surprising if there are no extra plans that one can connect to the non linearity resulting in this system being stable.

I mean one would naturally except that there would be more plants that we can connect to the non linearity and the resulting system is asymptotically stable. Now, what we are going to do is explore this situation where you have the non linearity given in a certain sector. One wants to characterizes all the all the transfer function which you can connect in this feedback loop with those non linearities such that the resulting system is asymptotically stable. Now, in order to do this what we would do is we will do some characterization of the various kinds of non linearities. So, let me let me begin by looking at certain classes of the non linearity and I will show that these non linearities can be transformed into a non linearity in the 0 infinity sector, let me let me make this clear by some examples.

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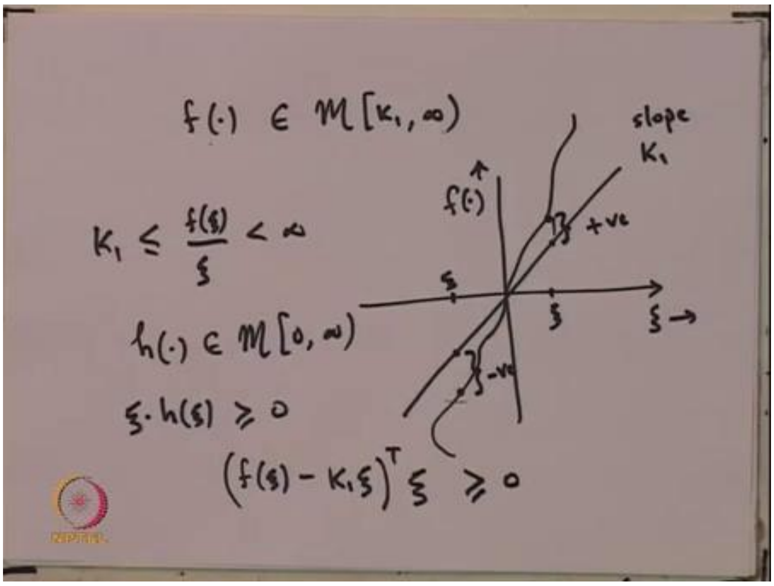

So, suppose you have a non linearity belonging to the k 1 infinity sector what do you mean by saying that a non linearity belongs to the k 1 infinity sector what we mean. So, here is oh God here is this line k 1, let me use a new sheet, so we want to talk about this f which is in the k 1 infinity sector. So, what you mean by that is, so this is line with slope k 1 and if non linearities in the k 1 infinity sector, which means it lies here and here could be something like that. In other words non linearities is such that f psi by psi where psi is the input of the non linearity and f psi is the output.

This is less than infinity, but greater than equal to k 1, now if you have a non linearity like this, we can convert this in to a non linearity in the zero infinity sector. Now, how does one convert this into a non linearity 0 infinity sector, well the if you have a non linearity in the 0 infinity sector. Then, the following is true psi times h of psi is greater than equal to 0, now for a non linearity that lies in this sector this is true, but this is true is the same f of psi minus k 1 of psi. So, for example if look at this particular psi, so this is f of psi and so f psi minus k 1 psi is this portion here which is positive.

On the other hand, if I take a psi which is negative, then f of psi is here and this s k 1 psi and therefore, this quantity here is in fact negative this quantity here is negative. So, if now this thin, now let me put a transpose is multiplied to psi then here you see you get a positive quantity multiplying positive. So, that resulting thin is positive in the left half you have negative thing multiplying negative. So, the resulting thing is positive, so for any non linearity which lies in the k 1 infinity sector, we can say that this condition is always satisfied, now if this condition is satisfied then this is what we do?

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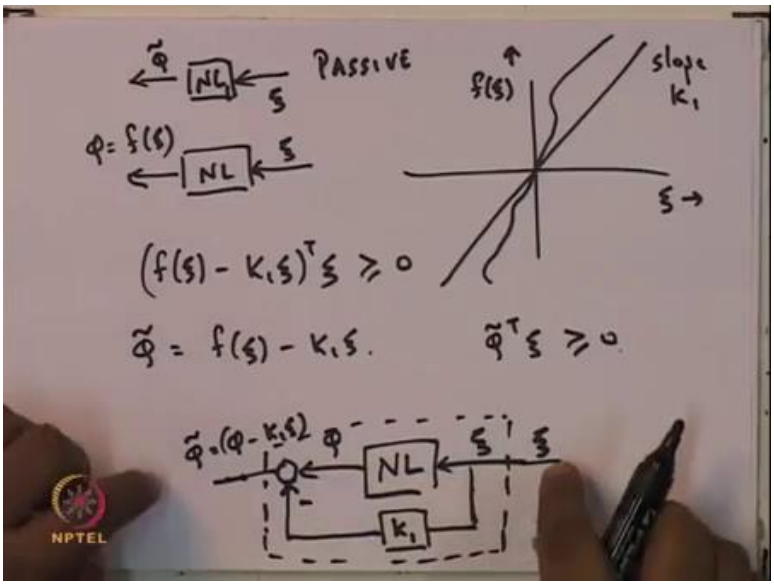

So, this is slope k 1 and you have an non linearity like that and what we have just written was at the input of the non linearity is psi. So, f of psi is the output of the non linearity, so if you think of this non linearity like this, now let me call the input psi and the output is f of psi. Then, one way we could characterize such a non linearity by this quadratic form which is f psi minus k 1 psi, which is transpose psi is greater than is equal to 0. Therefore, now if you think of this is the output psi if you think of a non linearity some new non linearity, which as the same psi as the input, but it has phi as the output, where this phi is really f of psi minus k 1 psi.

Then, this non linearity this new non linearity will have the property that phi transpose psi is greater than is equal to 0 and therefore is new non linearity, let me call it n l 1 is passive. Now, how does one obtain n l 1 from n l, well one way to obtain n l 1 from n l is the following, so you have the original non linearity and it has the input psi and if it has the input psi, it will give the output which let me now call it phi. You want the new non linearity which it does not give five has output, but gives five tilde as the output.

So, how to get five tilde as output you want use this same psi as the input well if i take a k one here, so k 1 time psi is what I will get and I use a feed forward, then what I will have here is phi tilde which is phi minus k 1 psi. Now, if I think of what is in this dotted box as my new nonlinearity it has as its input psi and it has its output phi tilde and the phi tilde transpose psi is greater than equal to 0. So, this resulting non linearity is in fact passive, therefore this resulting non linearity, if it is now connected in a feedback loop with a plant which is strictly positive real then the resulting system is asymptotically stable.

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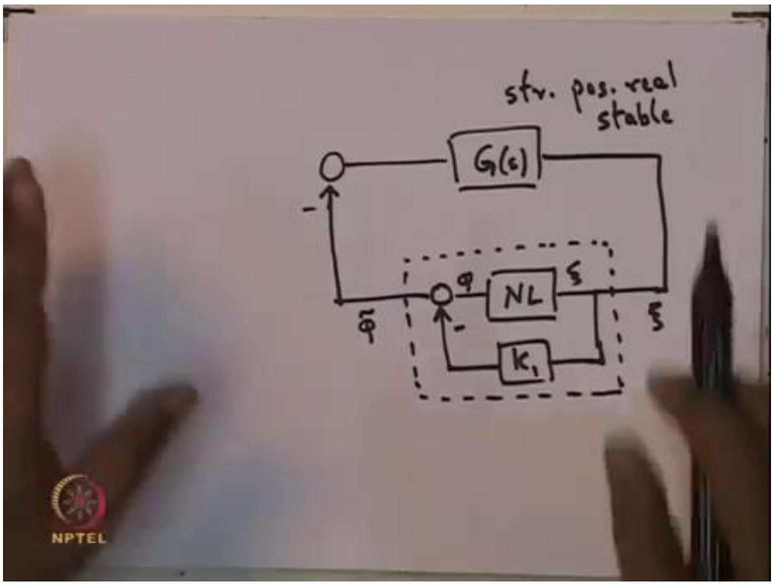

So, let me let me draw that so you have this non linearity, so here psi and you have k 1 of psi set forward. So, its till psi here so the original non linearity at this output phi, but this new non linearity has the output phi tilde. Now, this phi tilde is connected to some G s that is G s is strictly positive real and stable, then the resulting system is of course I am in using earlier results what we have concluded earlier using the positive real lemma and so on. We can conclude that this resulting closed loop system is asymptotically stable, but now we are interested in knowing not what can be connected to this new nonlinearity.

What can just be connected to this non linearity, what plants can be connected to this non linearity and the resulting system would be asymptotically stable. So, what I am now going to do is I am going to do a set of transformations and these transformations go under the name of loop transformations and by loop transformations. We can actually conclude or we can find much larger class of plans, which you can connect to non linearity and the resulting system asymptotically stable, so this is what we do.

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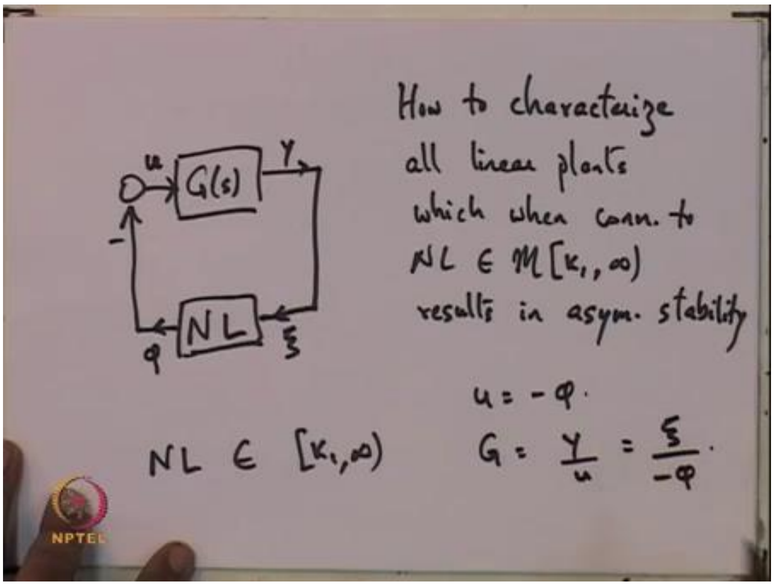

So, what we are going to do is so here is a linear plant and here is a non linearity and we are assuming that non linearity is in the k 1 infinity sector and we would like to know that when you do this feedback connection what are that linear plants. So, that question you would have ask is the following how to characterize all linear plants, which when connected to non linearity in the k 1 infinity sector results in asymptotic stability.

Now, let me call the input to the linear plant u and the output y and let me call the input to the non linearity psi and the output phi. Then, of course by the feedback connection then you know u is equal to minus phi and you also know that the transfer function G is really y by u, but that is the same as psi by minus phis.

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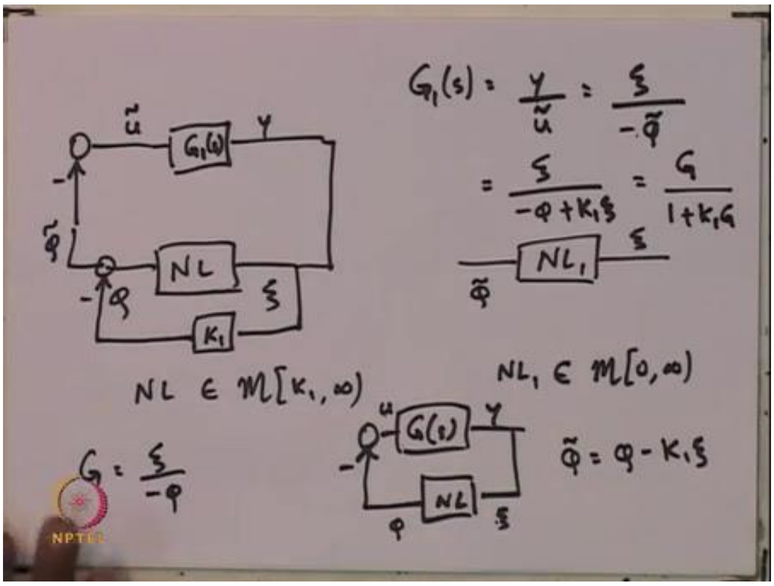

So, earlier we saw so given this non linearity with input psi output phi such that non linearity belongs to the k 1 infinity sector, you can get a new non linearity. So, let me call it n l 1 such that it has the same input, but now the output is phi tilde such that this n l one is really it belongs to the to the 0 infinity sector. Now, how does one get this n l 1 from this n l we just saw earlier that one day get n l 1 from this n l is by using this feed forward, where you take this k in k one and feed it there. Therefore, now what you have will be phi tilde, now you originally had a G of s here, which you have inter connected with this n l. So, what I am trying to say this you had this G s inter-connected to the non linearity n l and we call this u we call this y, we call this psi, we call this phi.

Now, one could modify this transfer function G of s into some new transfer function G one of a s such that this G 1 of s will take this modified input, but give the same output as the original G of s. So, what we are trying to say is modify this G of s in some way such that takes this u tilde as the input rather than the earlier u s t.

It gives same output as G of s would give for you, then we can calculate what this G 1 of s would be because G 1 of s G 1 the transfer function G 1 of s is given by y divided by u tilde, but this is the same as saying psi divided by, because y is equal to psi. Here, u tilde is equal to phi tilde minus phi tilde, but this minus phi tilde, we know is the same as psi this minus phi tilde is the same as minus phi. So, we had taken this phi tilde to be we had taken this phi tilde to be phi minus k 1 psi, so I am just substituting phi tilde.

So, i have minus phi plus k 1 psi, now if you remember in the last slide I had written that the original transfer function G is really psi by minus phi. So, dividing by minus phi in the numerator and the denominator, we get this G 1 is really the original G upon 1 plus k 1 G, so what we are trying to say is the following.

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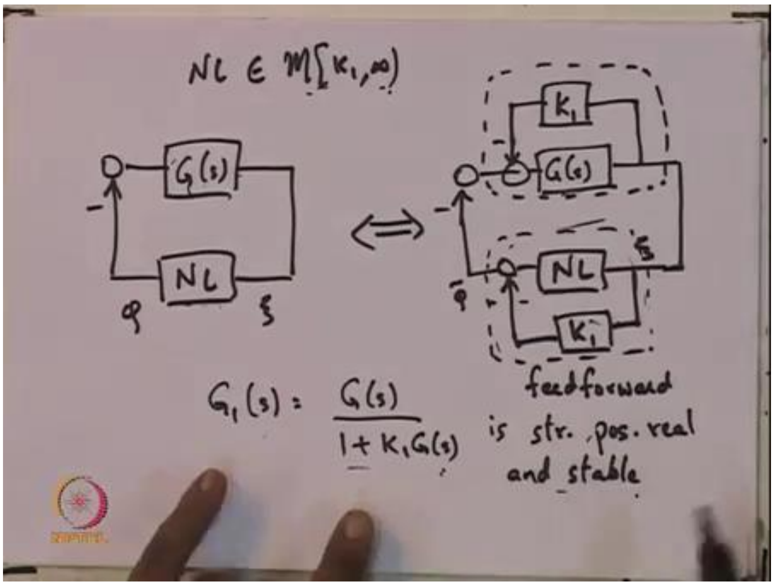

So, what we are trying to say is the following, so if you had a non linearity psi phi and you had this G of s, then this inter-connection is exactly the same as the following inter-connection where you have this non linearity the same non linearity. Before you keep the input to the non linearity exactly the same, but for the output what you do is you put k 1 here. So, you fed forward this k 1 and now you have this input phi tilde and so this new output phi tilde you feeding back.

Then, the resulting transfer function that you would have here which one call G 1 of s as we just derived in the last life would be G of s upon one plus k 1 G of s that is same as saying u of G of s. Then, you have one plus G of k of s what that means is you have this feedback structure G of s with fed back with k of sand this resulting system. So, when you put G s with this non linearity and this non linearity this non linearity is in the k 1 infinity sector and you have this non linearity. Now, you can convert this non linearity into something in the passive that means you can convert into something in the 0 infinity sector.

By doing this feed forward, so you have done a feed forward and what have you done you have forwarded the input to the output by this loop. So, along with the non linearity you have given a feed forward, but when you do this feed forward on this loop on the open loop what you do you could do a feedback with the same k 1 on the original plant that you had. Then, this resulting plant along with this new non linearity, so the new non linearity is what is inside the dotted box that I am putting along with this new linear plant. This is what there in the dotted box there these two this combination is exactly the same as this combination.

Now, this combination is exactly the same now by using the feed forward we had done is this non linearity in the k 1 infinity sector you converted into non linearity in the zero infinity sector. Therefore, you know that for anything in the 0 sector making use of the positive real lemma and the results about the passive systems and so on. This resulting system here if this is strictly positive real and stable then that along with this this resulting system is asymptotically stable, but that since this system is equivalent to this system that the same as saying this system is asymptotically stable. Therefore, for a non linearity in the k 1 infinity sector given a G s that G s with that non linearity in a closed loop system, this would result in asymptotically stability.

When you modify G to this thing that means you modify G to G 1 the G looks likes this G s upon 1 plus k 1 G s and if this is strictly positive real and stable, then the original system is going to be asymptotically stable. So, I hope this is clear, so what you are really doing is you doing some sort of a loop transformation. So, what you did is on the on the non linearity you gave a feed forward loop and as a result on the linear plant u get a feedback loop both these loops have the same gain.

So, this is a feed forward loop negative gain here you have feedback loop with negative feedback, now as a result this modified linear plant along with this modified non linearity, they both together. Now, in work the positive real lemma whichever one is applicable and then you can talk about the stability of that resulting system and this resulting system is exactly equivalent to this system.

Therefore, you can say something about all the transfer functions which when interconnected to non linearity in this particular sector will give asymptotic stability. So, what is the result the result is if you take any plant you do this transformation of the plant this transformation of the plant should result in this new plant G 1 of s is being strictly positive real and stable. If that is true, then the original plant of G of s with the non linearity in the k 1 infinity sector is going to give you the stability now that was one special case where you took a non linearity in the k 1 infinity sector.

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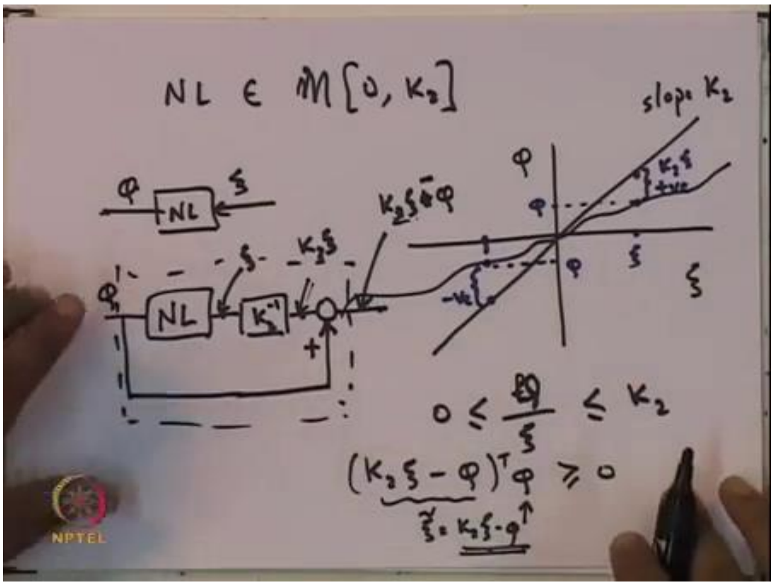

Now, one can also think of taking non linearity in the 0 k 2 sector, so if you take a non linearity in the 0 k 2 sector, so what will its characteristics look like if I call this psi the input and this the output phi and I draw this line which as slope k 2. Since the non linearity lies in the 0 k 2 sector, which means non linearity is something like this.

In other words phi by psi is less then equal to 0, 0 is less than equal to phi by psi, which is less than equal to equal to k 2. Now, if you have a non linearity of this kind, then this particular in equality I could rewrite this in the following way, if I take k 2 psi minus phi. So, suppose I take some point psi, so this here this point here is k 2 psi and corresponding five is this. So, this quantity here is positive and for this particular k psi was positive, now similarly, if I take psi to be negative, I get k 2 psi to be here and I get phi to be here. So, k 2 psi minus phi this is negative and when psi is negative, but more importantly k 2 psi minus phi this quantity here is positive.

So, I could write the following you see when psi is positive phi the corresponding phi is also positive and here when psi is negative the corresponded phi is also negative. So, I could write following down k 2 psi minus phi transpose multiplying phi the output when this is positive this is positive and when this is negative this quantity is negative this quantity is negative. So, this product is greater than equal to 0 and so what I have really done is I have looked at this nonlinearity and I have looked at certain quadratic in equality which gets satisfied by any non linearity, which lies in that sector.

Now, what I am going to do is I am going to use this non linearity this particular quadratic relation to modify my original non linearity in such a way that now it becomes a non linearity in the zero infinity sector. So, how do I do that, I do very similar to what I did is last time, so here is the non linearity here, let me assume as the input and here is the output phi. Now, what I am going to do is the last time what we did was we kept the input was same and modified the output and this time around what I am going to do is am going to keep the output the same, but am going to modify the input. So, the new input psi tilde is going to be equal to k 2 psi minus phi, how do I do this?

Well, what I could do is I could take this non linearity, the output still going to be same phi, but the input the new input is going to be psi tilde, but let me do the following let me put a gain here which is k 2 inverse. Now, if I give k 2 psi as the input here, then because of the k 2 inverse, I will get psi and then I have the original non linearity. So, what I am trying to do is I should get psi here such that I have the original non linearity, so the original non linearity I am multiplying before you reach the non linearity. I am multiplying the input k 2 inverse and so now I would use this output and I feed this back. Now, if I am to get psi here what I should get here is k 2 times psi and so what I should here should be k 2 psi plus phi.

So, I want k 2 psi minus phi here because that is what I want here and so what I will do is I will feed this back is the positive feedback. So, if I do k 2 psi minus phi, here if I have k 2 psi minus phi here and I am feeding back this phi this phi will cancel out that phi. I will have that k 2 phi here and the k 2 inverse will give me psi here, so have the original one non linearity modified in this particular way should give me a non linearity like this. This non linearity, now if I box this off this new non linearity has the same output as the original non linearity, but as a the input it has k 2 psi minus phi, so it looks like I am out of time. So, we will continue whatever we are doing in the next lecture.