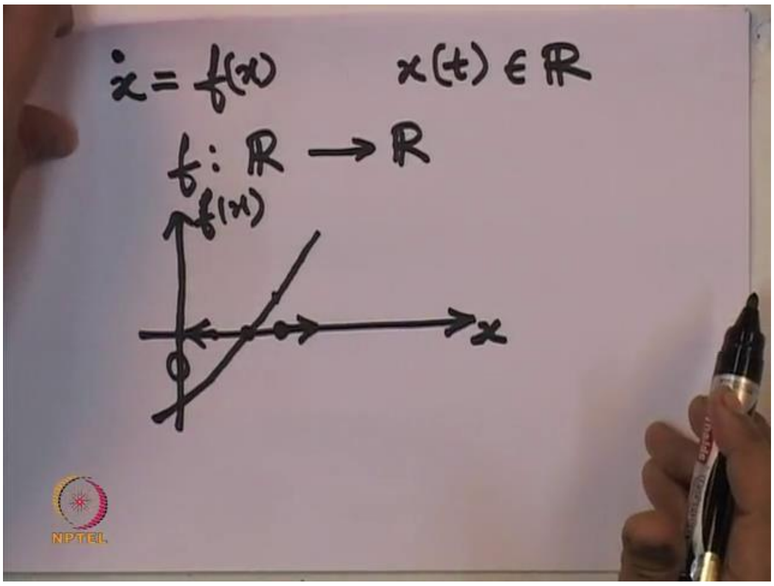
Nonlinear Dynamical Systems
Prof. Madhu. N. Belur and Prof. Harish. K. Pillai
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture - 2
First Order Systems

Hello everyone. Welcome to the second lecture on non-linear dynamical systems. Today, we will continue with first order systems. First order meaning the variable x has only one component, this is what we just begun last week. We will continue with this now.

(Refer Slide Time: 00:36)

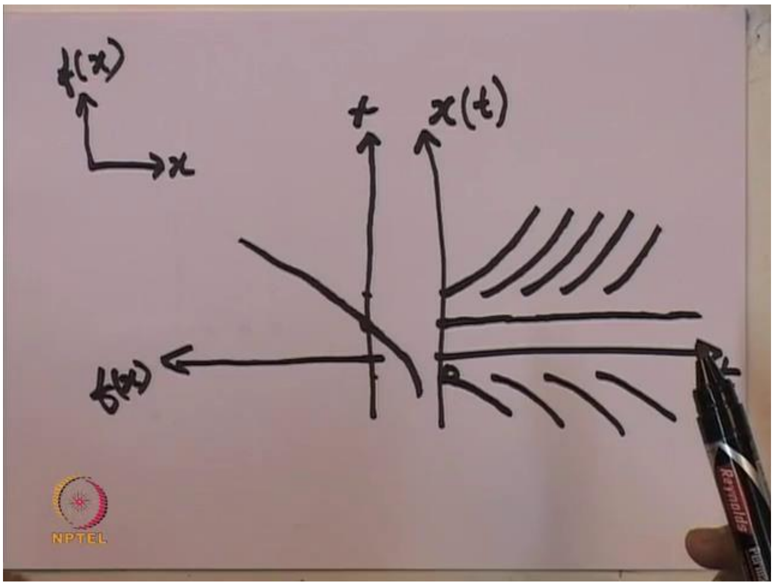


So, we will we were dealing with a case that we have the differential equation x dot equal to f of x, in which x at any time instant had only one component. In other words, it was a real number. In this situation f is a map from real numbers to real numbers. More generally it was a map from R n to R n, but because x has only one component we have a map f from R to R. So, for this differential equation, we saw that the variable x lives only on this line. So, at any time instant, suppose this is the origin at every point there is an arrow which is either towards the positive direction of x or negative direction of x, depending on whether x dot at this point is positive or x dot is negative.

So, whether it is positive or negative can be very conveniently be seen by plotting f here, this is incidentally possible only when x has only one component. So, we can draw a graph and suppose the graph of f versus x is like this. So, at this point we see that f is positive and since f of x is exactly x dot, we can see that because the value of f is positive at this value of x, we see that x is increasing. At this point because f is 0, it means rate of change of x is equal to 0 when x is at this point, and hence that is an equilibrium point if x starts at that point it remains there.

On the other hand if we are considering this point then, this point f is negative and hence this point x dot is negative at this point, and hence it is decreasing so we will draw an arrow in this direction. So, for the single component case for this scalar case when x has only one component that time this line x is filled up with arrows either to the right or to the left, the special case arrow has length 0 when there is no arrow. All these can be drawn by just seeing whether f is positive or equal to 0 or negative. See in other words we can plot a graph of f versus x, but since we are interested in seeing how x varies as a function of time.

(Refer Slide Time: 03:18)

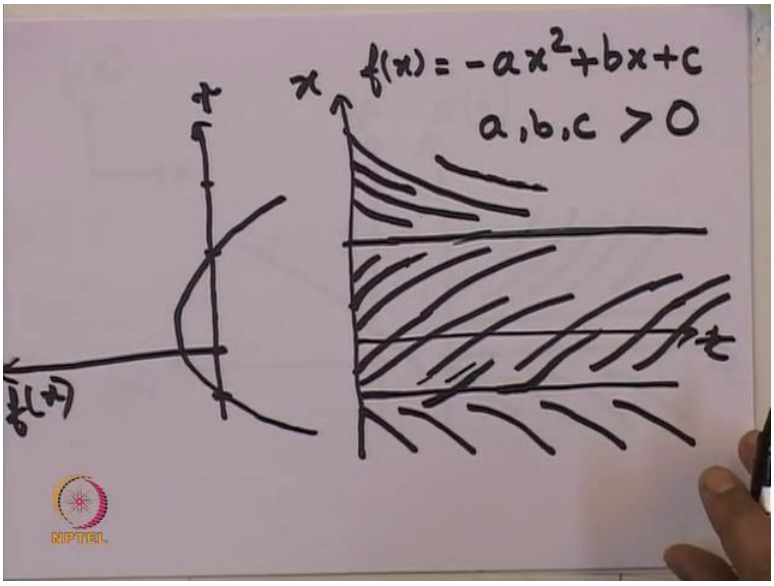


We are interested not just in f with respect to x, but we are also interested in a graph of x versus time because x, the variable x is evolving as a function of time. And in such a situation we would like to draw a graph of f versus x and x versus time and this is one convenient way to do it. So, in our earlier situation, we had x here, and f of x here that has been just rotated by an amount, so that this particular plot x is here and f of x is here denotes precisely this plot that is being pointed by my finger.

So, our previous example f was some graph like this, so we are able to see that this point is an equilibrium point and hence suppose in this graph this is time equal to 0 and this is increasing direction of t. Suppose, this value of x, we saw f of x is equal to 0. So, this is an equilibrium point. If x begins there, it just continues exactly at the same value why because at every point at every time instant t, x dot is equal to 0. In other words x is constant, it is equal to what constant? Precisely this value of x where f is equal to 0, and on the other hand where x is slightly higher value.

Suppose, here then we see that f is positive because f is positive, x dot is positive which means x is increasing as a function of time, if it is equal to this value at any time instant t, it just increases. How long it will increase that depends on how f varies with x. On the other hand, if we start with the value of x that is slightly below this particular equilibrium point, say here then x dot is negative x is going to decrease with time. So, please note that we have dependence of f with respect to x that tells us how x depends on t. Whether x is increasing like we saw here, whether x is decreasing as a function of time depends on whether f is positive or negative as a function of x. So, this we will see a little more in detail when we see an example.

(Refer Slide Time: 06:02)



So, consider the example f of x equal to minus a x square plus b x plus c where a, b and c are 3 real numbers. They are all chosen to be positive numbers, so for this situation when we draw a graph of f versus x. So, this is a little unconventional way of plotting a function, so we have plotted with the same orientation of the independent and dependent axis, x has been plotted here and the dependent variable f has been plotted in this direction.

So, this particular equation we can say, has 2 roots one positive and one negative because a and c have opposite signs. We know that the roots are real distinct and opposite signs and for very large values of x. We know that f eventually becomes negative for very large unbounded values of x both positive and negative, we see that f becomes negative. In other words the graph of f versus x is a curve. Like this where these are the roots as we see one is positive one is negative.

So, for this particular graph, if we are to plot a function of x versus time, here the vertical axis is x and here we have the independent variable time. So, we see that this equilibrium point of course, is going to remain constant as a function of time for slightly higher values of x, f is negative to the right of the x axis here means, f is negative, so x is decreasing as a function of time.

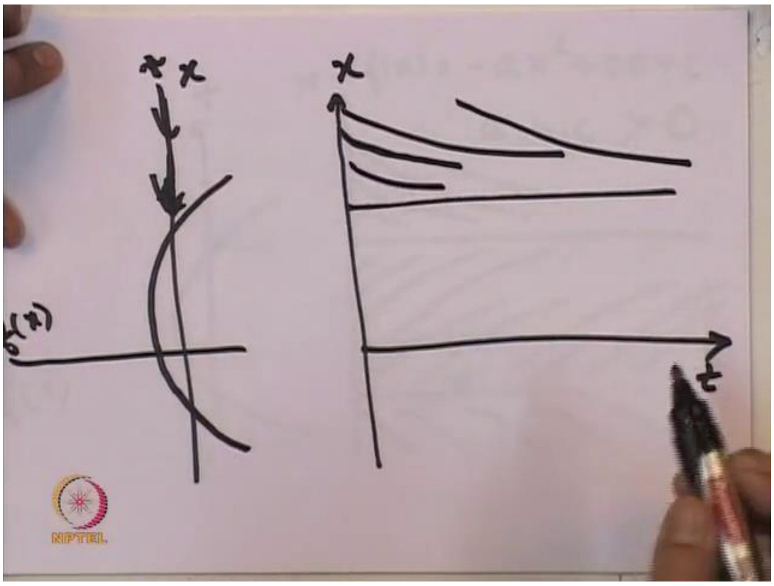
On the other hand for values slightly below t is equilibrium point we see that x dot is positive because f is positive and in other words it is increasing this is how the curves evolve as time increases. The time axis has a very well defined direction of increasing time which we want to say, time is progressing, this is the situation in our world. This other equilibrium point corresponds when f is equal to 0 again that is also being an equilibrium point x is constant because x dot is 0. So, we see that all this is cutting this t axis why because x is equal to 0 is not an equilibrium position. This particular line is denoting the t axis, it is not denoting an equilibrium point or nor a trajectory corresponding to an equilibrium point.

On the other hand because this value corresponds to an equilibrium point, this just remains constant as a function of time similarly this. And here again because x dot is negative we have this curves going away from the equilibrium point. So, this is how trajectories evolve as a function of time. So, we will investigate this particular example in more detail. We will check whether these equilibrium points are stable or unstable and we will also ask some important questions. What are the questions that we will ask; we will ask whether these particular trajectories, which are approaching these equilibrium solution, this equilibrium point do they come and meet it at a particular time instant. On the other hand, we will also ask here we see that these trajectories are going away from the equilibrium point.

We will ask the question is it possible that a trajectory here cuts and comes to this or is it possible that when going along this, we can move out and go here and reach the equilibrium point, these are the questions, we will analyze in detail. Another question, we will ask is at every point is it true that we have a trajectory that starts at every point x at t equal to 0 is it true that we have a trajectory starting from that point going in increasing direction decreasing direction or remaining constant, but this an existence question, does there exist a solution from every point x?

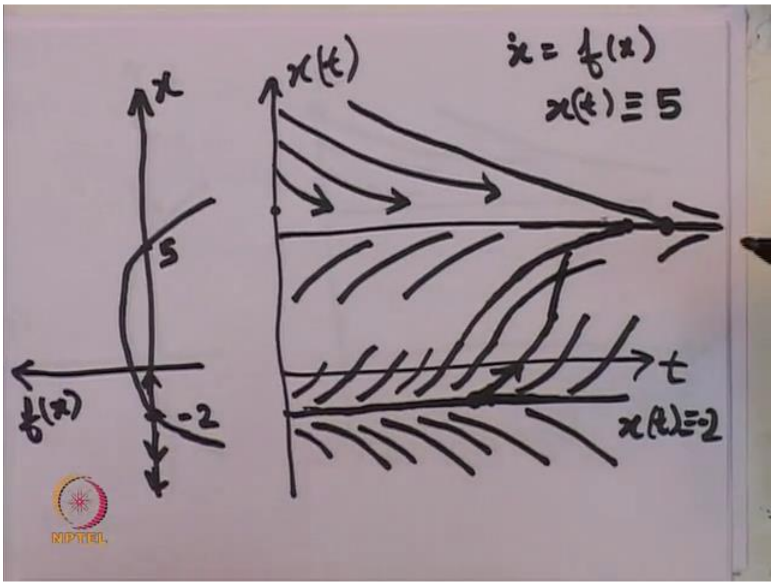
These questions we will answer in a little more detail, but we will stick to this example why because this particular quadratic equation with the leading term negative and the product of the constant and the leading term that coefficient having opposite sign is the situation we encounter in the Kalman filter case. In the continuous time Kalman filter in the recursive solution. We see that we encounter a differential equation that has this particular signs. Of course for the situation that the state variable has only one component x.

(Refer Slide Time: 11:22)



So, this particular example we see that here all the arrows are directed inwards. This is a graph, again f versus x and on the right hand side we have already plotted x versus time and we see that here because f was negative x dot was negative. So, all the trajectories are coming downwards.

(Refer Slide Time: 12:02)



Let us, just see this figure again because in the previous example we have already drawn intersecting trajectories, and the question is that can trajectories intersect for this particular question we need to draw his figure again. So, consider this figure in which this is the time axis and x varies as a function of time at any value of x whether it increases or decreases to decide that we have to go to this figure and check f is positive or negative. Why do we have to do that because we are studying this differential equation.

This differential equation let us go back to our previous example in which we had a quadratic equation with one root positive one root negative. As I said this corresponds to a continuous time Kalman filter differential equation, in which this equilibrium point correspond to the constant solution to the differential equation. Suppose, this value was equal to 5 and this one was equal to minus 2, then x of t equivalently equal to 5 is already a solution to the differential equation, why?

Because at x equal to 5 if you want to find whether x is increasing or decreasing then we substitute x is equal to 5 in this equation. We see f is equal to 0 why because we see that f x graph of f versus x cuts the x axis at x equal to 5, and hence x of t equivalently equal to 5 is the solution to the differential equation, but for values more than 5 we see that x dot is negative, hence it is all decreasing.

So, the important question is can this solution which is can this trajectory which is decreasing, can it meet the constant solution in finite time? Is it possible that a trajectory here only approaches this trajectory asymptotically or can it intersect at some time instant? This is one question we will ask, also we will ask here we see that for values slightly more than minus 2 x dot is positive and hence trajectories are increasing, increasing and going towards this because we see that there is no equilibrium point in the middle. This is, these are how important theorems especially in second order systems are arrived at.

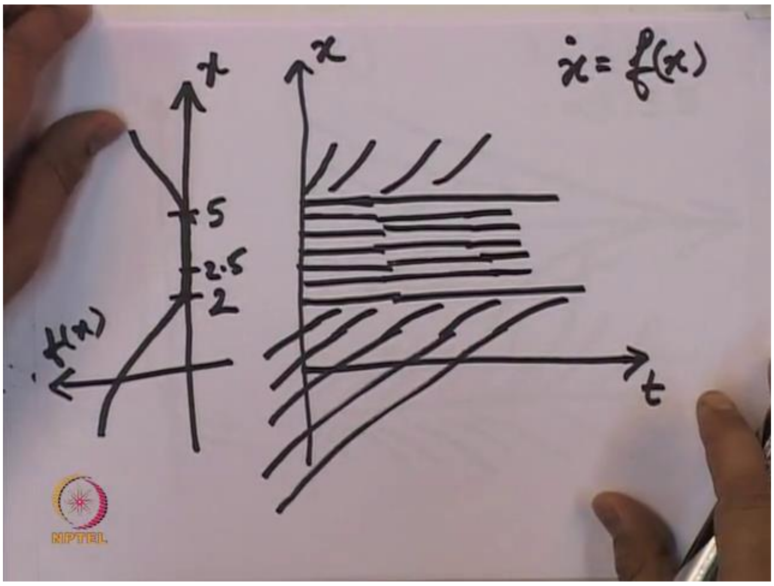
So, we see that all trajectories are going away from this equilibrium solution. What is that the equilibrium solution? The constant solution corresponding to x of t equivalently equal to minus 2, this constant solution or trajectory seem to be going away. We already decided that because arrows are directed away because every small perturbation takes solution away from the point minus 2. We decide that this equilibrium point is an unstable equilibrium point, and we can also see that from this particular phase curves from these solution curves sorry.

So, we see that all solutions are going away, but the equilibrium solution just itself remains constant along this. Is it possible that while it is remaining constant it go away from here. So, that at this particular point we have non-uniqueness of solution. There is one solution that remains constant, another solution that goes away. This is about intersection of trajectories. There is possibly one past up to here, but from here there are 2 futures. Since, there are 2 futures for this particular autonomous differential equation in which there are no inputs. We want to ask is the equilibrium point, an equilibrium point at all because we see that there is one solution that also comes out, which says that this value of 2 minus 2 was not an equilibrium point. Is that possible? This is the important question that one has to ask in encountering non-linear differential equations.

On the other hand we see that this particular solution which evolves goes towards this equilibrium solution. While it goes towards we want to ask does it intersect? Of course we know that once it intersects if at all it intersects, once we are on the equilibrium solution, there is a unique solution because for slightly higher values of x we are coming down for slightly lower values of x. We are also going up which means that we are approaching the equilibrium solution for higher and lower values of x, but while we are approaching is it possible that we intersect this equilibrium solution. This would be about non-uniqueness of the past of the solution when we are at a particular point.

Of course we have one equilibrium solution coming and meeting this time instant x value, but there is also this trajectory possibly. So, these are the questions that we will have to answer when studying non-linear differential equations. We will certainly answer this in more generality in more generality when x has n components. Another important issue is that we have these trajectories, is it possible that these 2 trajectories intersect here and then they criss-cross and go? So, these questions; one is going to see that this non-uniqueness will be ruled out as soon as we assume a Lipchitz condition on f.

(Refer Slide Time: 17:58)



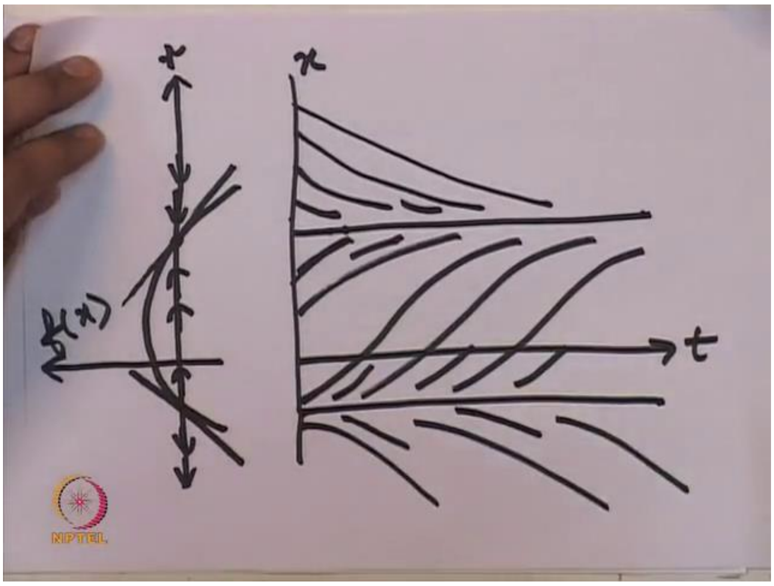
In a differential equation x dot is equal to f of x. We will define a particular property called Lipchitz property of a function. Once we impose the Lipchitz condition on f, we will see that solutions are unique both ion the past and future. However only for a small interval of time for a more longer duration of time to prove uniqueness would require more advanced properties and conditions on the function f. So, the next important question we will ask is the question, is the question of continuum of equilibrium points.

So, when would when would there be a continuum of equilibrium points if f where equal to 0, in this interval. What is this graph of? This is a graph of f versus x on the left hand side of this, we always plot f versus x and we will use that to plot x versus t. So, we see that because f is 0 from this value. Let us say, 2 up to 5 because the graph of f is touching the x axis for this entire interval. We see that from 2 up to 5 all solutions are just constant solutions why because x dot is equal to 0. At 5 also it is equal to 0 for slightly larger values of 5 we see that x dot is positive. In other words the trajectories are going away.

Also for slightly lower values of x, we see that x dot is again positive which means that the trajectories are increasing and hence approaching this particular equilibrium point. These trajectories could have existed for negative values of time also, this is the situation where we have a continuum, what is continuum? If 2.5 is an equilibrium point then very close to 2.5 let us say, 2.50001 is an equilibrium point.

In other words all the equilibrium points are sitting very close to each other, they are a connected set and they are a continuum, continuum of equilibrium point. This is the language we use to say that we have not just not isolated equilibrium points, but a continuum. So, we will also quickly see what linearization has got to do with solution to a differential equation and especially in the context of closed to an equilibrium point.

(Refer Slide Time: 20:52)



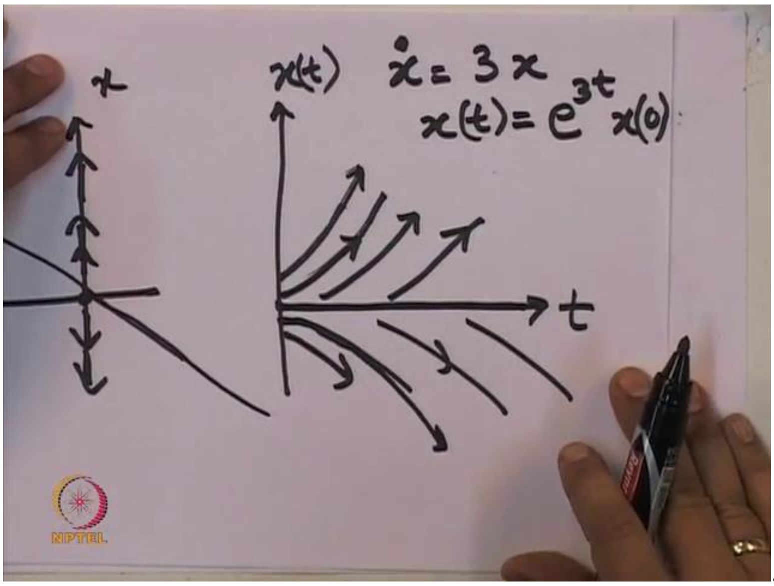
So, a good amount of information about stability can be seen by linearizing. So, suppose we consider our example, so we see that at this particular point whether f is increasing or decreasing. So, at this point we see that f is decreasing as a function of time as a function of x sorry. So, when we try to put a tangent to this curve at the point at the equilibrium point, we see that this tangent has negative slope, so for those who are having difficulty to answer this question with this axis tilted. Let me, just turn this page this is how we should be imagining.

So, this is a graph of f versus x and we see that at this equilibrium point is we draw a tangent to this particular curve f and this tangent has a slope, which is negative, which just means that f is decreasing as a function of x, on the other hand here this tangent is the line with positive slope. So, we are going to differentiate the function f naught at any point, but at the equilibrium point and the equilibrium point we see that this tangent is the line with positive slope, while this tangent is the line with negative slope. This positive or negative slope itself decided whether the equilibrium point is stable or unstable so that we can see also here. So, here we see that the arrows are all like this.

So, to see that the slope already gives the information whether this equilibrium point is stable or unstable and similarly, whether this is stable or unstable. We can see, we can as well analyze this equation this differential equation which corresponds to this line. So, consider the… Before see the differential equation corresponding to a line we will just complete the trajectories for this example. So, we will see that the equilibrium point remains constant similarly, this one also remains constant and here the trajectories are all decreasing and approaching this equilibrium point, here they are all increasing. So, I can be a little fast here because we have already seen this in more detail. This is an equilibrium point, an equilibrium point this is a graph of x versus time t.

So, the fact that this equilibrium point is stable why because trajectories close by are coming towards this, we would like to define this asymptotically stable in the sense of Lyapnov after a few lectures. On the other hand this one is unstable because there are small perturbations near it, which are going to take the trajectory away from this equilibrium point. And this much information we claim we can see already from the slope of the tangent of the function f with respect to x, at the point. Tangent drawn at which point of the curve? Tangent drawn at the equilibrium point because this tangent has positive slope, this particular equilibrium point trajectories are going away from it because these trajectories have negative slope. We will see that these trajectories are coming close by the tangent is going to decide whether the equilibrium points are stable or unstable. This we will see from studying the corresponding linear system.

(Refer Slide Time: 24:39)

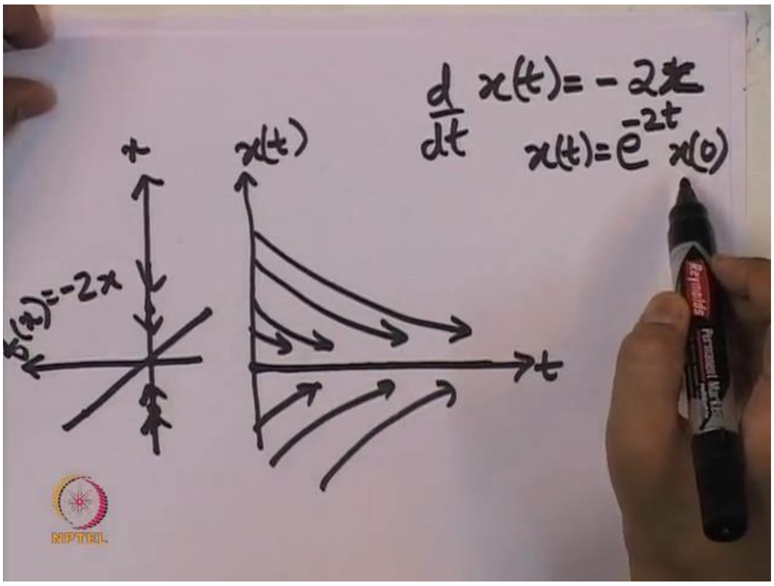


So, consider the example x dot is equal to 3 x, so here the line 3 x versus x on the other hand here is the graph of x versus time. So, we see that all trajectories are going to go away this, we are able to see, because the equilibrium point we are able to see because we can draw here the arrows of this function 3 x versus x. So, we see that for positive values of x, 3 x is positive this you could also consider drawing a graph of 3 x versus x before this rotated axis.

So, you can draw it in this particular situation and after you have drawn this because we are actually interested in the graph of x versus time. When we turn this, we see that this line with positive slope, slope 3 corresponds to this and we see that it is indeed unstable. And of course, we know that the solution to this is exactly x to the power x of t is equal to e to the power 3 t times the initial condition. So, we see that with this particular line has positive slope that positive slope 3 is going to come in here, so that f the initial condition is non-zero, then we see that e to the power 3 t becomes some number that is very large as t increase. And hence x to the power x of t is going to become unbounded. It will become positive and unbounded of x of 0 is positive, it will become negative and unbounded, if the initial condition was negative and that is indeed what we see.

For a negative initial condition the trajectories are going away, and for a positive initial condition the trajectories are going to become positive and unbounded. In any case it is going away from this equilibrium solution, which equilibrium solution? The initial condition corresponding to 0, in which situation we see that e to the power 3 t times 0 will always be equal to 0, no matter what time instant t we are interested in. So, we will quickly take an example where the slope is negative.

(Refer Slide Time: 27:25)

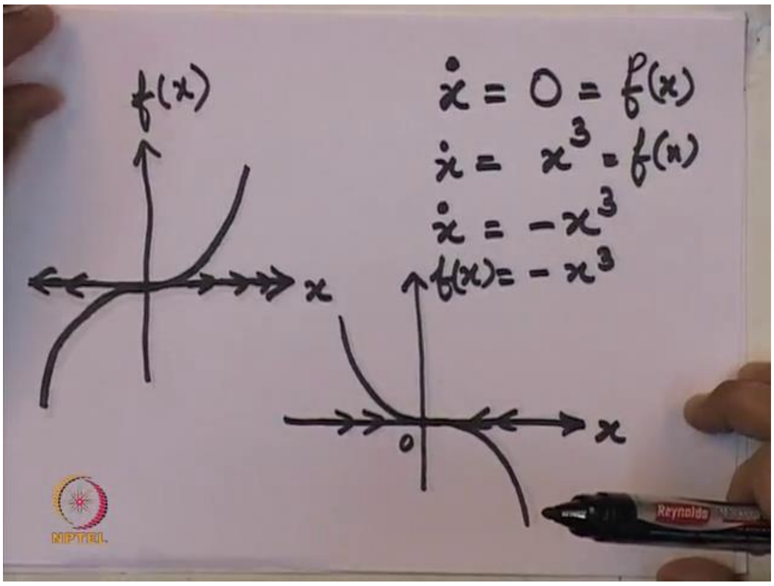


So, we take x of t is equal to sorry d by dt of x is equal to minus 2 times t, in which case we know the solution x of t is equal to e to the power minus 2 t times sorry for this mistake, we are doing x dot is equal to minus 2 times x. So, for this example we see x, x of t is equal to minus e to the power 2 times x of 0. And let us draw that same graph here, this is a line with negative slope with slope equal to minus 2 graph of f of x versus x in this case f of x is equal to minus 2 x.

So, we see that this is a line with negative slope and hence for positive value of x because f of x is negative all arrows are directed towards negative direction of x. In other x dot is decreasing on the other hand for when x is negative if you want to decide where to draw the arrows because f of x is positive, we draw it in positive direction of x. Now, we will study how the solutions look, for linear systems 0 has to be an equilibrium point we see that already here, this is how the trajectories are evolving.

So, this 0 is a stable equilibrium point why because close by trajectories are approaching towards it, it is asymptotically stable that we can see because this line has slope negative, which line? The graph f versus x and the point 0 in fact at every point, it is line with slope negative and that is deciding, it is deciding the exponent and because that correspond to a line with slope negative, we see that this is e to the power negative number times t and it is going to decrease in magnitude, no matter which initial condition x of 0 we start with. But then to see that when the slope is equal to 0, we will this our test is inconclusive to see that we will quickly see 3 differential equations…

(Refer Slide Time: 30:16)



x dots is equal to let us say, 0 x dot is equal to x cube and x dot equal to minus x cube. So, for these 3 differential equations, we will see that in each of the cases the tangent has slope 0 at the equilibrium point. So, we are drawing graph of f versus x in this particular example f is equal to 0, this is 0 function this is always equal to 0. So, all the points are equilibrium points and at any point the graph of f versus x has tangent equal to the x axis.

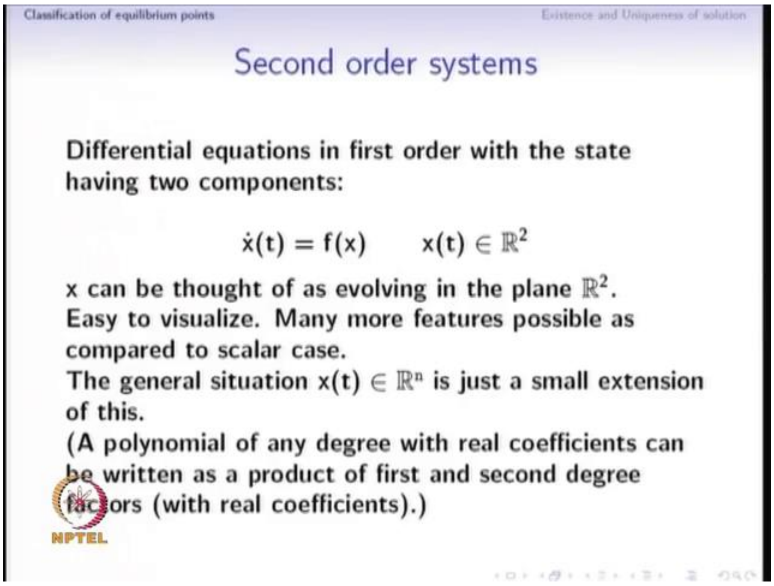
So, the tangent has slope 0 at every point and all the points are also equilibrium points. This is the conclusion we get for this linear system when the tangent itself is a graph f and both has slope 0, but this particular example x dot is equal to x cube when we draw f of x which is equal to x cube, we see that we get this curve. So, one can check that this particular curve is tangential to the x axis at this equilibrium point, what is the equilibrium point? Equate x cube to 0 and we see that x equal to 0 is the only solution only real solution. So, that tangent also at this point happens to have slope 0.

On the other hand we can see that the arrows for this example would always be directed away from the equilibrium point 0. However, the tangent having slope 0 is also the situation for the third example, where f of x equal to minus x cube for this example, here all the arrows are directed towards the equilibrium point. And hence this equilibrium point is a stable equilibrium point it is asymptotically stable all trajectories are eventually going to approach 0 as t tends to infinity, but here also the tangent has slope 0.

So, these 3 example on where x dot is equivalently equal to 0, second where x dot is equal to x cube. And third example where x dot is equal to minus x cube each of the 3 examples the tangent the curve f of x versus x at the equilibrium point has slope 0, but in one case all the points are equilibrium points. In one case the equilibrium point is unstable, while in the other case the equilibrium point is stable. So, these 3 situations can easily emerge when the tangent is having a slope 0.

So, this we will see is a very important situation where the general case also when the, when the linearization has Eigen values on the imaginary axis, the test of stability or instability using the linearization will end up being inconclusive. So, having seen this we will precede to the situation of second order systems, after having seen the first order case. What is first order? When there is when x has any one component, the scalar situation after having seen this we will proceed to the second order case here x has 2 components. The situation when x has 2 components has many different possibilities.

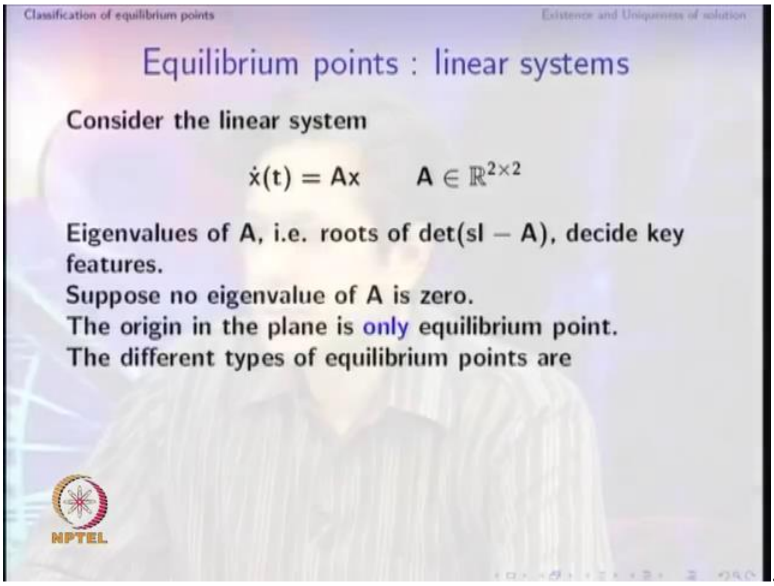
(Refer Slide Time: 34:04)



And these different possibilities will also throw some light about what can happen when x has more than 2 components. So, consider the differential equation x dot is equal to f of x, where x at any time instant has 2 components, so x of t is now in R 2 so we can think of x evolving in a plane R 2. So, this is relatively easy to visualize, but there are many more features that is possible. Let me emphasize, it is easier to visualize than the general case where x has n components, even though it is little more complicated than the one dimensional case where x had only one component. But going from one component to 2 component introduces so many possibilities, but all these possibilities can also happen in R n and compared to R n R 2 is easy to visualize.

More over the general situation is only a small extension of this. Why is it only a small extension? Because a polynomial of any degree with real coefficients can be written as a product of only first and second degree factors, if we want the coefficients to be real that is.

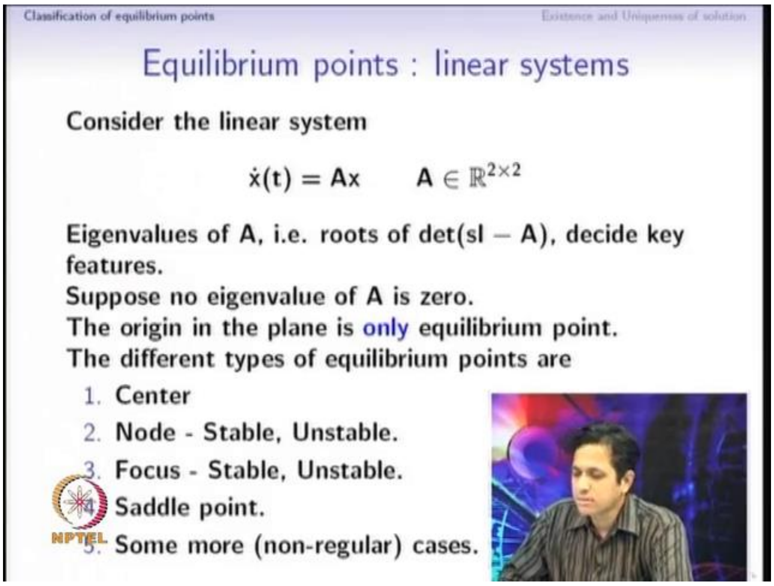
(Refer Slide Time: 35:22)



So, consider the linear system x dot is equal to a of x where A is A 2 by 2 matrix. So, we will start with linear systems, because we are in a qualitative study of the differential equation. Qualitative study means we want to only extract out certain features whether it is stable, whether it is asymptotically stable, whether the periodic orbits for studying this situation it turns out that, linear systems have a rich a rich set of possibilities. And that is the reason that we are starting with the case that a is a matrix A is A 2 by 2 matrix and x dot is equal to a of x.

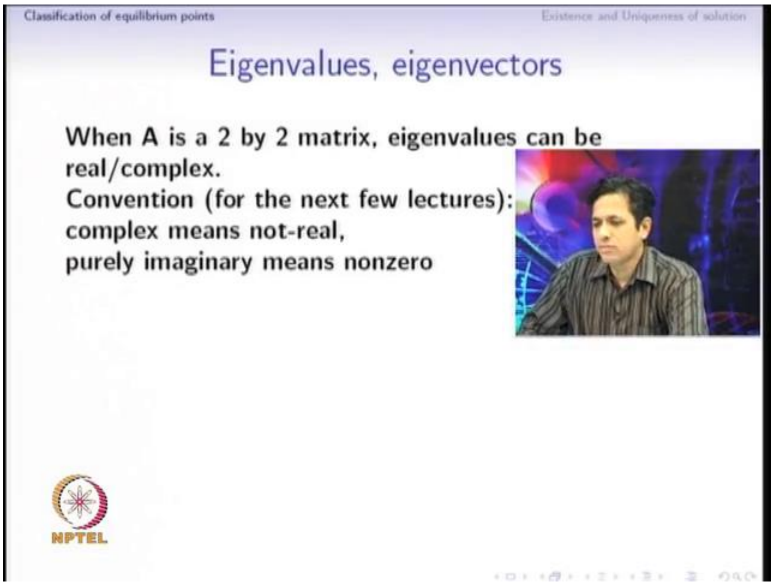
So, the Eigen values of a that is the roots of the determinant of s i minus a decide the key features for this case. So, let us begin by assuming that a has no Eigen values at 0. So, for this situation the origin in the plane is the only equilibrium point because A has no Eigen value at 0. It is non-singular A is non-singular and hence if f of x is equal to 0. It means x has to be equal to 0 that is what is being said by the origin is the only equilibrium point, but this equilibrium point can be of different types depending on different A.

(Refer Slide Time: 36:48)



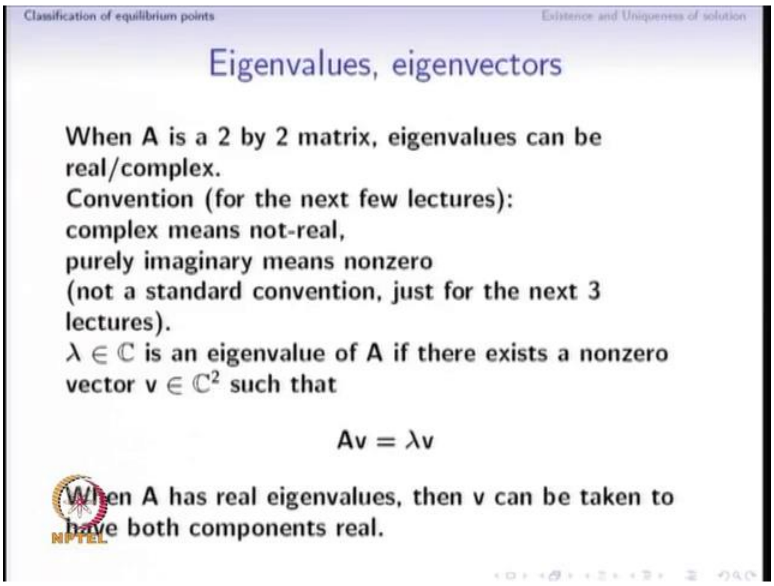
So, the different types we will see are the equilibrium point could be a center we will define what a center is? It could be a node in which case it could be a stable or unstable node it could be a focus, a stable or unstable focus, it could be a saddle point and for certain situations for more general for some non-regular cases it could be none of these which we will also see.

(Refer Slide Time: 37:11)



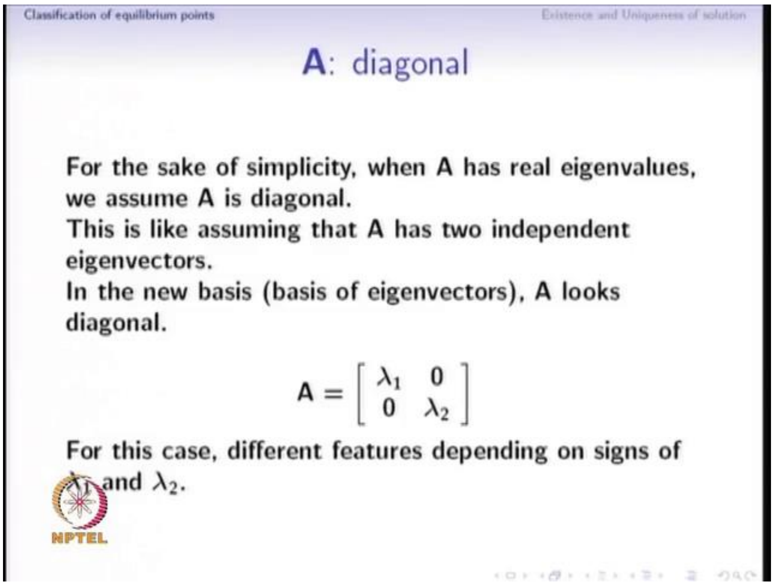
So, let us quickly review what Eigen values and Eigen vectors are? So, when A is a 2 by 2 matrix, the Eigen values can be real or complex. So, for the next few lectures we will use this convention that we will use complex if the Eigen values are not real we will also say if something is purely imaginary. Then, we will assume that it is non-zero why because if it is 0, then we will say it is 0 even though the origin is also on the imaginary axis, even though purely imaginary includes the situation that it is equal to 0. We will follow the convention in the next few lectures that purely imaginary means non-zero. So, this is not a standard convention just for the next few lectures.

(Refer Slide Time: 37:52)



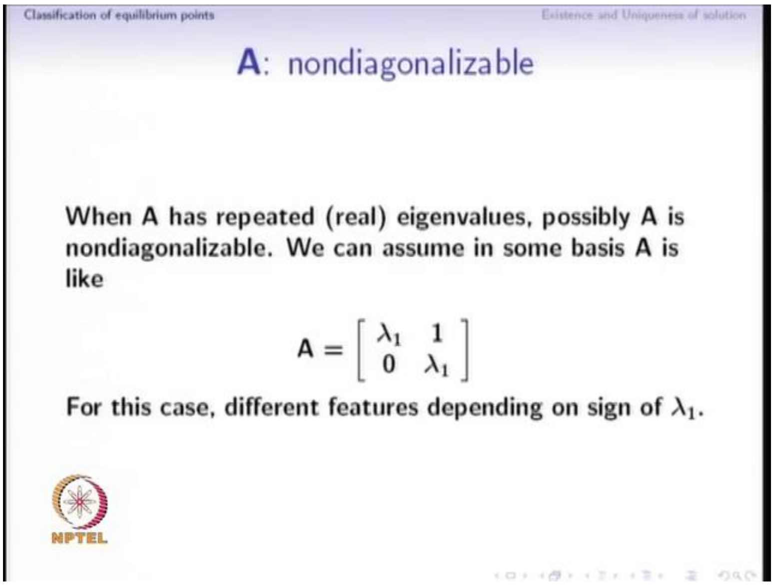
So, a complex number lambda is called an Eigen value of A if there exists a non-zero vector v such that A v is equal to lambda v. So, this equation, if we are able to solve for a non-zero vector v then that lambda is called an Eigen value and that v is called an Eigen vector. So, in general we might require the 2 components in v to both be complex, but when A has real Eigen values then we can ensure that the Eigen vector has both components real.

(Refer Slide Time: 38:31)



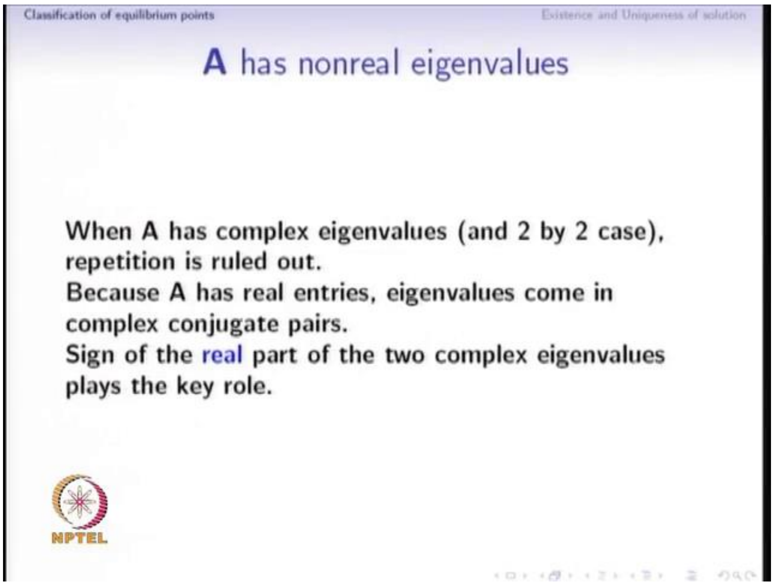
We will start with the situation that when A has real Eigen values A is already diagonal this is like assuming that A has 2 independent Eigen vectors, because it has 2 independent Eigen vectors in a new basis, in which basis in the basis comprising of the Eigen vectors A looks diagonal, which we say A is diagonalizable. So, in this new basis A already looks diagonal that is the reason that we are assuming this situation. So, in this case there are different features depending on the signs of lambda 1 and lambda 2.

(Refer Slide Time: 39:10)



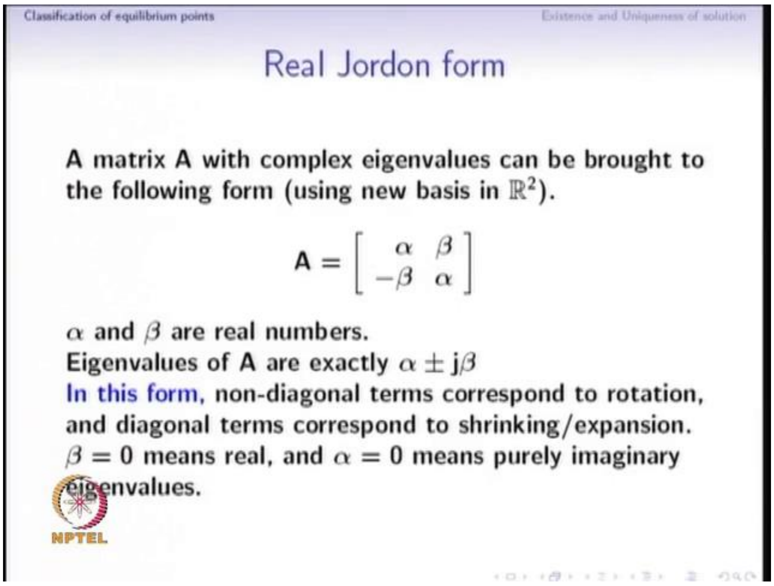
And when A is not diagonalizable then it is forced to have repeated Eigen values because A is 2 by 2 matrix. Then, they are forced to be real Eigen values, then in that situation we will assume that in some basis A is of this form lambda 1 1 0 lambda 1. This is the so called Jordan canonical form. Here, the different features depend on the sign of lambda 1.

(Refer Slide Time: 39:38)



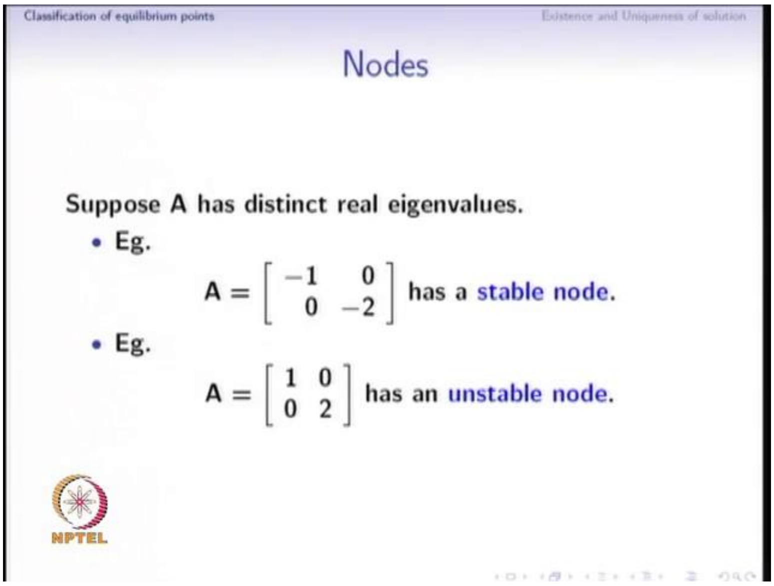
And when a has non-real Eigen values that means it has complex Eigen values then for the 2 by 2 case repetition of Eigen values is ruled out, why? Because they have to occur in conjugate pairs since A has real entries, the Eigen values have to occur in complex conjugate pairs that time the sign of the real part plays a key role the real part of the 2 complex Eigen values.

(Refer Slide Time: 40:02)



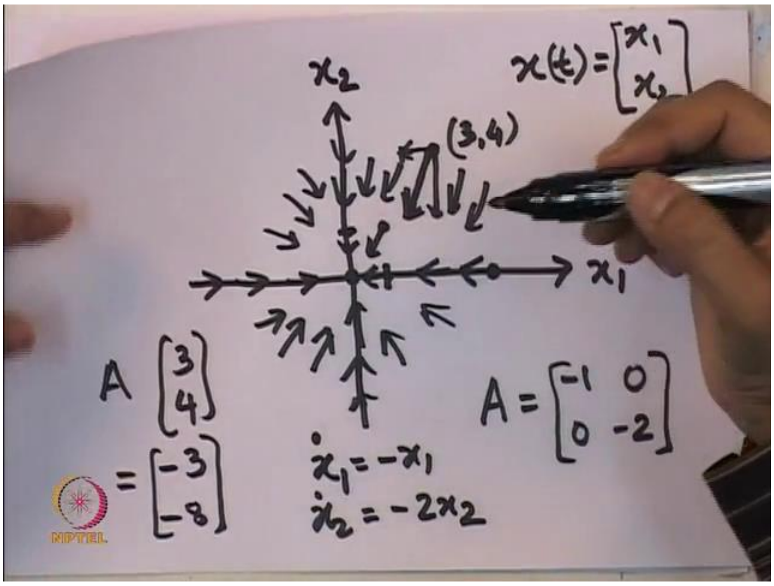
We will also need to know, what is real Jordan form? So, matrix A with complex Eigen values can at least be brought to the following form using a new basis in R 2. What form the diagonal elements have alpha, the number alpha and the off-diagonal elements have beta and minus beta alpha and beta are assumed to be real number. So, Eigen values of such a matrix A are exactly alpha plus beta alpha plus j beta and alpha minus j beta. So, in this form the non-diagonal terms beta correspond to rotation and the diagonal terms alpha correspond to whether the trajectories are shrinking on expanding. This is what we like to emphasize for the next few lectures. Also beta equal to 0 means at the 2 Eigen values are real and alpha equal to 0 means the 2 Eigen values are purely imaginary, they are plus minus j beta.

(Refer Slide Time: 41:07)



So, suppose A has distinct Eigen values distinct real Eigen values. So, let us see these cases one by one for example, take a equal to minus 1 0 0 minus 2. So, this is what we call a stable node. Why is this we will see in more detail, when we see a graph of how the trajectories evolve?

(Refer Slide Time: 41:26)



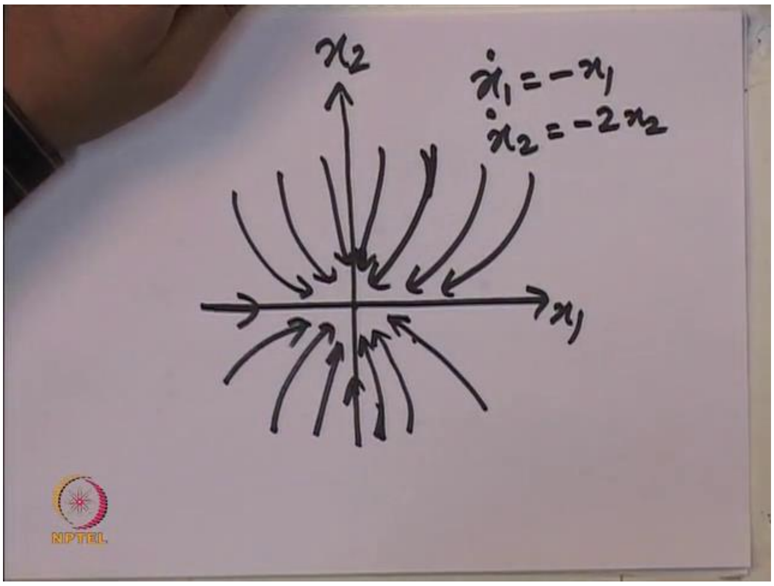
Now, x has 2 components hence the components are called x 1 and x 2 both are a function of time. So, when we are on the x 1 axis, we are dealing with the case when A is equal to minus 1 0 0 minus 2. This is the example we last saw on the slide. So, differential equation corresponding to this A because A is diagonal is a set of 2 decoupled differential equations x 1 dot is equal to minus x 1 and x 2 dot equal to minus 2 x 2. So, we see that if we are along the x 1 axis to be along the x 1 axis means x 2 component is equal to 0. Then, x 2 always remains 0, it continues to be 0. The origin itself of course is an equilibrium point because A is non-singular. The origin is a unique equilibrium point along the x 1 axis, we see that all arrows are directed towards the origin when we are along the x 1 axis that is what this differential equation says.

Similarly, along the x 2 axis also all arrows are directed towards the equilibrium point. More over these arrows along the x 1 and x 2 axis are parallel to the x 1 and x 2 axis themselves, and if we take a general point. Suppose, we take the point 3, 4 this general point. How do we determine the arrow at this point? So, we take the vector 3, 4 and make matrix A act on it. When we make matrix A act on 3, 4 vector we get exactly minus 3 minus 8. So, this point 3, 4 to determine where the arrow should go towards we will draw vector towards minus 3 minus 8 starting from this point 3, 4. So, the vector has it is some vector like this with 3 components in this direction, along the x direction and 8 components along the negative x 2 direction.

So, when we resolve this we get this vector. So, this is how all arrows are directed. So, at any point x 1 and x 2, whatever components we are given we can directly decide where the arrows lie and this is how we are able to fill the x 1, x 2 plane fill the plane R 2 with arrows. And we see that for this example all the arrows are directed towards the origin. Why they are directed towards the origin, at any point? How to draw the arrow? The x 1 component can be just determined by the x, the x 1 component of the arrow can be determined by just looking at this arrow here.

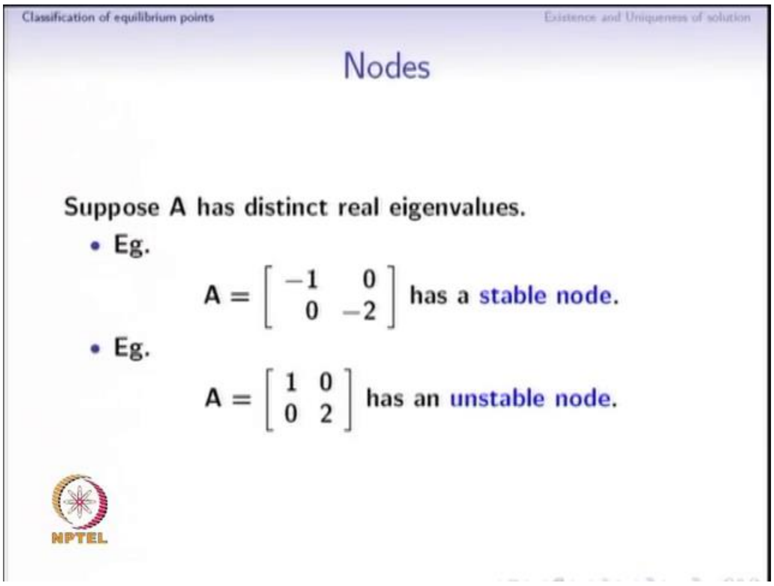
Similarly, the x 2 component of the arrow starting at this point can be determined by the arrow starting at this point and the next is just the sum of these both. This is how the arrows are drawn along the x 1, x 2 plane; on the x 1 x 2 plane. So, we see that all trajectories are directed towards the origin. Whether they are all, while the trajectories are coming towards the origin whether the arrows are directed towards the origin or not is an important issue. For this example we saw that the arrow had more direction along the x 2 axis, compared to the x 1 axis. And hence it was not directed towards the origin, but we see that the solutions will all converge towards the origin.

(Refer Slide Time: 45:31)



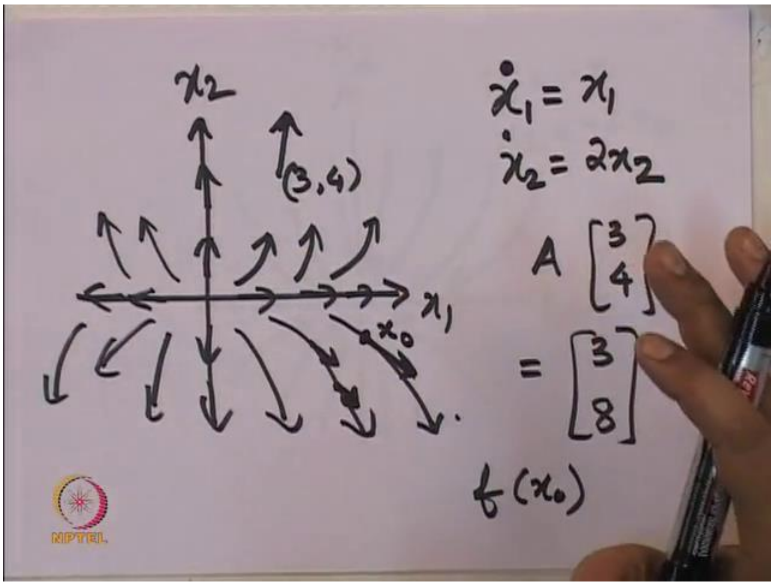
So, for the same example for which example x 1 dot is equal to minus x 1, x 2 dot is equal to minus 2 x 2. When we complete, when we draw a curve that is tangent to each of the arrows, we see that the curves look alike this solution curve. So, this we will see is the stable node. It is a node and all the trajectories are directed towards it that is why it is a stable node. The other situation is when A has 2 Eigen values negative and it is diagonalizable.

(Refer Slide Time: 46:34)



Let us go back to t is example when A has A's again a diagonal matrix and along the diagonal, there are 2 numbers which are both positive. This situation A is a unstable node let us see, this example just like we saw the previous case because it is diagonal.

(Refer Slide Time: 46:50)

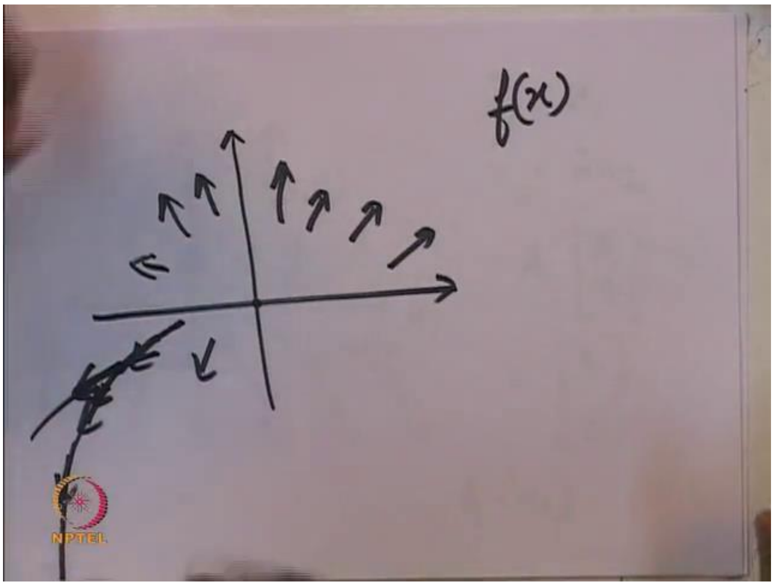


Because the matrix A diagonal x 1 dot is nothing but x 1 and x 2 dot is equal to 2 x 1 2 x 2. So, we have 2 decoupled set of equations that was the advantage of assuming A is diagonal here along the x 1 axis all arrows are directed axis, along the x 2 axis also all arrows are directed away and at any other point. Let us say, 3, 4 we can see that we can draw the arrow. How do we find the coordinates of the arrow? It is an arrow starting from the point 3, 4 directed towards which point that is to be evaluated by making the matrix A act on the vector 3, 4 at which point we are interested in finding the arrow.

So, this as we evaluated in the previous example is just 3, 8 negative of the previous situation. So, this how we are able to draw all arrows and then upon completing all these arrows into a solution we see that this is how the trajectories look. So, what is the trajectory? So, let us make note of an important property here, A trajectory is a curve such that at each point when we draw a tangent to this curve, then we get an arrow and that arrow is precisely f of x at that point.

So, suppose at this point we draw this particular tangent to this curve it is this particular arrow. Then, this arrow this vector starting at this point has some coordinates, what would the coordinates be? If this is the point x naught, it is precisely f of x naught which gives the coordinates of this vector, but starting at the point x naught. Conversely, if somebody gives us the vector field if they give us arrows drawn at various points. So, let us see this example.

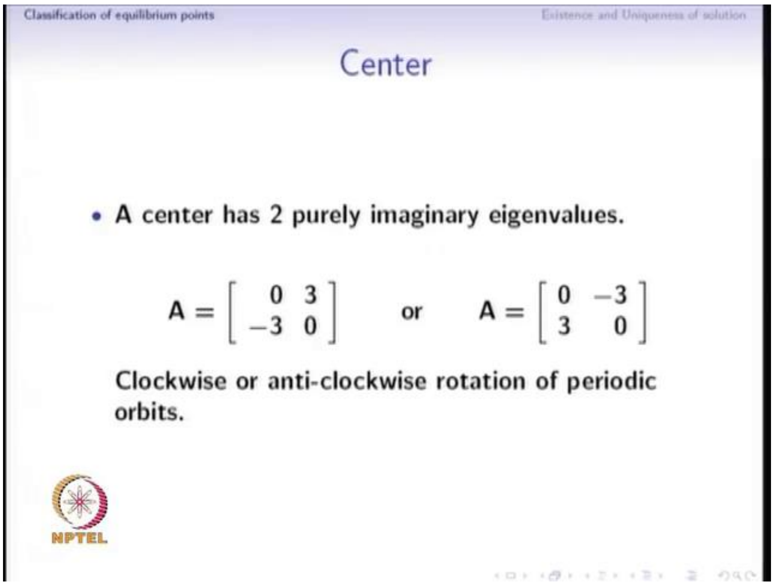
(Refer Slide Time: 49:12)



Somebody gives us arrows at various points, then what is the solution? It is we just go long these arrows and we complete all these arrows into a curve that is how the solution to a differential equation looks. We can now ask, is it possible that 2 solutions intersect? When would they intersect? At that point there are 2 different continuations, if there are 2 different continuations at this point there should be 2 arrows and that is not possible for a function, it appears, why? Because a function at every point x takes only one value one vector values it could be vector valued, but it can take only a unique value for each value of x. And hence it is not possible that at a particular point we have 2 arrows emanating out of that point this already lays to rest, the situation about intersection of trajectories.

However, from the origin there is no arrow it appears, but it turns out that we could have under non-Lipchitz property on the function f, we could have trajectories emanating out of the equilibrium point. So, that is how we should be understanding solutions to a differential equation that they are various curves, that are completion of that is a, if you had given with arrows at different points this curve is just connection of all these arrows to form a smooth curve. A smooth curve such that tangent to the curve at any point is the arrow that was specified.

(Refer Slide Time: 50:59)



So, with this we would the situation about center the equilibrium point being the center is an example, we would study in more detail from the next lecture onwards.

Thank you very much.