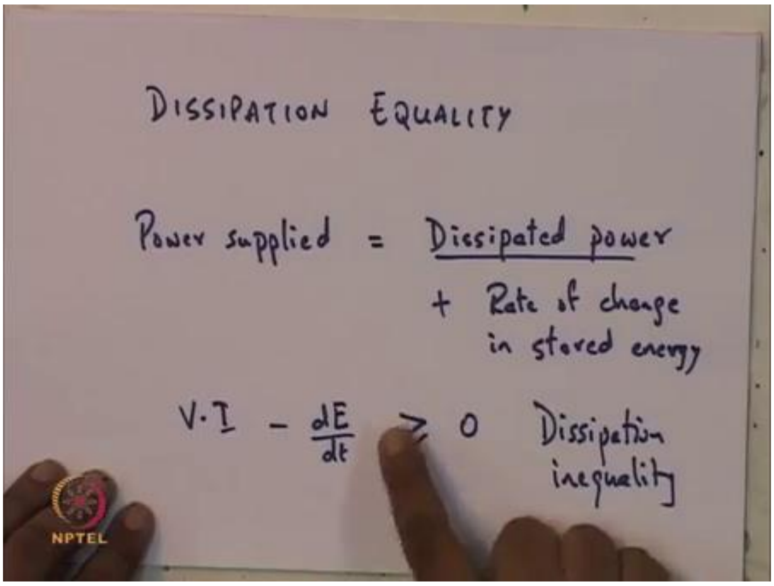
Nonlinear Dynamical Systems
Prof. Madhu N. Belur and Harish K. Pillai
Department of Electrical Engineering
Indian Institution of Technology, Bombay

Lecture - 17
Passive filters:
Dissipation Equality
Positive Real Lemma

So, in the last class I was trying to motivate what we mean by passive circuits, and the sort of conclusions that we arrived at was that passive circuits is something, where the amount of energy that is supplied to the circuits, this is always positive. And the amount of energy that is apply to the circuits is bifurcated in two ways a part of it goes into stored energy, which get stored by the energy storing elements in the circuits. The other parts get dissipate, so let me write out an equation that sort of captures this idea of passivity.

(Refer Slide Time: 01:14)



So, this equation is what is called dissipation equality and what it says is that power supplied is equal to dissipated power plus the rate of change in stored energy. Well, as far as dissipated power is concerned, this dissipated power is always going to be something which is positive. Therefore, I could take this rate of change stored energy to the other side and I can always say that power supply minus the rate of change in stored energy is some quantity, which is always going to be positive.

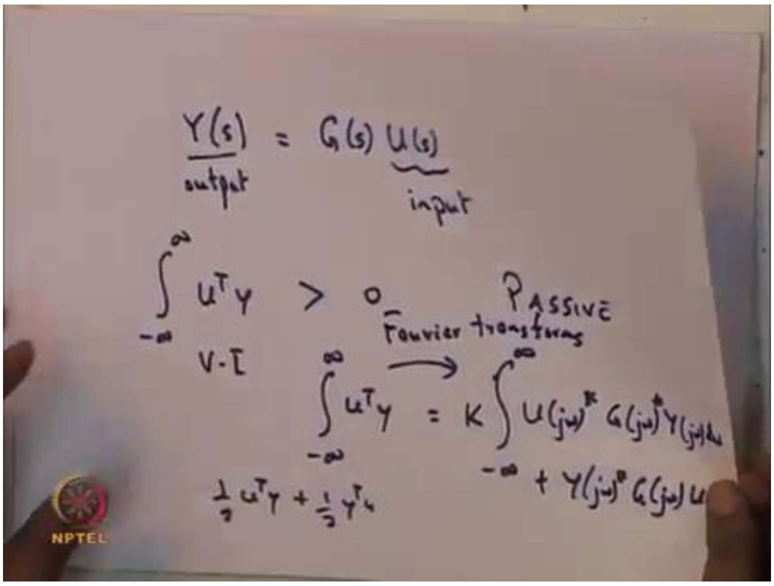
Now, in case of circuit, of course what we can do is, we can write this down in the power supply V dot I and stored energy is denoted by E then d E d t. So, v dot minus d E d t is greater than equal to 0, this greater than equal to 0, essentially this in fact equal to the dissipated power. The dissipated power is a always positive quantity, so V dot I minus d E d t is greater than equal to 20, this is in fact is called the dissipation inequality and this dissipation inequality is always going to hold true for circuits which are passive.

Now, how do we characterize circuits, which are passive? Now importantly the circuit the examples of the circuits that we were looking at it has one input and one output and typical in circuits. The number of input is equal to a number of outputs because we think of circuits in terms of ports and each port has a voltage and the current. So, one of them either voltage or the current you think of input and other one is the output.

So, the circuits are the special kinds of systems that number of input to equal a number of outputs. So, of course till now in the example that we were looking at single input single output case, but you could also look at the multi input multi output case. there, you have a whole vector of motivation current of various ports forming the input and the another whole vector of the corresponding currents voltages on those ports is the output. Then, just like we talk about the V dot I, you could just talk about the inner product of the input and output giving you the energy supplied.

There would be a storage function depending upon the number of depending upon the various energy storing elements which are there in the system. One could write out the dissipation inequality, which is the inner product of the input and output minus the rate of change of the stored energy. This is always going to be greater than equal to 0 because this quantity in fact is going to be the dissipated power and the dissipated power is always a positive quality, so let me just constrain myself to a single input and single output case.

(Refer Slide Time: 05:22)

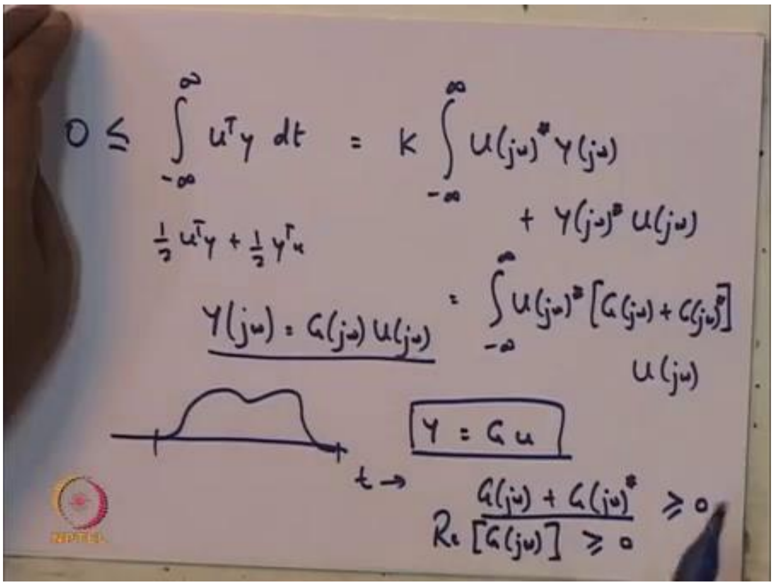


Let us say the single input and single output situation is given in using the transfer function Y s is equal to some G s U s where Y is the output and U is the input. Now, earlier what we had said was that U transpose Y this is the same in the case of the electrical circuit v dot i is exactly the same as U transpose y. We are saying that U transpose Y integral for minus infinity plus infinity this particular quantity should be greater than 0. Then, the system is passive, now if you going to take the integral from minus infinity plus infinity of U transpose Y, one could take the Fourier transforms and one takes the Fourier transforms.

This is the saying as there would be some scaling factor k and minus infinity plus infinity of U j omega star times and G j omega g j omega star Y j omega, but of course U transpose y. I can download half of U transpose Y plus half of Y transpose u, so it a sort of making it symmetric, therefore along with the expression, this other expression also which is Y j omega star G j omega U j omega.

So, this integral of course, this d omega, so this integral would be equivalent to this integral after having taken the Fourier transforms. So, we have taken Fourier transforms, now if you look at this expression, this is the same, well I think I have made a mistake sorry, so let me do it again.

(Refer Slide Time: 07:56)



So, if you looking at this from minus infinity to infinity of U transpose Y d t, this is in the time domain this is equal to some constant of proportionality. Of course, the U transpose Y the U transpose y. I could write it out as a half U transpose Y plus a half Y transpose u, now I could pass over using Fourier transforms into the into the j omega into the omega domain. I will have again integral minus infinity of U j omega star Y j omega plus Y j omega star U j omega. So, last time the mistake I made was I put this G j omega in between, now I can bring in this G j omega by substituting this Y once you have taken the transforms Y j omega is really the G j omega times the U j omega.

So, substituting this in there, we would end up with integral minus infinity to infinity plus I am just forgetting the proportionality constant, you have U j omega star multiplying G j omega plus G j omega star or multiplying the U j omega. Now, of course when I write down the assumptions that I am going in the assumptions following, for example, we will assume that U is a compactly supported trajectory.

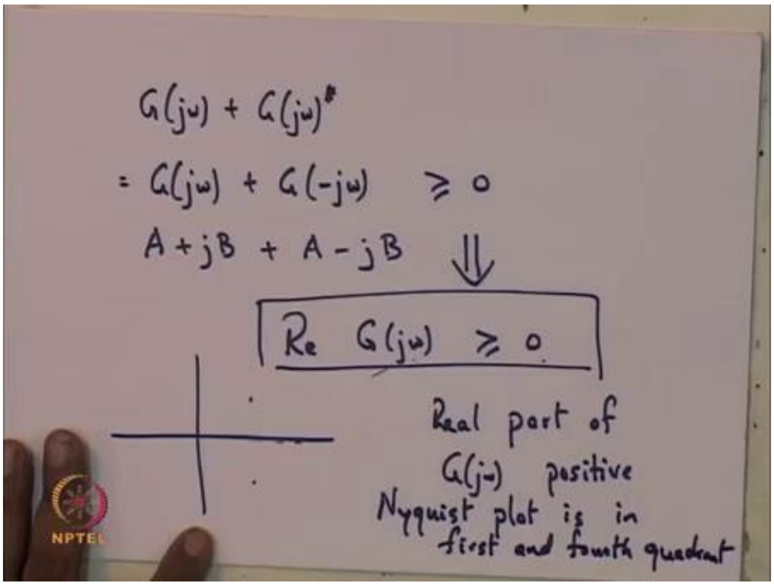
That means if this is the time axis U is non zero only over a compact set, now if U is non zero over a compact set, then it is you need to take the Fourier transform. Now, it makes sense to take the Fourier transform, now if U is compactly supported because the transfer function that tells us that Y is equal to G times u, this is the equation. Therefore, the Fourier transform of Y also makes sense and if U is compactly supported in the circuit.

We would assume we can expect that Y is also compactly supported and so then it makes sense to take the Fourier transforms.

Now, if you take the Fourier transforms the Fourier transform of Y and the Fourier transform of U is related in this way. So, once you have this expression, you substitute and you get this and now this is this integral is greater than equal to 0. So, if you look at this last expression whatever U you choose this expression must be positive and now one can show that this expression can only be positive if G j omega plus G j omega star is greater than equal to 0. This here is really the real part of G j omega and so the real part of G j omega must be greater than equal to 0.

So, this condition is a condition for passivity the condition for passivity translates and here of course, we have consider single input single output case. So, in single input single output case Y and the input and output related through transform function G of s and then what this translates to by the set of manipulation is the set of real part of G j omega must be greater than equal than 0. So, let us look at situations that the real part of G j Omega is greater than equal to 0, what is does mean?

(Refer Slide Time: 12:20)

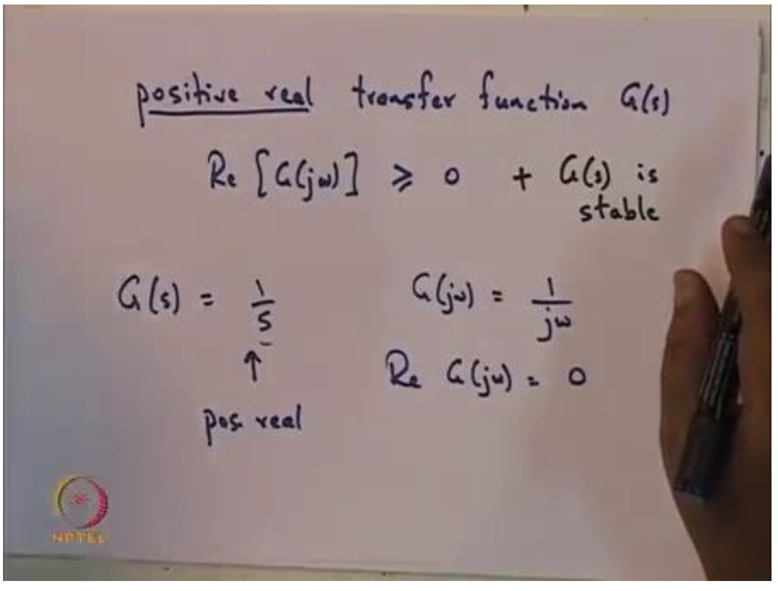


Now, you see expression we had is G j omega plus G j omega star, but this is a same as same G j omega plus G minus j omega. Now, if G j omega is can be written as a plus j b then this will turn out to be a minus j b, so the imaginary part gets cancelled out and your just left the real part. So, this being greater than equal to 0 is same as saying that the real part of the transfer function G j omega must be greater than equal to 0, how to translate this condition into something more meaningful for us. One way we can translate this condition into something more meaningful is you see the image of G j omega is an micro spot.

So, saying that real part of G j omega is greater than equal to 0 is the same as saying that the Nyquist plot of the transfer function should lie in the first and second quadrant because then the real part of G j omega is going to be positive. So, what this translates to whatever we have been doing till now, this result we have got what it translates to if the real part of the transfer function G j omega should be positive. What that means is that the Nyquist plot is in first and fourth quadrant first and the fourth quadrant, but as it turns out this Nyquist plot being in the first and fourth quadrant is not a complete characterization of passivity.

That is because mathematically the Nyquist plot lying in the first and the fourth quadrant may not necessarily translate into that particular condition that we had, where you have a storage function and supply function. That interpretation that you have, first let us look at some examples of transfer functions, which are positive real, now sorry I did not mention that. So, those transfer functions whose Nyquist plot lie in the first and fourth quadrant well one definition for that is positive real.

(Refer Slide Time: 15:46)



So, positive real transfer function G s we search that the real part for G j omega is greater than equal to zero, I cannot claim that this is the definition of positive real because its depending upon various books the definition of positive real changes. Now, the real interpretation that we had for passivity the one would like to say that positive real functions is equivalent to passive, but that is actually strictly not true.

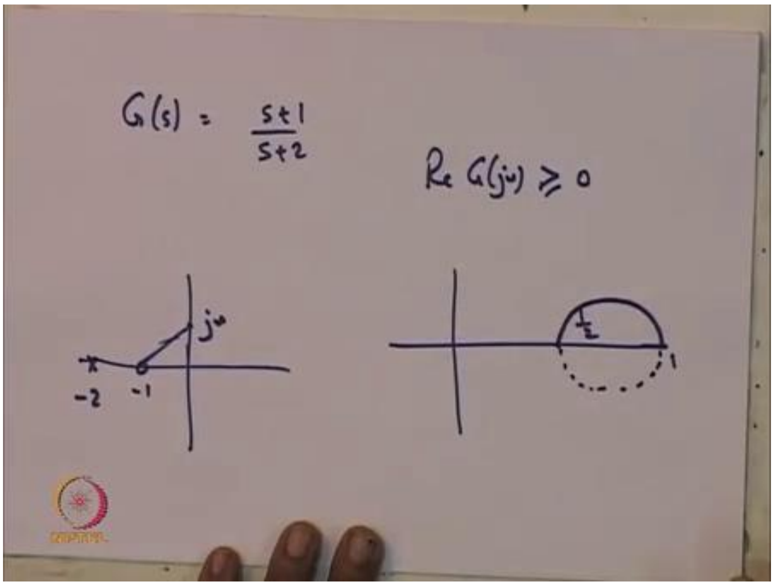
If the definition of the positive real to be that the real part that the Nyquist plot lies first and fourth quadrant, then one can show that there are some transfer functions, which will satisfy the conditions of positive reality that are not actually passive. So, it turns out in many books they use the definition of the positive reality to be the fact that Nyquist plot lies in the first and the fourth quadrant.

In many other books, the definition for positive reality says that the real part of the transfer function in evaluate that means Nyquist plot of lies in the first of the fourth quadrant that in addition the transfer function is stable. Now, I would give some examples to try and tell you what exactly the difference between defining positive reality is. In this way and defining positive reality in in the sense of this plus G s being stable, so let us look at some examples suppose we look at some transfer function G s equal to 1 by s.

Now, this transfer function is it positive real, well if you take just this definition then G of j omega is 1 upon j omega. Therefore, the real part G of j omega is equal to 0 and so the real part is greater than equal to 0, so one can say this transfer function is positive real. If one uses the definition that positive real means this plus G s stable one uses that one uses this definition. Then, again this particular transfer function is positive real because real part is actually 0 and this is also stable means s being a pole at 0.

If you think of that stability a marginal stability whatever is then one can still continue to call this positive real. So, the definition of positive reality could be just this or could be this along with G s be stable, again you know as I said earlier it depends on the books that you follow some books use only this definition plus. Most books use this definition the fact G s is stable, so let me use a few more examples to show Y there is a difference between just having this condition which is the condition that got, earlier equations that also having this condition that G s is stable.

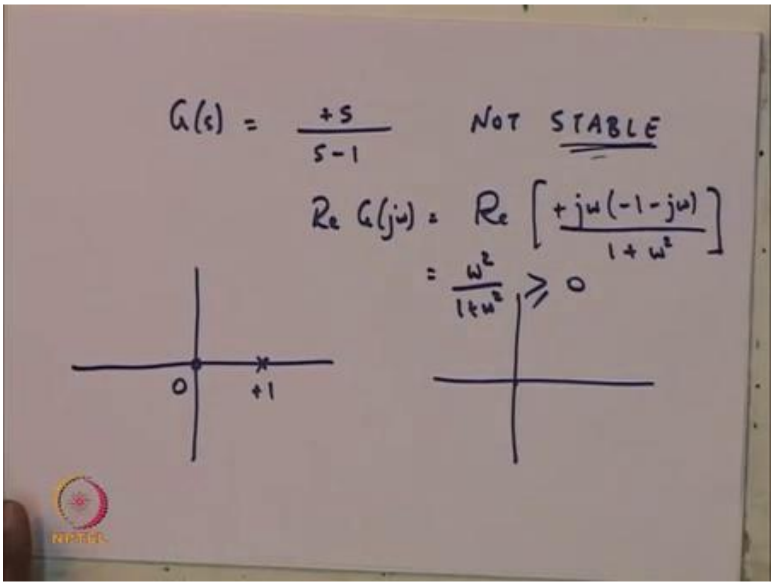
(Refer Slide Time: 20:00)



So, let me take another transfer function G s is s plus one upon s plus 2. Now if you look at the Nyquist plot of this, the Nyquist plot of this at s is equal to 0 is at half and then so this particular transfer function in as a 0 at minus 1 and it has a pole at minus 2. So, as you go up you find at any particular G omega at this angle, which is large, then this angle and this going is positive and as j omega that tends to infinity, this finally tends to 1, so you end up with a Nyquist plot, which looks like this to one and then then you look at the rest of it this is what you get.

It is clear then the real part of G j omega is greater than equal to 0, so by the definition the real part of G j omega is greater than equal to 0 transfer function is positive real. Of course, if you also put in the fact that it should be stable while this transfer function is also stable, therefore by both the definitions, this transfer function is positive real.

(Refer Slide Time: 21:35)



Let me now take another transfer function which is G s is let us say s minus 1, now this transfer function of course is not stable. Now, if you want to draw the Nyquist plot of this it has a pole at minus 1 at plus 1 and it has a 0 at the origin. Now, if you plot this thing as you go up, so at omega equal to 0 this gives you 0 and then as you G omega what you get is the real part of G j omega is equal to the real part of minus j omega, multiplying by minus 1 minus G omega upon 1 plus of omega squared. Then, this term turns out to be perhaps I should use a minus, I should just put plus, so if I take this transfer function s upon is minus 1, this is not stable.

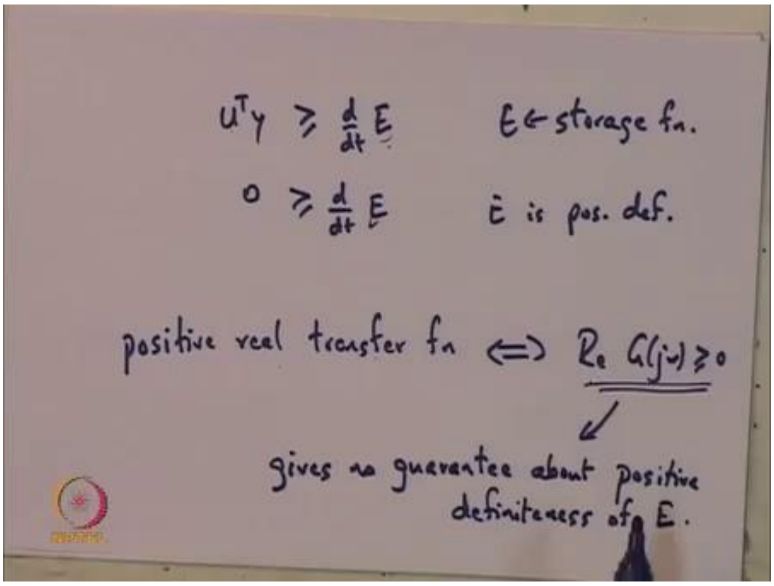
When you calculate this is plus and so you get one plus single squared and so in the numerator, so the real part will turn out to the omega squared upon 1 plus squared and this quantity here greater than 0 of all omega greater then equal to 0 for all omega. So, if you use the definition that the real part of the transfer functions is greater than equal to 0, then this thing is positive real this transfer function is positive real, but in an addition you also put in the condition of stability this is not stable.

Therefore, this transfer function is not positive real, now the question is why was this condition is stability brought in to associate the transfer functions with passivity, now that is completely dependent on what one would call the storage functions.

So, it turns out that if you have a transfer function like this you are not going, it would not be possible to synthesize a circuit which has this transfer function using purely passive elements. The reason for that is because the G of s is not stable, as a result the associated storage function that you would get that you can get for this particular transfer function that storage function is not going to be positive definite. So, let me explain what I mean by that, so earlier I had said that the storage function is the bit like a Lyapunav function.

That means if you set the input to 0 and then you look at the system, then the storage function plays the role of the Lyapunav function, now the Lyapunav function of course has to satisfy certain conditions. One of the condition is the Lyapunav function must be a positive definite function and its derivative must be negative definite, now the fact that the derivative is negative definite will get satisfied because of dissipation inequality.

(Refer Slide Time: 25:56)

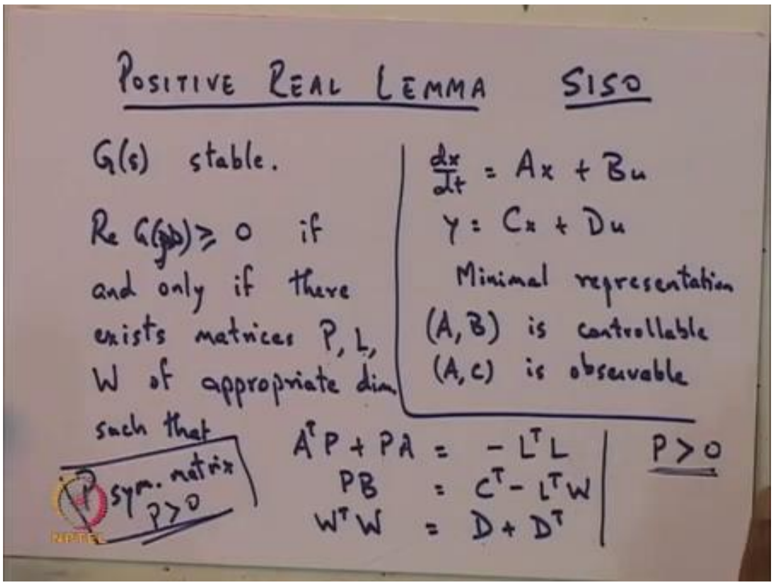


So, just recall that the desipation inequality gives us something like this U transpose Y is greater than equal to d d t of E where E is the storage function, but now what does it mean if the input is 0. Of course, this quantity is 0 and so 0 is greater than d t of e, which means for using system with 0 input this E can act like a Lyapunav function provided of course E is positive definite.

So, E is positive definite, here this thing essentially tells us that the derivative of E is negative semi definite and as a result the resulting system, the system that you get by setting the input to 0 is a system, which is stable. Therefore, it is important that this storage function that you get should be positive definite, now then we use the definition of positive real transfer function to be equivalent to the real part of the G j omega greater then equal to 0.

Then, this condition alone gives us no guarantee about the positive definite of this storage function you said this condition gives no guarantee about positive definiteness of e. So, that is why a many people would like to call other transfer function positive real when not only this condition is satisfied, but it also guarantees that this storage function could result in the system is a positive definite storage function. Now, in fact the guarantee for that is given by a famous lemma, which is called the positive real lemma, so let me stay the positive real lemma.

(Refer Slide Time: 28:42)



So, the positive real lemma, so let us take a transfer functions G of s which is stable and let us assume that this is transfer function this written in as space state model and so you have equations d x d t equal to a x plus b ay equal to c x plus d u.

Let us assume that representation is a minimal representation, so minimal representation essentially guarantees that a b is controllableand a c is observable. So, what we are saying is we start out with transfer function of course I am just restricting myself. Now, in single input and single output case, you are taking a transfer function, which is stable look at the states place representation and the states place representation. We are looking at is a minimal representation, which means the a b c d matrices such that a d is controllable and a c is observable.

Then, the real part of G s is greater than equal to 0, so G s is already stable that we have assumed then real part of G s is greater than equal to 0 if only there exists matrices p l w of appropriate dimensions. So, what we are saying is that the real part of G s is greater, sorry the real part of G j omega is greater than equal to 0. That means the Nyquist plot is going to lie in the first and the fourth quadrant if and only if there exists matrices p l and w of appropriate dimensions such that the following three equations hold.

The three equations are a transpose p and p a is equal to minus l transpose l p times b is equal to c transpose minus l transpose w and w transpose w is equal to d plus b transpose. Let us look at this three equations and think about it for a minute, so what we are saying is the positive real lemma what it saying is suppose if you start off with G s which is stable and look at the states place representation, which is a minimal representation. This essentially and equivalent to a b controllable and a c observable, then the real part of G j omega is greater than equal to 0.

In other words, Nyquist plot lies in the first and fourth quadrant if and only if these matrices is p l and w of appropriate dimensions such that which is the system matrix. Here, a transpose p plus p a of course incidentally this matrix p is going to be asymmetric matrix. So, p is going to be a symmetric matrix, therefore a transpose p plus p a is going to be symmetric and what we are saying is that this matrix is going to be minus l transpose l.

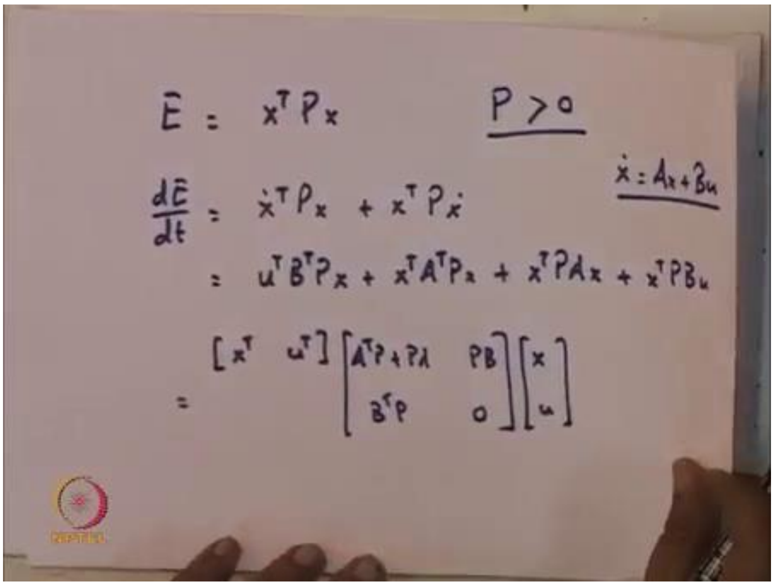
Now, if you take any matrix l and if l transpose l that is going to be positive semi definite I mean it is guaranteed to semi definite if you could be positive definite. If l had the full appropriate rank, but it is guaranteed to be a positive semi definite what the first equation is saying is that a transfer there is some matrix symmetric matrix p such that a transpose p plus p a is negative semi definite.

Now, this of course already connects p and l, because p as a the next equation connects p and l and w through b and c like b and c come from the state representation. It says p times b equal to c transpose l transpose w and this w is something purely depends on d, so if you d plus b transpose, so this is a this is going to be a symmetric matrix and that symmetric matrix is exactly the same as the w transpose w. So, the positive real lemma is something that holds only if you assume G of s stable if you take away this assumption.

That means you do not assume G s as stable, then this condition that real part of G j omega is greater than equal to 0 if and only if these things that this does not hold any more. Now, the point is that this p essentially defines it essentially defines the storage function, so there is one more connection that I need to add which is there exists matrices p l and w with p being positive definite. Now, if the condition G s is stable is put in, then this p is positive definite, but if this condition G s is stable is not put in.

Then, this p is not guaranteed to positive definite, so the positive real lemma says that if you take a G s which is stable and this is a states representation it is in the minimal representation. So, ab is controllable and a c is observable than the real part of G j omega is greater than equal to 0 if and only if there exists matrices p which is a symmetric matrix and positive definite and to other matrices l and w to appropriate dimension such that this all this is true. So, p is a symmetric matrix and p is greater than equal to 0, then all these equations are satisfied, what does all this mean? I mean I have just stated a lemma all positive real lemma, but what does this finally mean, what this finally, means is that you couldthink of this storage function for the system.

(Refer Slide Time: 37:08)

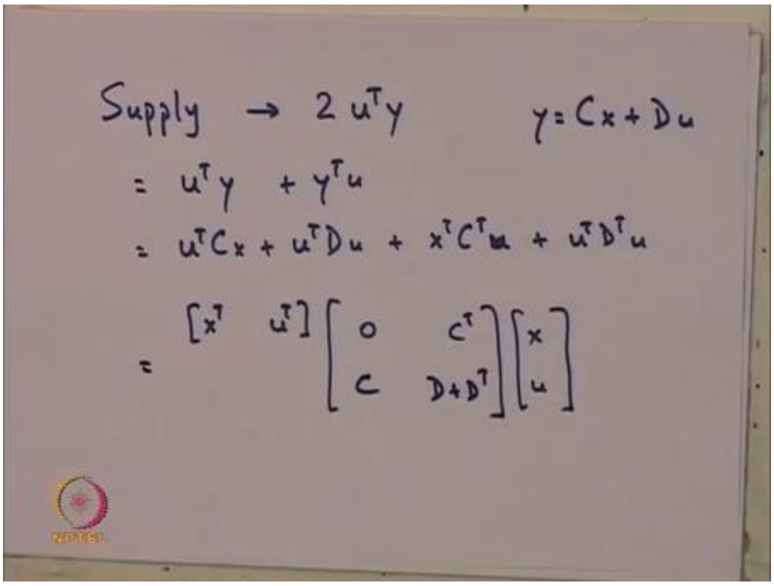


So, this storage function E one could think of this Storage function as x transpose p x where p came from that positive from the earlier lemma. So, p is greater than 0, then we talking about so if p is a positive definite, then we are talking about a storage function which is positive definite. Now, if you have the storage function and you look at d E d td

E d t is equal to x dot transpose p x plus x transpose p x dot for x dot, we substitute a transpose b because we have this state equation in x dot is a x is b u. So, if I substitute this in for the x dot and you would end up withu transpose b transpose p x plus x transpose a transpose p x plus x transpose p a x plus x transpose p b u.

I will write this in a more compact form x transpose U transpose, then I have matrix here and then I have x and U and here I would have a transpose b plus p a, then I would have p b here I would have b transpose d here 0 here. So, this expression is the same as this, so what we have got is explain for d E d t by just using the state equation and using that p that had appeared in the positive real life and I get this.

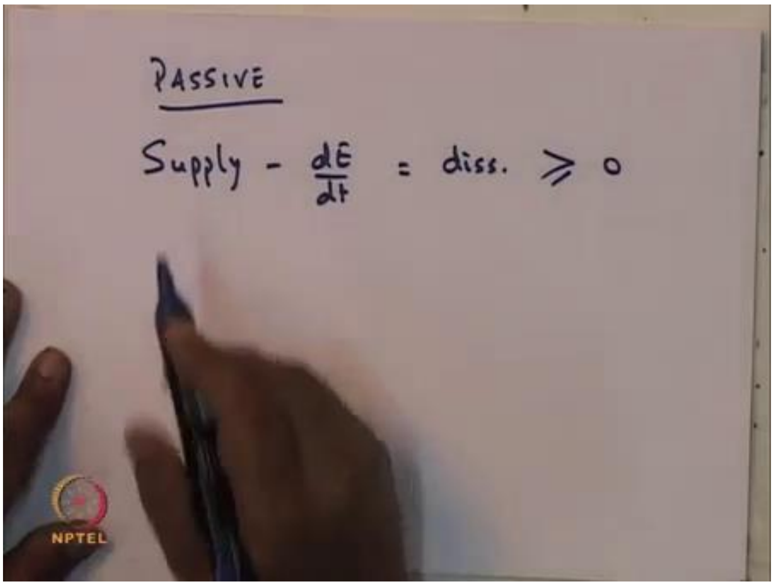
(Refer Slide Time: 39:22)



Now, throughout we have been saying that the supply is given by U transpose Y, so a moment here you see I have taken this storage function to be x transpose p x. Ideally, this is a quadratic one would put a half here and so what would happen is there would be a half appearing in all these terms, because of the half that I have put there. So, I will compensate for that by taking the supply to be 2 times U transpose Y and you would see that this is with constants, it really does not matter that much. Now, the supply is 2 times U transpose Y, but from the state space equations, we know that Y is c x plus d u.

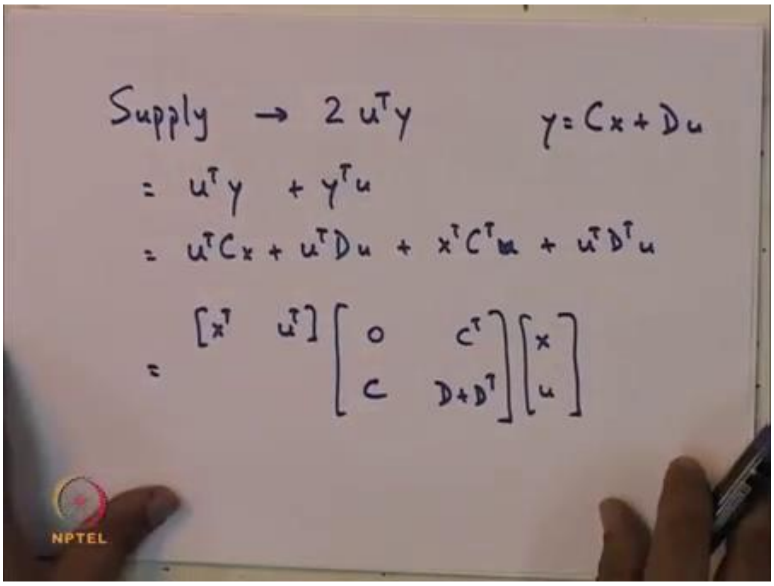
So, the supply which is U transpose Y I write it in the symmetric form, so I write it as U transpose Y plus Y transpose u. So, I plug in this c x plus d U in here and I would end up getting U transpose c x plus U transpose d U plus x transpose c transpose x U plus U transpose d transpose uwhich again just like earlier I would write down as x transpose U transpose some matrix x u. There is no x transpose that term in not there I have x transpose c transpose U and I have seen here and here I have d plus d transpose, so the supply turns out to be the expression here.

(Refer Slide Time: 41:44)



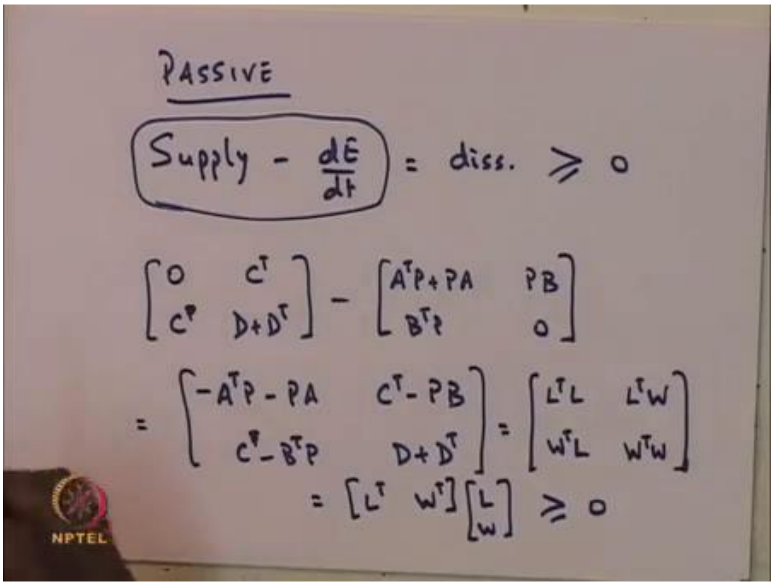
Now, for passive systems, we said supply minus the rate of change of energy this must be equal to dissipate, but this dissipation is of course greater then equal to 0.

(Refer Slide Time: 41:57)



Now, we found in an expression for the supply, so if x transpose U transpose that this matrix x U this vector we are using only the either side of this matrix, so I will just use this matrix and sort of suppress this x u.

(Refer Slide Time: 42:18)



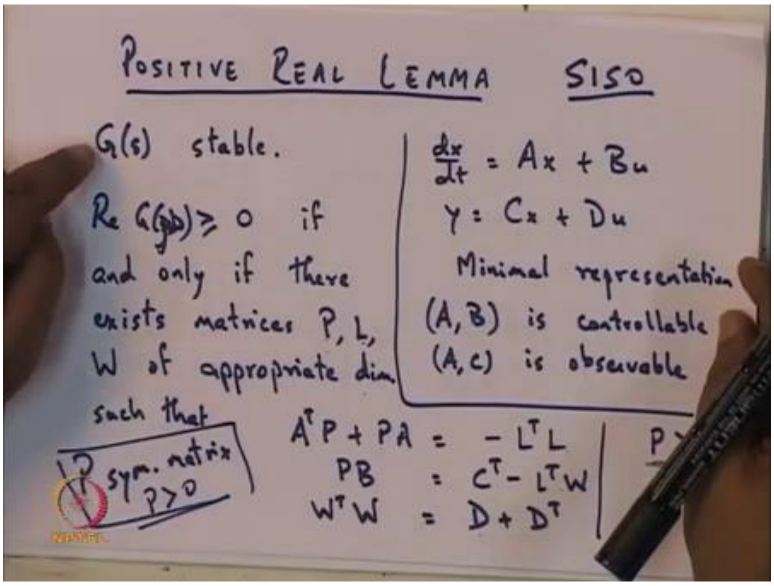
So, if I write down the supply matrix I get 0 transpose c transpose d plus d transpose minus now d E d t again I had a similar expression using x transpose and U transpose which must be this. So, if d E d t is particular expression, so I can subtract this expression from the earlier expression, so what I would get is minus a transpose p plus p a p b b transpose p and here 0. This whole matrix put together pre multiplied by x transpose U transpose multiplied by x U this must be what I have written down here is supply minus d E d t.

So, this whole thing must be equal to dissipative dissipated energy and so this matrix must be a positive definite matrix is what we should have if the given system was passive. Now, if you put this together then the resulting matrix that you will have the first entry is going to be minus a transpose p minus p a.

Let me write it down minus a transpose minus p a, then here I will have C transpose minus p b here ill have c transpose, no both of them or not c transpose one of them the bottom one is c. So, this is mistake c minus b transpose p and here I have d plus d transpose, so this particular matrices is the same as this, now I am going to use the expressions that I have derived that I had written down in the positive real lemma. You see minus a transpose p minus p a is look at the positive real lemma and a transpose p plus p a is minus l transpose l.

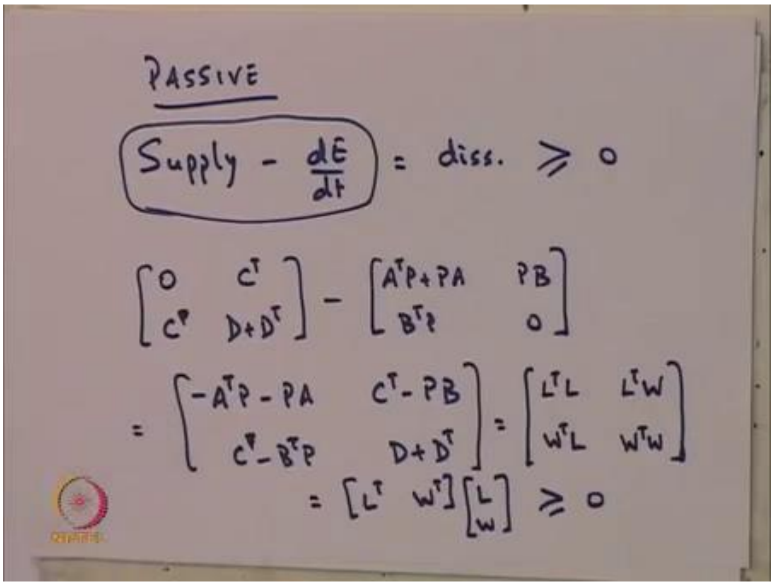
So, the negative of this is l transpose l, so for this I could write down l transpose to know if I look at the second equation its say c transpose minus p b is l transpose w. So, for c transpose w is p bi could write down l transpose w and this is just the transpose that this is w transpose l and then the last equation tells us w transpose w is d plus d transpose. So, this I could write as w transpose w, but what this matrix is this is nothing, but l transpose w transpose multiplying l w and this is certainly greater then equal to 0 because this essentially like squaring a matrix.

(Refer Slide Time: 45:36)



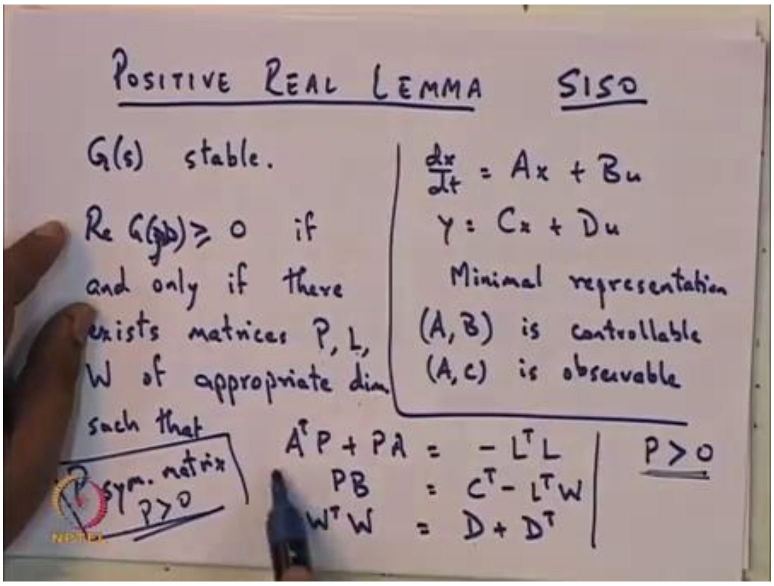
So, what we are essentially saying is if you go back to the positive Real lemma, then you are saying G s is stable and this is true. Then, it says that there exists all these matrices p which is asymmetric matrix greater than 0 such that these conditions and the other two matrices l and w such that all these conditions are satisfied. Now, what I have just shown here is if you start by considering E this storage function if you start like considering the storage function to be x transpose p x by p is obtained from that positive real lemma.

(Refer Slide Time: 46:21)



Then, if you think of supply as U transpose Y, then when you take supply minus the rate of change of storage, then the end of getting this equation and manipulate that if that the that thing is going to be greater than equal to 0.

(Refer Slide Time: 46:44)



So, what we are essentially shown is that in the positive real lemma if all these conditions hold. Then, if all these condition hold, then you have passivity, but as far as this lemma is concerned we have not shown either way what we have shown is if these conditions are satisfied, then the system is passive. Now, I seem to be out of time for this lecture and show in the next class I would show the positive real lemma and the implication the necessity and the sufficiency I will prove in the next class.