Nonlinear Dynamical Systems
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Lecture - 16
Interconnection between Linear & Non-Linearity, Passive Filters

So, you would have already seen how to handle non-linear systems, which do not have an input. Now, a very clever thing, which was done, was that when u try to analyse these non-linear systems which do not have an input, you can actually decompose that into two separate things. One a linear system and the other you pack all the non-linearity together. Then when you want to talk about the stability of the equilibrium point of the original non-linear system, that is equivalent to talking about the stability of this interconnection between the linear system and that non-linearity. So, let me try and explain what I mean by this.

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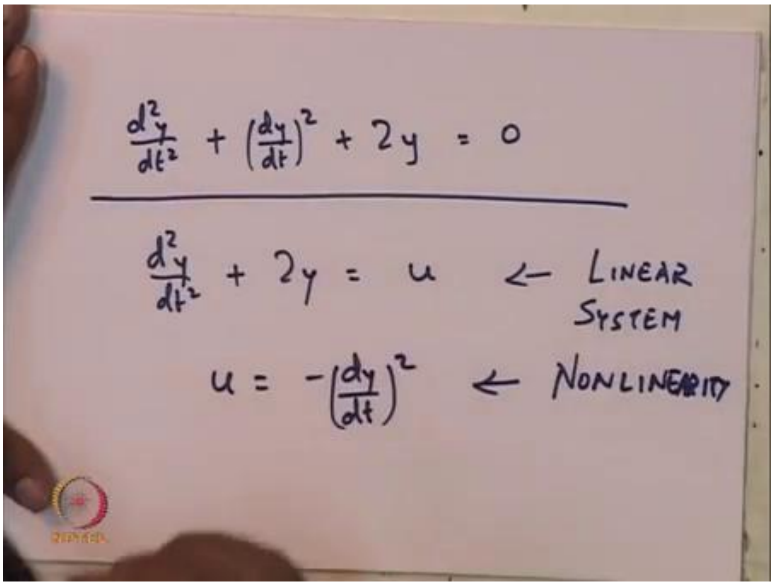

Suppose you were to look at an equation like let us say the second derivative of y plus, let us say the first derivative of y squared plus, let us say 2 times y equal to 0. Now, one wants to analyse this system. This system is a system without any inputs. There are well known methods of trying to analyse this system, but what I was just talking about, was this clever way of thinking about this system as an interconnection between a linear system and a nonlinearity.

Now, let me try and give you how that is done. So, think of this equation d 2 y d t squared plus 2 y equal to u, this now is a linear system. Now, this is exactly like this equation except that you could think of this d y d t squared as minus u. So, let me put that as the next equation which is u equal to minus d y d t squared, this is a non-linearity. Now, the interconnection of these two systems, then would give us exactly the same as this original non-linear system. So, let us look at what this interconnection would look like.

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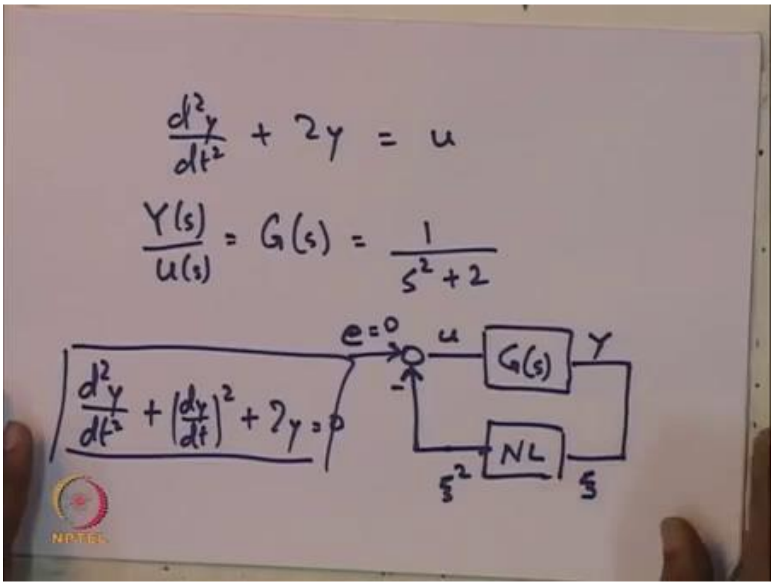

So, if you look at that system d 2 y d t squared plus 2 y equal to u, this linear system which has transfer function given by y s by u s. This is the transfer function G s which is equal to 1 upon s squared plus 2. Now, let us think of this G s in this way and here is the input that you give which is u and the output that you get for this which is Y. You take this y and put it to a nonlinearity and this nonlinearity has the characteristics that, if the input to the nonlinearity is psi the output is psi squared.

Now, in this case if we... Now, feedback this thing with a negative feedback, then if this e which is the input of this feedback loop, if this e is set to 0. Then what we would be looking at, is exactly the original equation that we had which is d 2 y d t squared plus d y d t the whole thing squared plus 2 y equal to 0. So, talking about this close loop system being asymptotically stable for example, is exactly the same as saying that the original non-linear system is asymptotically stable.

Now of course, in this system we can clearly make out that, this system would have the, I mean if you draw face claim portrait of the system, then it origin is an equilibrium point. Now, the origin is the equilibrium point if one wants to find out whether this origin is, let us say asymptotically stable, that is the same as in this feedback system, whether that given system this feedback system is asymptotically stable.

So, if you set up this feedback system and set y with some value and then you let this system evolve. If this system evolves finally, settling down to the value y equal to 0 u equal to 0. Then what that would mean is this system asymptotically stable and what that effectively means is the original non-linear system is effectively stable. Now, this clever trick is, I mean once this clever was discovered it was used again and again. So, the non-linear systems were analysed by looking at a splitting it up into a linear system and a non-linear part and looking at the feedback connection of this linear system and this non-linear part. Now, once that was done, then people came up with some sort of a conjecture. Now, so let me give you what this conjecture was all about.

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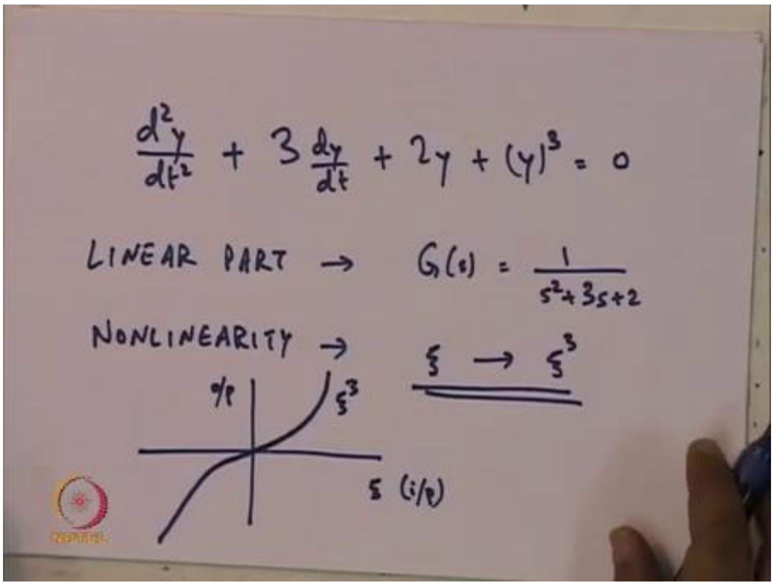

So, suppose you had a system like let us say similar to the earlier one, but d 2 d y squared by plus let us say 3 times d y d t plus let us say 2 times y plus y cubed equal to 0. So, this now can be split up into a linear system. So, the linear part would be given by a transfer function, which is 1 upon s squared plus 3 s plus 2. The non-linearity that you would interconnect with this linear system is given by something were if psi is the input the output is phi cubed.

Now, if one looks at the nonlinearity carefully this non-linearity. So, suppose I think of psi which is the input to the non-linearity, then the output to the non-linearity is here and this looks like something like that, that is psi cubed. Now, the conjecture, which was made was made by a person called Aizerman. Aizerman made the following conjecture that, suppose you have a nonlinearity and let us assume that this nonlinearity is memory less.

So, what do I mean by memory less what I mean by memory less is that this nonlinearity it only the output of the non-linearity only depends upon the instantaneous input. Now, the earlier nonlinearity that we looked at where the input was psi and the output was psi cubed. This output was independent of what the past values of inputs were. So, it is sort of the output only dependent on instantaneous input. So, one could think of this nonlinearity has a memory less nonlinearity.

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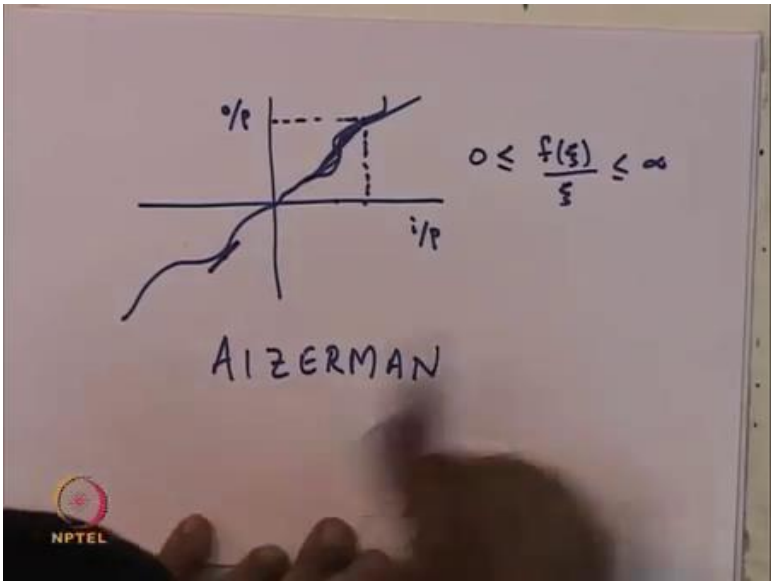

So, typically if you look at a nonlinearity which is memory less, it might be you know something like that. So, this here is input to the nonlinearity and this is the output to the nonlinearity. So, when you have this as the input then the output is this value here. Since it just depends on the. So, this curve of course, is a slightly twisted curve. So, it looks like you know with this input, there might be multiple values, but let us assume that the curve was straight enough so that such a situation never allows.

So, the output purely depends upon the instantaneous values of input. Now, the conjecture was made by this person Aizerman and his conjecture was the following. You see if you look at this nonlinearity at each point, there is some tangent to the curve. Now, one way that you could talk about this nonlinearity is you could limit the nonlinearities.

So, this particular for example, one could say that 0 is less than or equal to f of psi by psi which is less than equal to infinity. So, this is a nonlinearity whose slope instantaneous slopes lie between 0 and infinity. Of course, I am sorry, but initial curve probably had had slopes, which were greater than infinity, but this one is some trajectory and this is a nonlinearity which has it slopes lying between 0 and infinity. Now, if one looks at such nonlinearities.

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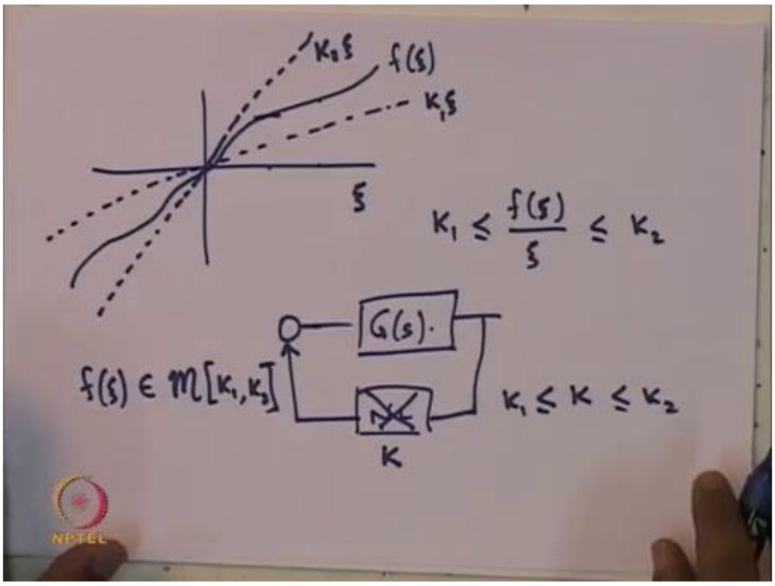

So, let me give some more examples of similar nonlinearities. So, you could and have a non-linearity like that and then clearly all the slopes of this non-linearity lie between this line. So, if I call this input psi. So, this line is k time psi or k 1 times psi and there is another slope here. So, this is k 2 times psi. So, this particular nonlinearity one could say satisfies the characteristics f psi by psi is less than is greater than this particular slope k 1 and is less than this other slope k 2.

Now, suppose we interconnect this non-linearity. So, this non-linearity is interconnected with a linear plant G s in this feedback structure. Then suppose this linear plant was such if instead of the non-linearity, you use the linear gain k and this k was between k 1 and k 2. The resulting linear system was stable, then Aizerman's conjecture was that, if you put the nonlinearity instead of that linear gain, then the resulting system would also be stable would be asymptotically stable.

So, the idea being that, if you are at this point and this was the input, then there is some slope here correct. Now, this slope when you put that linearity, the resulting system is stable or asymptotically stable. Now, if the resulting system is stable then the guess was that, for the nonlinearity because locally it is like this linearity here. Therefore, it will behave nicely. Now, as a result of course, the system evolves and you go to some other psi and at that psi there is some other slope.

Now, this slope when you put in as linear thing, then again you get an asymptotically stable system asymptotically stable linear system. Now, working in this way, the guess was that if you had a non-linearity which lay in some sector. So, this particular non-linearity f of psi is set to lie in the sector. So, it is a non-linearity which lies in sector k 1 k 2. So, if you have a nonlinearity lies in the sector k 1 k 2, then if the linear plant is such that, when you put feedback gain, any gain between k 1 and k 2 and the resulting system is asymptotically stable. Then if instead of that linear feedback you put the nonlinearity in there, then the resulting system is going to be asymptotically stable. So, this was Aizerman's conjecture. So, perhaps I should just sort of formally write it down.

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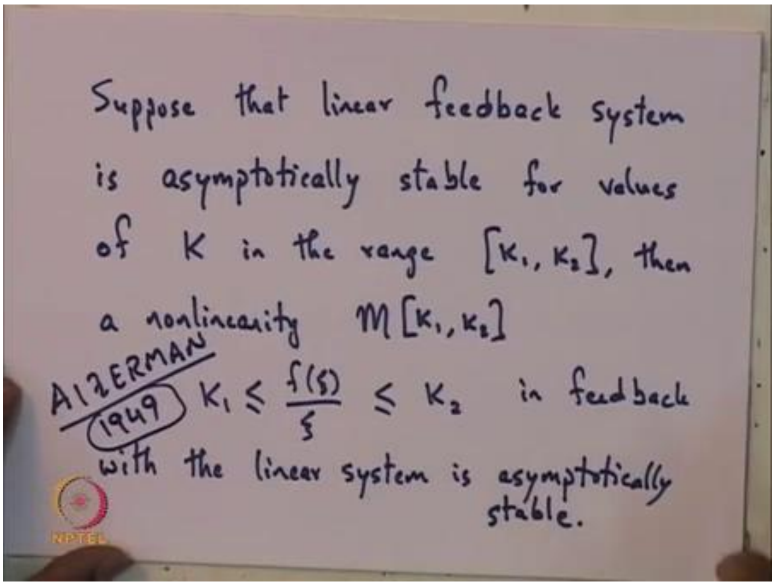

So, suppose that the linear feedback system is asymptotically stable for values of k in the range k 1 to k 2. Then a nonlinearity that belongs to this sector, which is equivalent to saying that the nonlinearity f psi by psi, this is less and I mean k 1 is less and equal to this and which is less and equal to k 2. Then a nonlinearity of this kind in feedback with the linear system is asymptotically stable.

Now, this is Aizerman's conjecture and this was given roughly in 1949. Now, initially it was not clear whether this way of looking at nonlinearities as approximation of linear systems is going to work. In fact in the literature for some time this used to be call the method of linearization because the idea was that, you have this nonlinearity and at instantaneous at various instance for given input there is an output and there is also the slope of the output, according to the characteristics of the nonlinearity.

Then Aizerman's conjecture essentially said that, if for all those slopes that you get in the non-linear I mean if for all of those, the nonlinearity is in that particular sector and for those gains it turns out that the given close loop system is stable, then the close loop system is stable if you put in the nonlinearity. It turns out that Aizerman's conjecture is false and this was proved by a several people. Over the years several people prove this thing.

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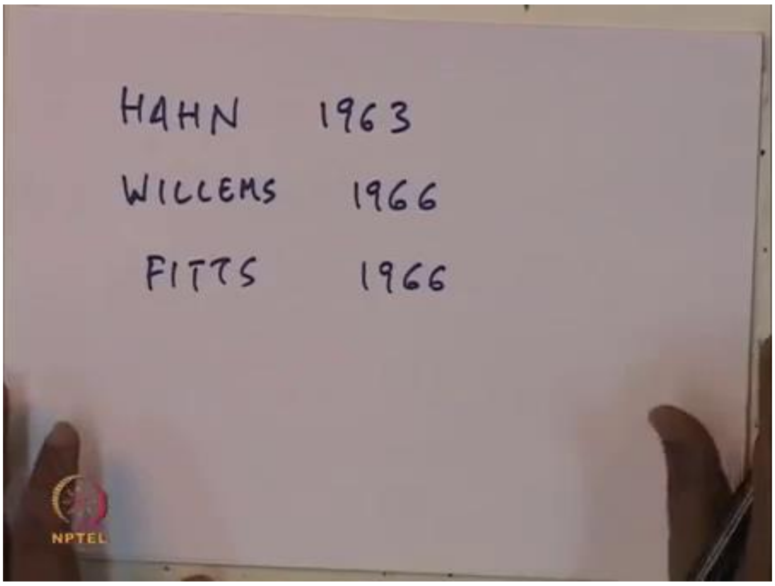

Some of the more famous names include Hahn, who did this in a 1963, then Willems. So, Willems assured a contour example that this does not hold. Willems did this in 1966, then there was Fitts, who did this also in 1966. So, they constructed all sorts of examples which showed that, this conjecture of Aizerman is not correct. So, let me now give an example of a system where this actually does not hold.

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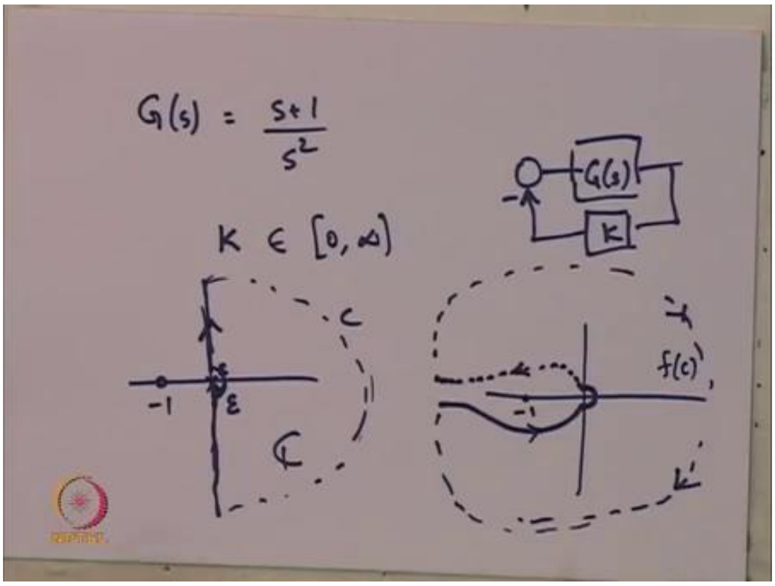

So, for this system I consider the linear plant G s to be s plus 1 upon s squared. So, if you take this linear plant in a feedback structure, then it should be clear that, for all k for all feedback k in 0 to infinity. This given feedback system is stable that means, if you take G s and you give a feedback which is a linear gain k, then the resulting system is stable.

We can see this in several ways I mean we have already looked at this nyquist criterion. So, perhaps we can draw the nyquist plot of this G s. So, if this is b complex plane and we want to draw the nyquist plot here, then as you start moving up, let us say from a small value epsilon you start moving up. For this small value epsilon you will get something here and you will a curve like that. So, this goes right up to the top and then you have this infinitely large circle and corresponding to that you will get something here like this.

Then you come up the negative slope and when you come up the negative slope you essentially have the reflection of the original one and of course, this particular thing has double pole at the origin and has a 0 at minus 1. So, because of that to avoid this double pole we can draw a small circle here of radius epsilon. When you draw this small circle, when you look at its image here you will end up getting something like this.

Now, this contour c will travels in this way that means in the clockwise direction. As a result the contour that we got here using the nyquist plot was something like this. Then the critical point is the point minus 1 and we find that this nyquist plot, this contour the image of the contour f of c, this does not enclose minus 1 and therefore, the resulting system is stable for all gains, all gains from 0 to infinity.

Now, now what I am going to is I am going to demonstrate a particular nonlinearity which lies in the sector 0 to infinity. Now, what do we mean by saying that a nonlinearity lies in the sector from 0 to infinity. So, earlier I talked about the slope, but it is not really the slope that we are looking at because the slope could very well be negative, but what when we are looking at f psi by psi.

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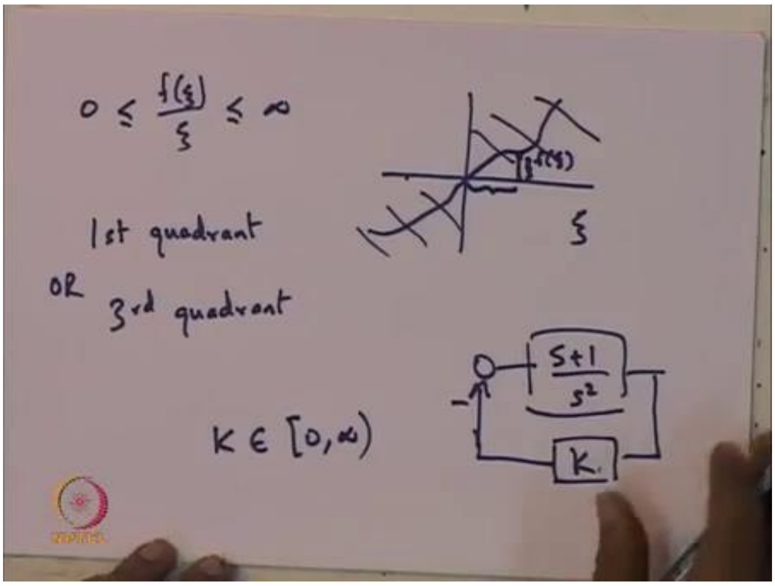

This is psi, and let us say the nonlinearity is like that. So, let us say this was f psi, then when we are looking at this f psi by psi, we really looking at this divided by this. So, when you say f psi by psi lies between 0 and infinity, what you are really saying, is that the characteristics of this nonlinearity should lie in the first quadrant. That means when the input is positive the output is positive or the third quadrant, that means when the input is negative the output is negative.

So, the first quadrant or third quadrant. Now, that the linear system that we looked at s plus 1 upon s squared, we saw that this particular linear system is stable, when you give when you put this in a feedback connection with this k lying between 0 and infinity. Therefore, if now because this is true and if Aizerman's conjecture where true. If we put in a nonlinearity here whose characteristics lie in the first quadrant and the third quadrant, then the resulting system should be asymptotically stable. Now, what I am going to do is I am going to demonstrate, a particular nonlinearity, which lies in the first quadrant and the third quadrant, but the resulting system is not stable. So, here is the nonlinearity. So, the nonlinearity is given by the following equation.

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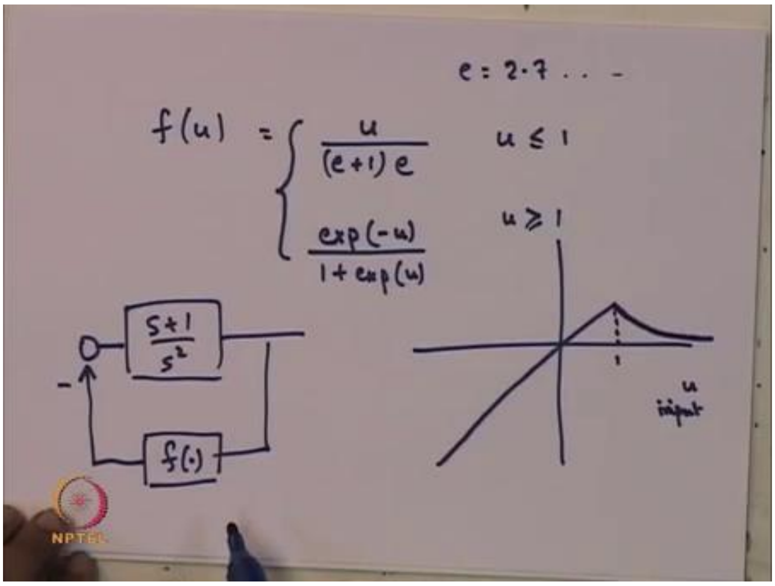

So, f of u. So, if u is an input to the nonlinearity, f of u is given to be u upon e plus 1 times e, here e of course, is a number e. So, e is that 2.7 something. So, this the equation for f of u, for u less and equal to 1, but this if you look at this this is like very linear. So, if you are going to draw the characteristics, you are going to get a line with slope 1 upon e plus 1 by e.

This goes right up to 1. So, this being the input u. So, this is the first part. Now, the second part for u greater than or equal to 1 what you have is this is exponential of minus u divided by 1 plus exponential of u. So, here the characteristic that you are going to get is like that. Now, if you look at this nonlinearity, it lies in the third quadrant and the first quadrant. So, this nonlinearity given by this f of u in feedback connection with that plant. So, s plus 1 upon s squared and here you have the nonlinearity f.

By Aizerman's conjecture because s plus 1 by s squared is stable when you put any feedback k lying in the range from 0 to infinity. Therefore, this linear plant with this nonlinearity defined in this way should result in asymptotic stability. Now, what I am going to show is that this resulting system is not asymptotically stable I am going to demonstrate a particular solution to this particular system, which is actually growing with time. If you have a trajectory, which is perpetually growing with time, obviously that system cannot be asymptotically stable.

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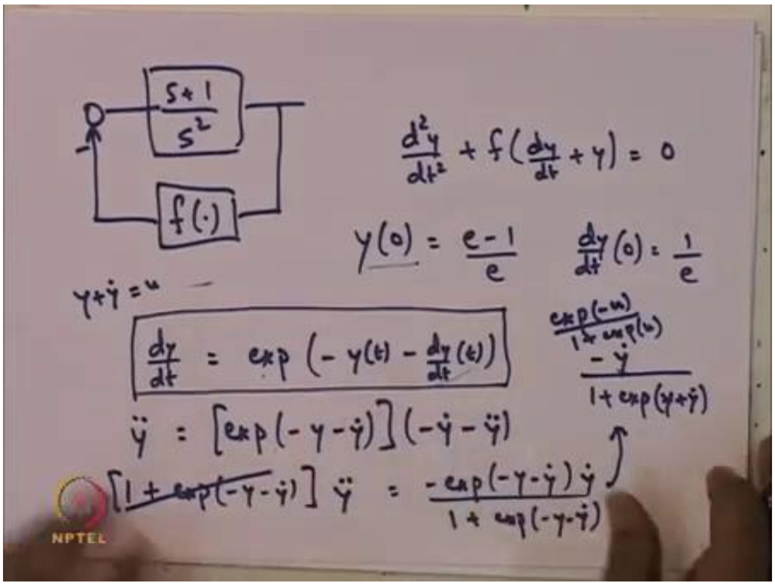

So, here is the particular solution that I am going to give you. You see when you put this s plus 1 upon s squared and you feedback through this nonlinearity, then you know I reverse the earlier trick, this is the same as looking at a system with the equation d 2 y d t squared plus f of d y d t plus y equal to 0. So, looking at this system is the same as looking at this system. Now, for this system I look at a particular solution, which is given by.

So, y of 0 is specified as e minus 1 upon e and d y d t at 0 is specified as 1 by e. So, one wants to solve those differential equation with the initial conditions, y at 0 is e minus 1 by e and d y d t at 0 is 1 by e. So, I claim that for these initial conditions solution for this is given by the following. So, d y d t is equal to exponential of minus y t minus d y d t. So, I am claiming that this particular thing this particular equation gives us a solution to this original system with initial conditions y at 0 being this and d y d t at 0 being this.

So, how do we show that this is a solution well let us differentiate it. So, y double dot that is a second derivative this is equal to exponential minus y minus y dot and then I have to take the derivatives of these. So, that the derivatives of these will give me minus y dot minus y double dot. So, this y double dot I take to the other side. I end up with 1 plus exponential minus y minus y dot, the whole thing multiplying y double dot is equal to exponential minus y minus y dot times. So, there is a minus times y dot.

So, y double dot is going to be I take this below 1 plus exponential minus y minus y dot, but. Now, if you think about this, I sort of this exponential of some negative quantity as 1 upon the exponential of the positive quantity. So, if I normalise this I mean, if I simplify this I will end up with minus y dot upon 1 plus exponential minus sorry y plus y dot.

For y dot we had said that this is a solutions for y dot, we can substitute this. If you now think of y plus y dot as u, then this expression is the same as the exponential of minus u upon 1 plus exponential of u. This minus sign is appearing essential because when you take the f to the other side that is a negative. So, this particular solution is indeed a solution to this differential equation with initial conditions given by, y at 0 is e minus 1 by e and d y d t at 0 is 1 by e.

Now, if you look at this solution, then it should be clear that, once this and this are the initial conditions, this exponential of some quantity. So, d y d t is going to increase. So, when d y d t increases, I mean d y d t is positive. So, d y d t is positive means y increases. So, y increases d y d t increases and this exponential. So, it will continue to have d y d t to be positive and therefore, y would continue to increase as time goes on.

Which essential means that this system is not asymptotically stable. So, this particular example is an example which tells us that the Aizerman's conjecture is wrong. Now, at this stage, there was Aizerman's conjecture, which was promising, but it was shown that this Aizerman's conjecture is false. Now, what can we do about analysing general systems and what sort of general theory can be obtained which at least makes use of Aizerman's conjecture and Aizerman's idea.

Now, we are going to talk about some things which essentially uses Aizerman's idea, but gets passed the conjecture and gives an actual answer as to which systems, when you interconnect you will end up with something which is asymptotically stable. Now, the inspiration for this thing comes this particular technique, comes from electrical engineering and it comes from electrical circuits, which are called passive.

So, what do we mean by passive circuits. Now, you must be knowing very well, that if you have a circuit made up of passive elements, whatever that passive element is, if you have a circuit made up of passive elements then the resulting, I mean that then this circuit is passive. This really does not give you an answer all it is says is that I am using the same word again and again. I am saying if it is made up of passive elements then the resulting circuit is passive what exactly is passive. Now, a sort of very layman kind of definition for a passive system is a following. A passive system is something that does not generate energy. So, what it means is if you have a passive circuit and you supply energy to passive circuit, then that energy either get stored in the circuit or it gets dissipated or lost, but a passive circuit does not generate its own energy. So, maybe we start by looking at some examples to get a generic idea of what passivity is...

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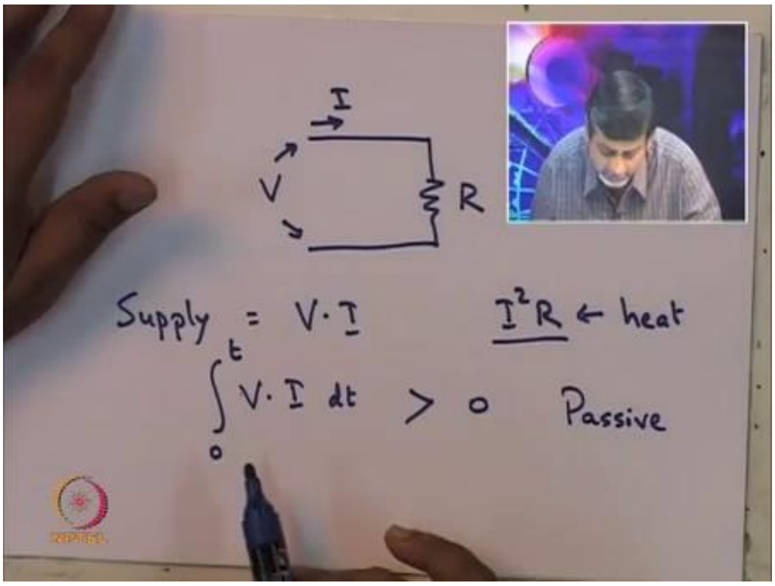

So, suppose we look at a circuit which consist of just a resistance. So, the voltage that we apply. Let me call it V and the current flowing in, let me call it I. Therefore, the power that is fed into this socket or this supply is V times I. Now, when you supply this, this current I goes to this resistance and it gets dissipated as I squared R, that is heat or something.

So, if you look at V dot I in this particular circuit and you look at the overall long period of time that means say starting from time t equal to 0, to sometime t, then this resulting thing is always going to be greater than 0. Now, one could use this as the definitions of passive. So, when we talk about a passive system, then what we are talking about is that the amount of energy supplied to the system is positive. Of course, this definition of passivity may not, I mean this may not always strictly hold the moment we use some element like a capacitance or an inductance. So, let me try and explain what I mean by that.

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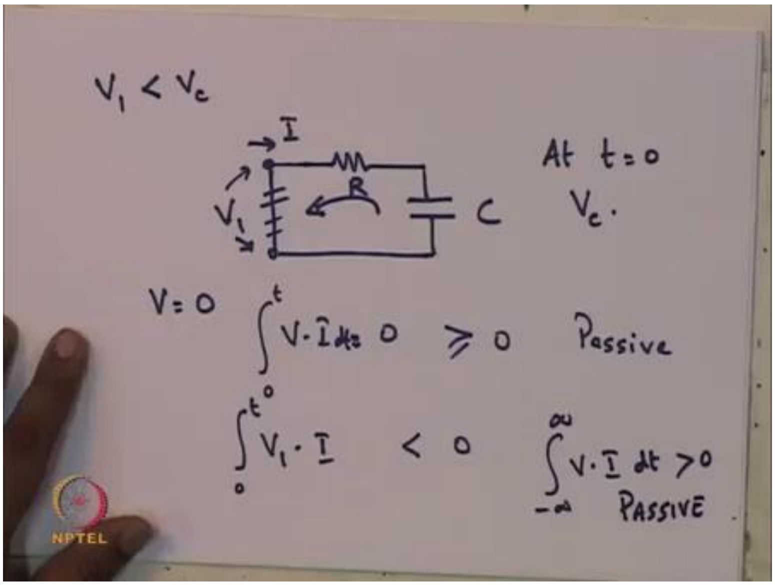

So, suppose now instead of resistance you have a capacitance. Suppose, you also have a resistance here and then we apply of voltage V and the current going in is let us say I. Now, let me also additionally assume that this capacitance is charged. So, at time t equal to 0 the charge in the capacitance is V c. Then let me short this, if I short this then V that means the terminal voltage is 0. Therefore, if you want to talk about supply of energy well the energy that is supplied in this V dot I, which is 0, but of course, in this circuit we know something is happening. Of course, even in this particular case if you have of course, take the integral from 0 to t of V I d t this is 0 which is greater than equal to 0. So, we could still continue to call it passive, but now suppose, we apply of voltage V.

So, instead of short let us assume that we apply of voltage V 1, where V 1 is less than V c. Now, if V 1 is less than V c, then what is going to happen is from the capacitor current will flow this way. Now, if we look at the supply, it is going to be V 1 multiplied by I, but I the actual current which is flowing is in the opposite direction to the convention that we use for I.

So, this quantity is here is negative. So, when you take this integral you are going to end up with something less than 0, but of course you would already know that we call a capacitance, a passive element. Because the capacitance does not generate energy of it is of own, but what is happening in this particular situation is that, this capacitance the time t equal to 0 was already charged some voltage V c.

So, this stored energy is been dissipated. So, if we want to also incorporate this stored energy one way we could do this is this instead of taking this integral from 0 to t we take the integral from minus infinity to plus infinity. If we take the integral from minus infinity to plus infinity and if you take the integral from minus infinity to plus infinity of V dot I d t. Then any circuit for which this integral V dot I, this is the power supply to this circuit from minus infinity to plus infinity.

This is greater than 0, then we can call this circuit passive, all right. So, how do we how do we capture this particular the notion of passivity. So, the basic idea is this. So, suppose we start off with a circuit and we assume the circuit is at rest, what do I mean by circuit is at rest. So, circuit could have several energy storing devices like capacitors and inductors. The circuit is at rest by saying that the circuit is at rest what I mean, is that all the capacitors are discharged.

Similarly, all the inductors have no current flowing through them. So, there is no stored energy in the circuit. That is what I mean when I say a circuit is at rest. Now, suppose to that circuit you now supply the certain amount of energy. If you supply certain amount of energy, what do you think would happen in the circuit. Well, this energy which comes in to the in to the circuit part of it of course, will get dissipated in the form of heat, in the resistances, but a part of it might get stored, either as electrical energy in a capacitance or as magnetic energy in an inductance.

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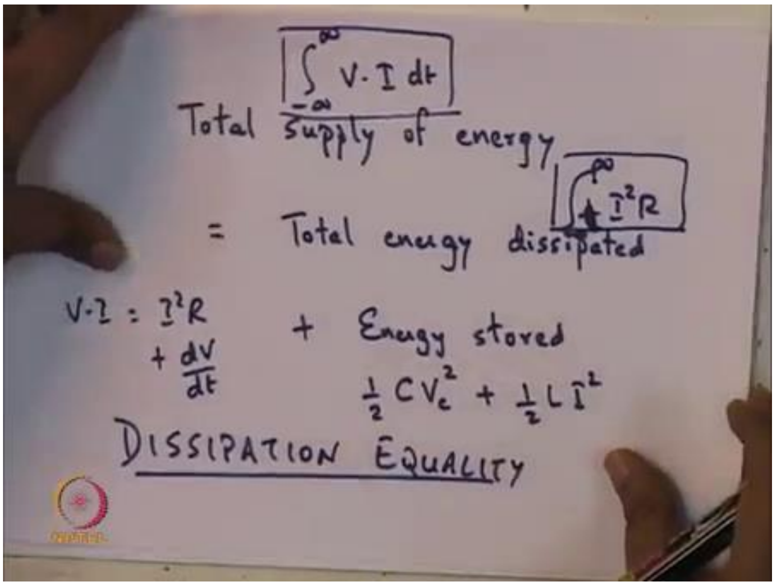

So, what, in very broad terms what we can say is that the total supply of energy. This would be equal to total energy dissipated plus energy stored all right. Now, the total energy, the total supply of energy of course, the expression for this would be V dot I. So, V dot I is a power d t, if we integrate this from minus infinity to infinity. So, this is the mathematical expression for the total supply of energy.

The total energy dissipated, well the energy dissipated or the power that is dissipated is given by half is given by I squared R. So, if you integrate this from minus infinity to infinity that gives you the total I guess there is no half here it just I squared R I squared R is the total amount of energy that is dissipated and you put this integral from minus infinity to infinity.

This is the mathematical expression for the total energy dissipated. What about the total energy that stored well, it could be stored in a capacitance and the energy stored in the capacitance this something like this. There could be energy stored in the inductors, it could be something like that. So, mathematically this is the expression that we will get the total supply. So, integral is equal to this integral which is the total energy dissipated plus the energy stored which is this.

Of course, this is an integral forms, so this equation that I have written down can be called dissipation equality. I can state this same dissipation equality, this is the integral form I can state ITT into differential form, which means I just take the derivatives. Each of this cases and I would end up with an expression like V dot I is equal to I squared R plus d V d t that d V d t is the rate of change of energy stored in this circuit.

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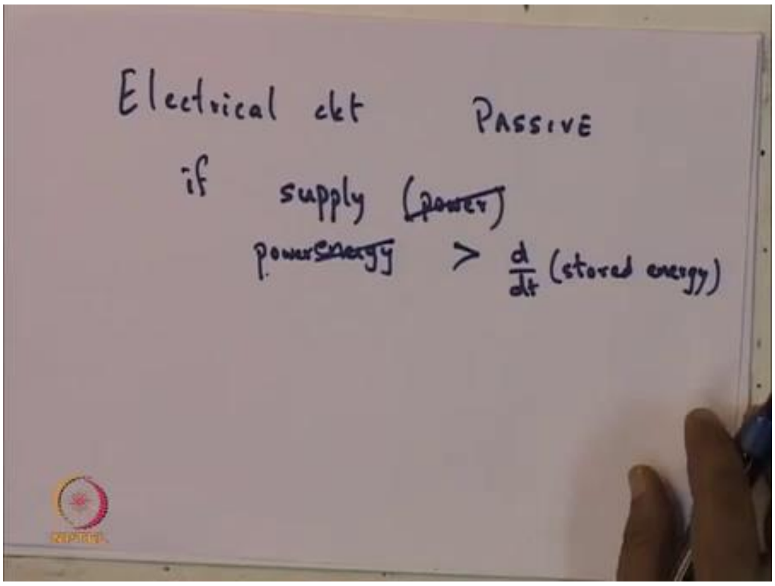

So, then coming back to an electrical circuit we call an electrical circuit, passive if the supply of power is. Now, the supply in terms of energy, of course is this an integrated part, power is the derivative part. So, perhaps let me talk about it as the supply in terms of energy, the energy supplied is greater than the rate of change of stored energy. So, of course, this is the rate of change of stored energy.

So, actually I am talking about the power. So, this supply power is greater than rate of change of stored energy, then such a electrical circuit is called passive. Now, this stored energy is actually quiet important. Of course, electrical circuit, circuits which have an input and an output and. So, when you talk about the stored energy this stored energy is something that you have in a in a system with an input and an output. This stored energy plays exactly the same role as a lyapunov function plays in a system without inputs. So, let me use electrical circuit example to show that this stored energy plays exactly the same role as the lyapunov function in a system without inputs.

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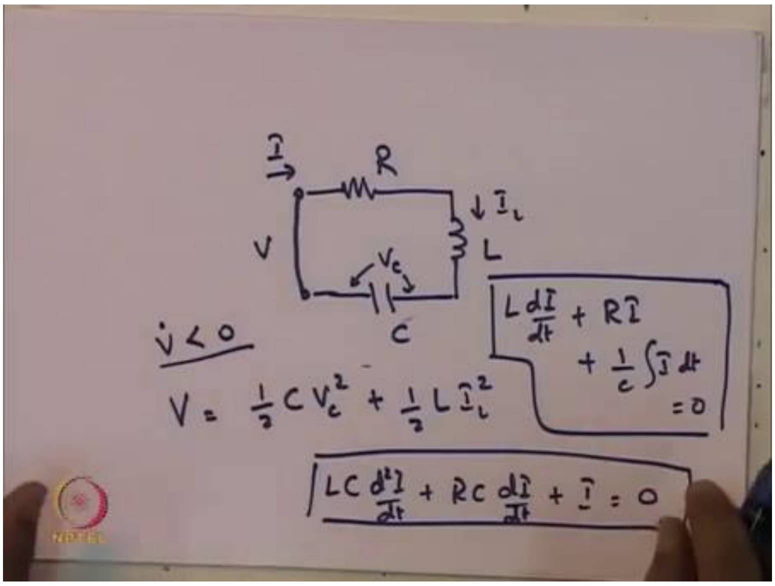

So, let us consider an R L C circuit. So, this is R, this is L, this is C. Let us assume that there is some stored energy, which is a half C V c squared plus a half L I l squared. Now, of course, this is system, which has an input and an output. Of course, one could think of the voltage as one could say that voltage is the input and this is the voltage that we applying. The output is the current, one could also think of the current I mean one could think of the current.

So, here on the current pushing in being pushed in and the voltage that is generated as the output. So, it really does not matter whether it thinks voltage as the input or current as the input, but here let us assume the voltage is the input and the current is the output. So, this now is a circuit with an input and an output and there is a stored energy which depends on the current I L and the voltage across the capacitor V c.

Now, here the input is V. If you set the input to 0, that means just you short here. Then we have an autonomous system, in the sense this is the system without any input. Now, when this is the system without any input what is going to happen well, what is going to happen is this going to be oscillation setup in this circuit.

During these oscillations, the electrical energy which is stored in the capacitor gets converted into magnetic energy in the inductor through half the cycle, but of course, the current this transfer is taking place through the current because of which there is some energy when dissipated. Then there magnetic energy in the inductor is going to return that magnetic energy which convert that magnetic energy back to electrical energy stored in the capacitance.

Again some more dissipation is going to take place and in this way this keeps oscillating until all the energy which was stored in the capacitor and the all the magnetic energy which is stored in the inductor gets dissipated. Then you would have no current in this circuit, but now if you write down these equations the equations, for this this particular oscillation that takes place.

Well one way to write out this you just think of the current in the circuit. You can write down write down the entropy in terms of the current. So, you will get the voltage drop here to be L d I d t plus the voltage drop here to be R times I, plus the voltage drop here to be 1 by c integral I d t, this is equal to 0. Well this is the second order differential equation and this second order differential equation I mean. So, take derivative once and you will end up with L c, the second derivative of I plus R c times the derivative of I plus I equal to 0.

Now, for this particular differential equation, if you use this V as the lyapunov function, then this lyapunov function is positive definite. If you take derivative of this lyapunov function you would find that the derivative of this lyapunov function is negative, which essentially proves that this circuit is asymptotically stable, when you convert it into a system without inputs. So, let me stop here right now and we will carry on about this in the next lecture.