Nonlinear Dynamical Systems
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Lecture - 15
Nyquist Plots and Nyquist Criterion for Stability

Hello everybody. My name is Harish Pillai, and I am from the department of electrical engineering IIT Bombay. I am giving this set of lectures along with Prof. Madhu belur, from our department and these are lectures on non-linear dynamical systems. Madhu belur probably has already covered some bits of non-linear systems. Today I am going to start one part of this course, and we will begin by reviewing some concepts that we have from linear systems. Specifically, this is the concept of a Nyquist plots and Nyquist criterion for stability. So, let us just revisit this Nyquist plots and Nyquist criterion for stability, which is of course very much based on principles of complex analysis, complex number theory. So, let us begin by explaining what the Nyquist plot is…

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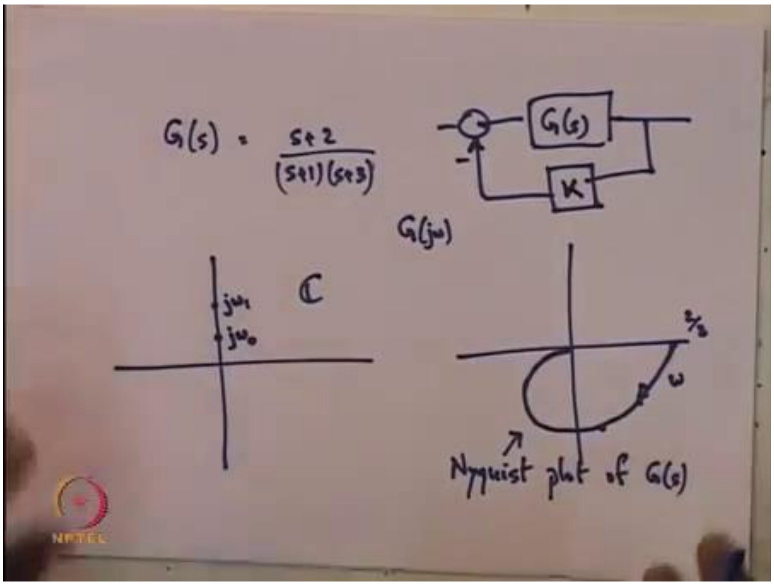

So, suppose we are given some transfer function G s. Let us say s plus 2 upon s plus 1 into s plus 3, then the Nyquist plot is drawn in the following way. So, one looks at the complex plain, and then for any imaginary value j omega, we calculate what G of j omega is, and this gives us a point. So evaluating this at some G omega naught gives us this point we evaluate it at some other j omega 1, they will give us some other point, and so on, so as a result you get a whole set of these points, and then when you join all of these together then you get the Nyquist plots.

So, for example, in this particular case if you evaluate at omega equal to 0, you would get 2 divided by 3. So, you have some point on the real line two-third. It starts from there, and then it turns out that in this particular case you will end up with a plot which looks like this, and it is parametric. So, as omega increases, you keep travelling down this way until you reach infinity, because when you substitute omega equal to infinity into this expression, you end up with 0.

So, this is the Nyquist plot. So, this is the Nyquist plot, of G s. So, given a transfer function we can draw this Nyquist plot. Now, this Nyquist plot, the importance of this Nyquist plot is in the fact that, you can use this Nyquist plot to talk about the stability of not only this transfer function, but also the transfer functions that you would get when you put this transfer function in a feedback structure of this form. Where, you have some plant here and you have let us say negative feedback.

So, when you have transfer and you know this feedback structure will have a net effective transfer function, and you can talk about the stability of this feedback structure also, by looking at this Nyquist plot. Now, how we can talk about this stability of this feedback structure especially, when on the feedback path you have a pure gain K, is what I will explain right now, and in order to explain that we will first begin by looking at the expression for this feedback.

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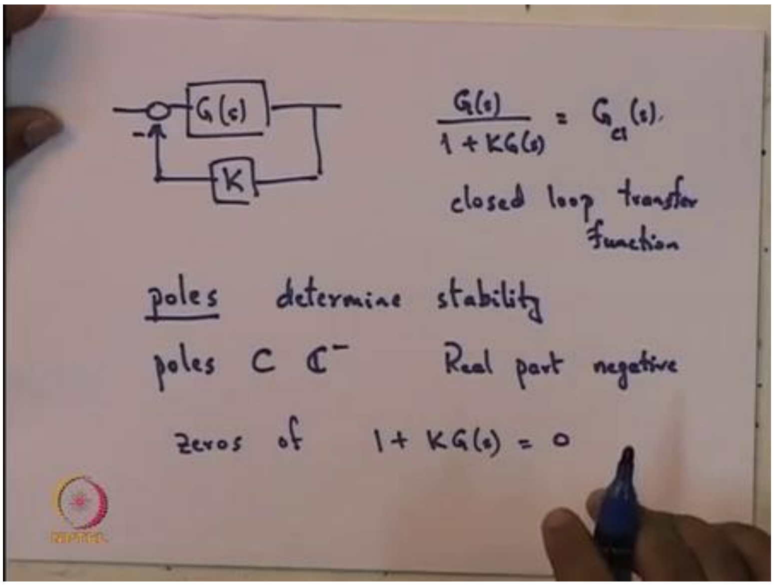

So, suppose you have this transfer function G of s, and we have the feedback block K. And we want to now, look at the effective transfer function, then the effective transfer function is given by G s upon 1 plus KG s. So, this is the G close loop, so this is the close loop transfer function. Now, if you look at this close look transfer function, when we are talking about stability, what we are concerned about is the poles. The poles are determent stability. Now, for example, if we are looking at continuous time systems, then the poles, if the poles lie in the open left half plane that means, the poles have real part negative.

So, if the poles lie in the open left half plane which means, the real part of the poles are all negative, then we know that the transfer function is stable. Now, given G, we can talk about the stability of just G without any feedback, and that would be determined by the poles of G. But, if we are looking at this particular close loop transfer function, then the poles of this close loop transfer function is really the 0s of the equation 1 plus KG s equal to 0.

So, all those values of s for which, 1 plus KG s is equal to 0, are the 0s of this equation, and the 0s of this equation are precisely the poles of the close loop transfer function, because this particular thing appears in the denominator of the close loop transfer function. So, we can say something about the stability of the close loop transfer function by looking at this 0s of 1 plus KG s.

Now, how we can look at the 0s of 1 plus KG s, and how that would help us predict the stability of the close loop transfer function. Knowing something about the open loop transfer function, is what this Nyquist, this Nyquist criterion is all about, I mean this Nyquist method of determining stability is all about.

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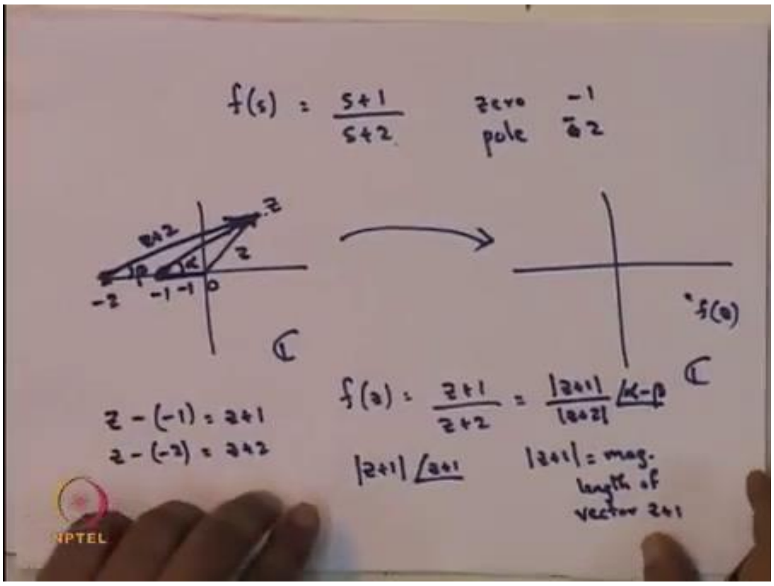

So, now, I would go back to some basics from complex analysis, and for that let us now look at some general function of the form let us say, s plus let me say s plus 1 upon s plus 2. Let us take some function like this, and let us look at this particular map, from the complex plane, to the complex plane. There, if you take any point z, that point z will map to a point, which is f of z ok. Now, let us see how we would calculate f of z given this z, and given this is transfer function. Now, if you look at this transfer function, this transfer function has a 0 at minus 1. And, so let me denote that by a 0, this z minus 1, and it has a pole, at plus 2 which let me denote by this cross, oh sorry, pole at minus 2, which I denote by the cross here.

Now, in order to evaluate this, we would essentially be evaluating f of z is, z plus 1 upon z plus 2. What is z plus 1? We look at those z plus 1 in the following way. Consider this vector going from 0 to z. This vector I can think of as a vector z. Similarly, consider this vector going from 0 to minus 1, I can think of this vector as minus 1. So, now if I take the vector z and subtract minus 1 from that, I would effectively get z plus 1, which is in the numerator of f z. Now, that I could show with the help of this particular vector.

So, effectively calculating this z plus 1 is like taking this vector from the 0 to the point z. In the same fashion if I do z minus, minus 2, I can evaluate z plus 2 and the z plus 2 is this vector from minus 2 to z. So, I could call this z plus 2. Now, when you evaluate this, it is essentially like evaluating this two complex numbers z plus 1 and z plus 2. Now, z plus 1 which, I have denoted by this vector, is really a complex number. And that complex number z plus 1 can really be thought of as the magnitude, and with an angle of z plus 1. Now, this magnitude, this magnitude is the length of the vector z plus 1.

So, the length of this vector gives us this magnitude. And the angle is this angle here which let me denote by alpha. Similarly, the magnitude of z plus 2, this vector gives a magnitude of z plus 2, and the angle is this angle beta. So, if we have to evaluate f z, that is the same as draw these two vectors and then whatever is the magnitude of this vector, write that in the numerator. The magnitude of this vector, write that in the denominator. And, then the angle the net angle that you will have for f of z is going to be the angle of the numerator, which is alpha minus the angle of the denominator which is beta.

So, for a given point z if we have to calculate f z essentially, what we do is you draw these vectors, from the poles and the 0s. The poles and the 0s in this particular case there is only one 0 and 1 pole, you draw vectors to that z from the poles on the 0s. Then you take the magnitude of those vectors, from the 0s and put them all in the numerator. The magnitude of these vectors from the poles, put them all in the denominator and, then for the angle what you do is you take all the angles, define by the vectors, coming from the 0s and put them all with a plus sign. And those coming from the poles, put them all with the minus sign, and you have effectively got a complex number which is f of z.

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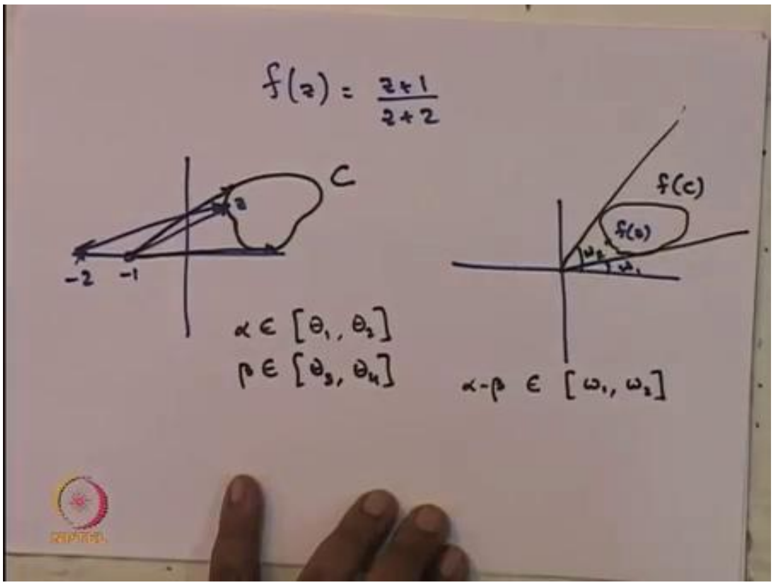

So, now what we essentially saw was, how to evaluate, how to evaluate f of z for this particular transfer function z plus 1 upon z plus 2 given some particular point z. Now, what we are going to do is, talk about something more general that happens those so it is a same transfer function, so it has a 0 at minus 1 and it has a pole at minus 2. Now, we do the following suppose, we want to evaluate this f of z, but we want to evaluate it on a contour which looks like this.

So, let me call this contour c. And now, we want to evaluate f of z at each one of these points and draw there it goes to here. If we do this, then we might end up getting something like this, which I could call f of c. Now, for each point z of course, we evaluate f of z in the way we talked about earlier that means, we draw this particular for this point z, we draw these particular two vectors and we look at the magnitude of this vector divided by the magnitude of this vector. And then you look at these angles alpha and beta, and the difference of alpha and beta will define what is the angle of the point f of z, which probably may be out here, in this point is z.

Now, what I want all of you to observe is the following, when we go around this contour z, and you look at all these vectors that would come from minus 1 to this various points in the contour. Then, one could easily see that, there would be some point here for which this angle will be the largest, and there will be some point here for which this angle is the smallest. And, so the angle alpha that you get here, one could say lies between this angle and this angle for all the point z on this contour.

And, so let me call these angles, theta 1 to be the smallest angle, and theta 2 to be this largest angle. So, alpha is an angle which lies between theta 1 and theta 2. Similarly, beta is going to be an angle between, you can imagine that, there will be a largest beta and there will be a smallest beta and, so I can say beta is an angle, which lies between the angle let us say theta 3 and theta 4. Now, as a result of this, the angle of the image is going to be constrained because these angles are constrained.

So, that angle one can say, the resulting angle alpha minus beta is always going to lie between some angles, let me call it omega 1, and some other angle omega 2. So, this alpha minus beta is some angle which lies between omega 1 and omega 2, but what that really means is from the origin, if I draw a line with angle omega 1, and I draw another line with angle with omega 2. Then, the image of this contour will like completely within the sector define by omega 1 and omega 2. So, what we did earlier was a, we looked at some generic curve.

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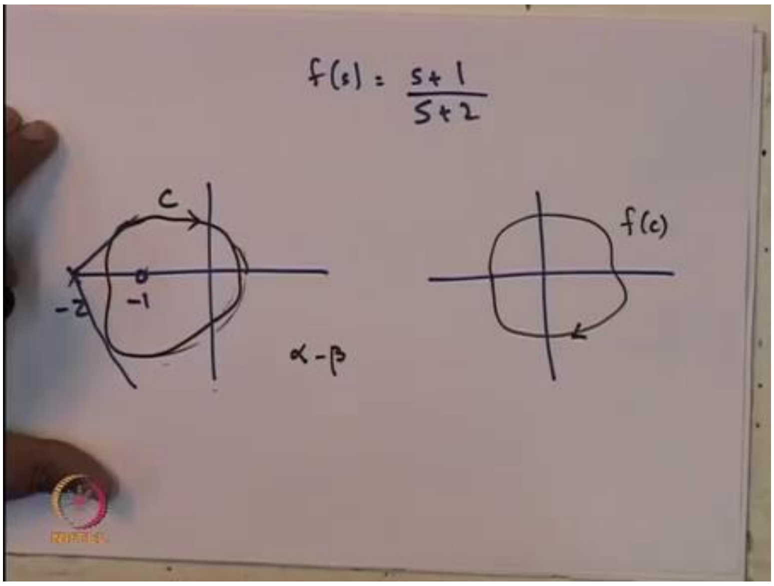

Now, we look at some other curve, so it is the same transfer function s plus 1 upon and s plus 2, and we have the 0 at minus 1, and the pole at minus 2, but this time we are going to consider a contour which lets say looks like this.

Now, when the contour look like this, it is clear that when you look at angles from the 0, the angles could be any angle from 0 right down to 360 degrees. Whereas, when you look at the angle from the pole, it gets constrained between these two angles. As a result, when you look at the image of this contour on the other side, it turns out that the image would enclose the origin.

Now, along the contour suppose, you travel in this way that means, in the clockwise direction, then it would turn out that in this particular case if this is the contour c, then the image contour f of c would be such that, f of c would go around the origin once. And the reason for this is also cleared to see, because as for as the angle from the 0 is concerned; that means, the angle alpha, this alpha is going to vary, say if you start from this point and you go around, it is going to start from 0, and it is going to go minus 45, minus 90, minus 180, minus 270 and so on right to minus 360.

So, as you go around the contour, this alpha varies from minus 1 going more negative until it reaches minus 360 degrees. On the other hand, the angle corresponding to the pole is going to start at 0 minus. So, it increases up to this maximum and, then it decreases down to 0, then it increases, or I mean depending on the how you want to look at the angle. It increases up to this point and, then again it comes down back to 0. So, as a result alpha, you go from you know minus some small value right down to minus 360, but this angle beta you only go from minus, you tend up to this point, the maximum minus and then you go up to this point which is the maximum plus.

So, as a result when you are looking at alpha minus beta, it is going to go through a whole cycle from minus 1 go right round to minus 360, which is why you have a clockwise circulation around the origin. Now, we can look at some other contour, and this other contour I want to draw in this particular fashion.

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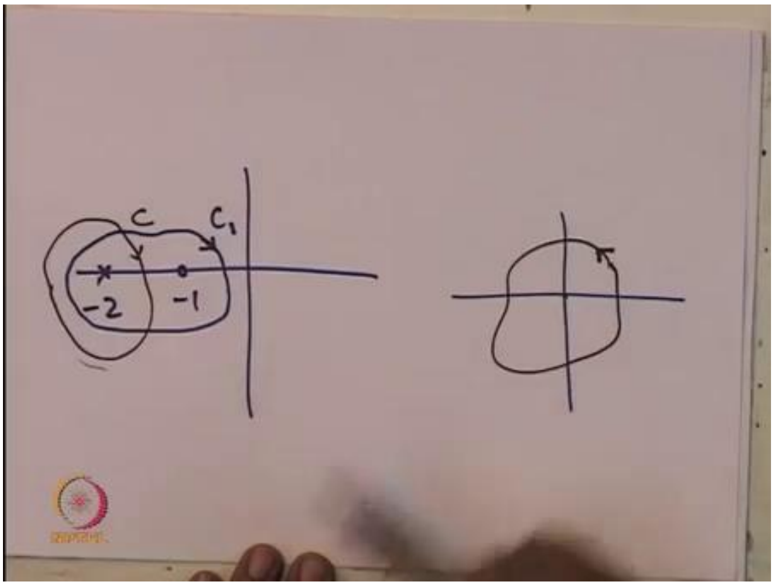

So, it is the same transfer function, again this 0 is at minus 1, the pole is at minus 2, and this time I take a contour like this. Now, if you take a contour like this, and you travel in the clockwise direction around this contour, it turns out that the image of this is going to be a contour which encircles the origin. But, this time travels in the anticlockwise direction. So, ok and the reasoning for this is also just like the reasoning that, we gave earlier. If you notice the angle that you will get at the 0 is going to bounded, between some maximum value and some minimum value whereas, the angle corresponding to the pole, is going to go right round 360 degrees.

So, as a result you will end up because, of this angle subtended by the pole you will end up going around the origin, this time in an anti-clockwise direction. Now, it will not be surprising, that if you consider a contour like this which includes both the 0 and the pole, and you go around this contour let me call it c 1. If you go around this contour, then it turns out that this contour will not encircle the 0 or if it does encircle the 0, it will go round it clockwise and anti-clockwise.

So, the number of clockwise revolutions are the number of anti-clockwise revolution of this particular contour image, is going to be 0. So, this particular property that comes from complex analysis, is what we are going to use, in order to predict whether a given transfer function is stable or not. So, generically, so let me talk about generic situation.

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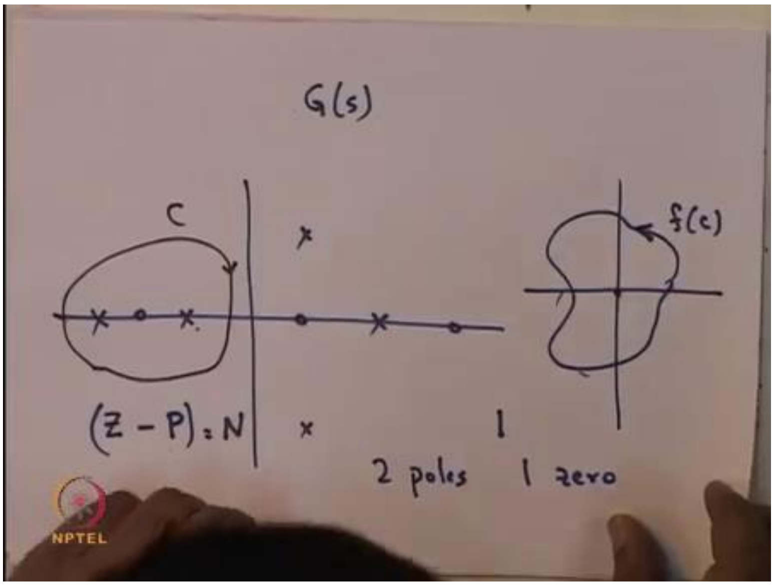

So suppose, you have a transfer function G s and suppose it has let us say, all these are poles, and suppose these are 0s. And now, if you take a contour like that, this contour and travel along this contour in the clockwise direction. This contour contains within it two poles and one 0 then when you look at the image of this contour, you will end up getting an image. Let me call it f of c, which encloses the origin and this is going in the clockwise direction,, so this will go in the anti-clockwise direction, and it will enclose the origin in the anti-clockwise direction once.

And the reason you get this 1 is because, when you look at this contour, it encloses two poles and one 0. Every time it encloses a 0, the contour, the image contour travels around the origin in the clockwise direction. And every time it encloses a pole, it travels around the origin in the anti-clockwise direction. So since, this contour encloses two poles and one 0 therefore, it will travel around the origin, once in the clockwise direction and twice in the anti-clockwise direction.

And so the net number of times that it goes around the origin is once in the anti-clockwise direction, and the typical formula that you give for this is, given a contour if the number of 0s, it encloses z and the number of poles it encloses is p, then the image contour will go around Z minus P times. Z minus P is equal to N where, N is the number of times that the image f c will go around the origin in the same direction as the direction of the, of the original contour.

So, if the original contour was in the clockwise direction, f of c, if it is going to be in the clockwise direction then Z minus P will be a positive number, and if Z minus P is a negative number, and the original thing was going in the clockwise direction, then the image will go in the anticlockwise direction. Now, how are we going to make use of all this, in order to talk about the stability of the close loop system, well the way we are going to make use of it in the close loop system is the following.

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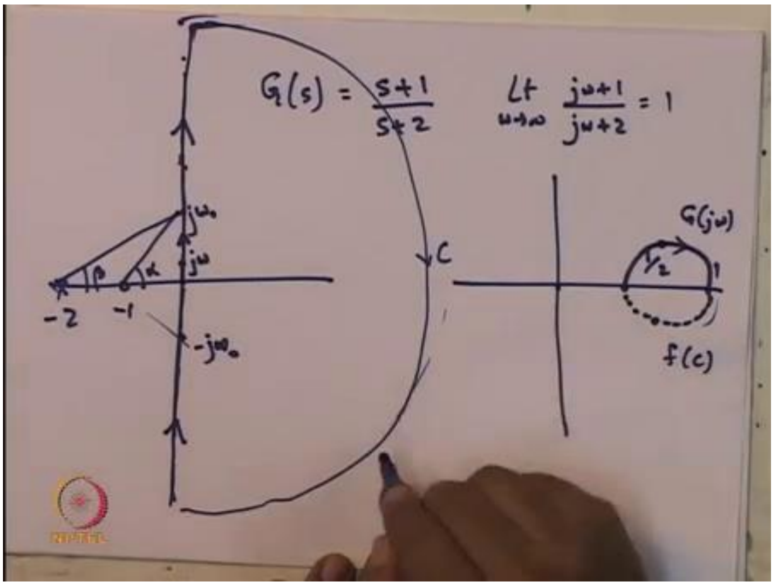

Let us review G s, some transfer function, which is given to us, and let us look at what the Nyquist plot is. So, the Nyquist plot is, you travel a long this portion, and for each j omega, you plot what it comes to. So, for example, if you take G s to be again, let us take the same thing s plus 1 upon s plus 2. Then as you go up this j omega axis, you will get the Nyquist plot.

And in this particular case, when you evaluate at omega equal to 0, you get half and, then as you go up the angles, so here is the 0. And here is the pole and these two angles keep increasing until when you reach infinity, this angle is 90 degrees this angle is also a 90 degrees, so the net angle is 0. And as for as the magnitude is concerned when you reach infinity in limit you know, so limit omega tending to infinity of j omega plus 1 upon j omega plus 2, this turns out to be 1.

So, it will start at half and it will end at 1. Now, as you are going up the j omega, let us say if you take some omega not here j omega not, then you see that this angle alpha is larger than this angle beta and. So, alpha minus beta is going to be a positive angle and, so here you will end up with some curve which looks like that. So, this is the Nyquist plot going in this direction G j omega, and this G j omega is the plot as you plot the imaginary axis just the positive part of the imaginary axis.

Now, in the same way you can plot negative part of the imaginary axis, and you would end up getting a curve which is symmetric to this, but in the negative part essentially because the all the angles, the magnitude will remain the same for j omega naught and minus j omega naught. The magnitudes will remain the same, but angles are exactly the negative. So, if the plot in the Nyquist plot for omega not is here, then for minus omega naught it will be just negative, as for as the angle is concerned, but the magnitude is going to be exactly the same.

So, now let us assume that we look at this go up this thing, and then you take this extremely large radius, at which you travels from the plus infinity right down to minus infinity and this, so this now we think of as a contour through which we are going in the clockwise direction. Now, if you think of this as the contour through which we are going in the clockwise direction, then out here what you effectively will have is, some closed curves like this, which is the image of this contour, so this is the contour c and this here is f of c.

That means you take this, which corresponds this, and then this whole thing which, essentially will correspond to hanging around near the point 1 because, they will all be points at the infinity, and when you take this limit they all turn out to be 1. So, it is just standing still at this point and, then this negative part is just returning along this path and you get back here when you are a the origin here. In case so you have this as f of c for the contour c that you have drawn in this way.

Now, obviously this f of c does not enclose the origin, and what that means is, the 0s on the poles of the original transfer function, either they all lie outside this portion, or the number of 0s and poles that lie in this portion are precisely equal, so that the clockwise and the anticlockwise things have cancel themselves.

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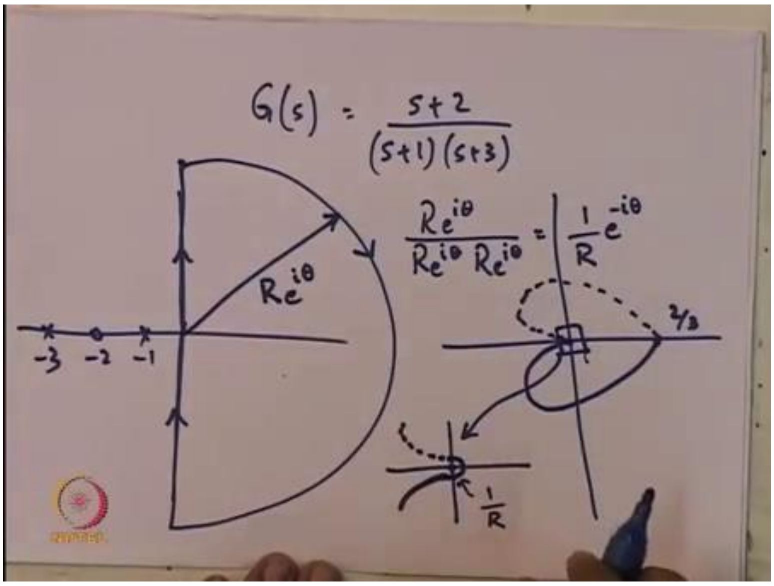

And therefore, there is no enclosing of the origin. Now, suppose you look at a more complicated transfer function like s plus 2 up on s plus 1 to s plus 3. So, you have minus 1, a 0 at the minus 2, and at the pole at minus 3. Now, when you take this particular contour, then the Nyquist plot of this of course, as we had said earlier will start at two-thirds, that means, we are just looking at this positive part. And finally, you will end up with a curve which looks like that. And, so it will reach the origin when omega tends to infinity because there is one s on the top and there are two s below.

So, when you take the limit, it goes to 0. So, when you go through this huge radius throughout here, we will assume the value of s is tending to infinity as a result, the magnitude will continue to be 0. And then when you look at this negative part you will get the negative of this particular Nyquist plot that has been drawn and, so you end up with this. Now, when you look at this figure, it looks as if the origin has been enclosed in the image contour, but in reality the origin is not enclosed in the enclosed by this curve because one has to magnify this origin to see what is happening here. You see as omega tends to infinity.

So, this is the magnified portion here is the omega, and does omega tends to infinity, this portion of the curve looks like this and this other portion of the curve, looks like this. There is this other portion that we want to plot. Let me assume that this is some circle with radius R, so this R is a very large number, and I will evaluate this transfer function for this R, then going along this if curve is like looking at R e i theta where, this angle theta varies from plus 90 degrees to minus 90 degrees. If I have to evaluate this, for this G s, then the evaluation will give me R e i theta plus 2, but this 2 is negligible in comparison to R.

So, I will just throw it away, divided by, for this again I will have R e i theta, this 1 I am throwing away because, this is negligible compare to this, and for this again I will have R e i theta. If I think of this, put them all together, I will have 1 by R because, this 1 R in the numerator 2 in the denominator and I will have e to the minus i theta. Now, this 1 by R, it depends on what this R, value of R is. When, R is very large 1 by R of course, very small, so it is somewhere near the origin, but it has not yet it 0. And I have e to the minus i theta and this theta, has we said on the contour, the theta is varying from plus 90 to minus 90.

Therefore, minus i theta will vary from minus 90 to plus 90 and, so what you will have here is a circle with the radius, the radius of the circle is 1 by R, and it goes in this way. So, this portion if you magnify, you will end up getting something like this and, so this curve though it looks as if the origin has been enclosed, really the origin has not been enclosed, because if we look at the magnified picture it look like this. So, in this particular case the origin is not enclosed and it is clear from whatever we have discussed we have that the origin will not be enclosed essentially, because within this contour none of the poles are 0s of the transfer function is present.

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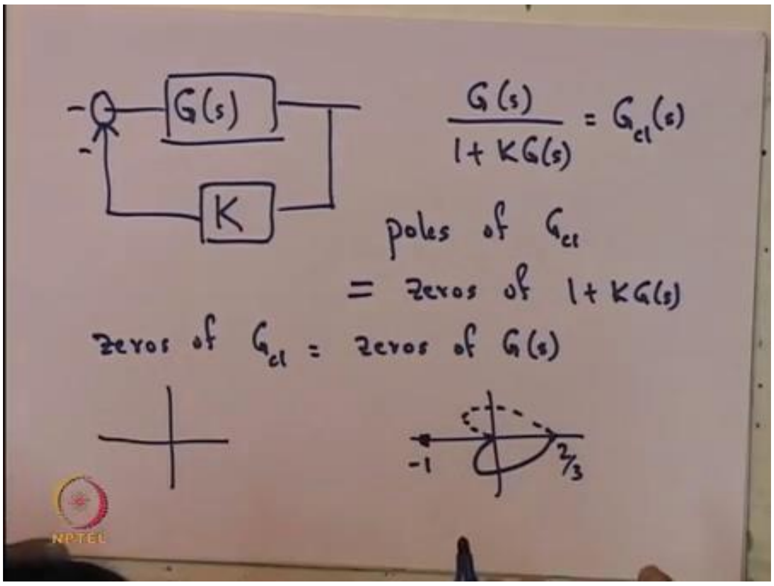

Suppose, we are now going to look at the feedback situation where, you have this gain K, and you want to know whether this is stable. We already said that the closed look transfer function is going to look like G s upon 1 plus KG s. And therefore, the poles of, so let me call this G cl, the closed look transfer function let me call this G cl. The poles of G cl is exactly the same as the 0s, of the equation 1 plus KG s. And the 0s of G cl is the same as the 0s of the original transfer function G s.

So, now if once again we want to use the Nyquist plot, and suppose we have the Nyquist plot of g j omega. Let us take the particular transfer function, which we have looked at earlier and, so the Nyquist plot looks like that and so with it is reflection it looks like that, this is two-third and so on. Now, from here what we can conclude is that, in the right half plain there are no 0s or poles of the original open look transfer function G s. Now, as far as the, so suppose, instead of thinking of this Nyquist plot, as the Nyquist plot for G s, what we can now do is, you see this 0s of 1 plus KG s is the same as the number of poles of G cl. So, the number of times that this particular curve encloses the special point minus 1, will tell us the number of 0s of 1 plus KG s, because of the, because of the following thing. Let me let me try and explain that.

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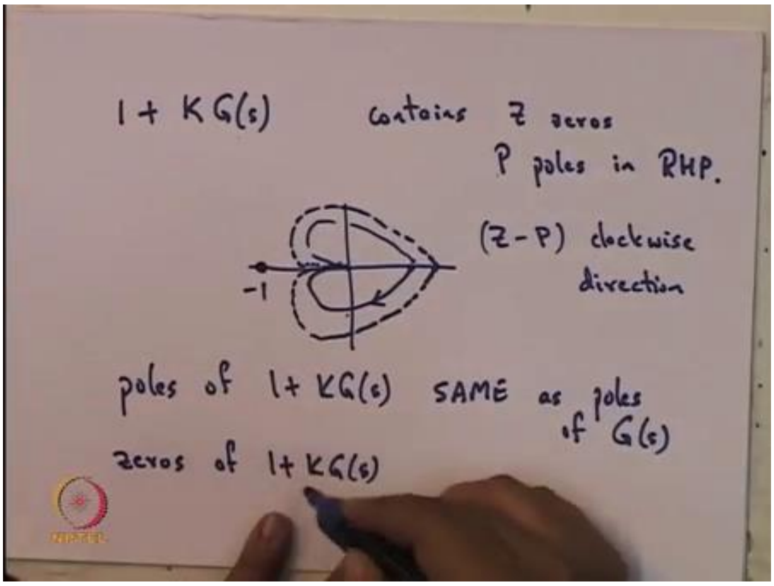

So suppose, G s is given to us and suppose, we know that G s contains z 0s and p poles in the right half plain. So, what that would mean, is that when you look at the Nyquist plot of G s. So, let me think of the Nyquist plot as something like that. Now, it will enclose the origin and suppose, the Nyquist plot was going like that in the clockwise direction, then it will enclose the origin z minus p times in the clockwise direction.

So, if G has contains z 0s and p poles in the right half plain, I mean if the 0s in the poles of G s lie in the left half plain of course, this number and this number both 0 and, so this result in Nyquist plot does not enclose the origin. Now, if we take K times G s, the only change is going to be all these points are going to get stretched out. So, if they get stretched out why some amount K, you are going to end up with a curve, that looks perhaps like that. Now, if you look at 1 plus KG s, this is as if you have changed the origin and, so you will no longer be thinking about.

So, the 0s the poles of this transfer function 1 plus KG s. So, let me write it down. So, poles of 1 plus KG s is the same as poles of G s, 0s on the other hand of 1 plus KG s well, what about the 0s of 1 plus KG s. Can we detect anything about the 0s of 1 plus KG s from this Nyquist plot? The one thing we can do is, we look at this result in Nyquist plot and how many times it encloses minus 1.

Now, if it encloses minus 1 in some direction, then what it means, I meant in the clockwise direction for example, then what it means is the 0s of 1 plus KG s minus the poles of 1 plus KG s that many is that means, z 0s of 1 plus KG s minus p poles of 1 plus KG s are present in the right half plain.

So, this 1 plus KG s gives us some idea about how many 0s, I mean how many times the Nyquist plot encircles the minus 1 gives us some idea about the 0s of 1 plus KG s. Because we already know about the poles of G s, which is the same as the poles of 1 plus KG s, and it is essentially this trick that we use for the close loop transfer function also.

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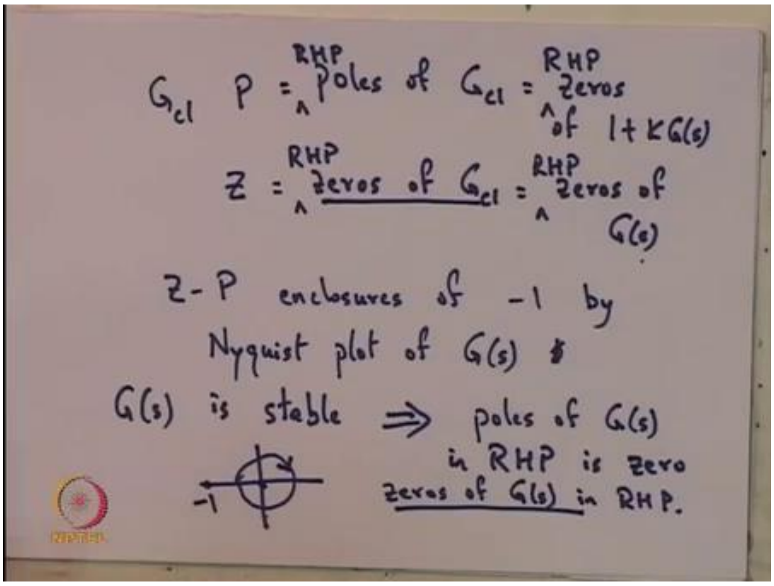

Because, for the close loop transfer function G cl, we observe the following poles of G cl equals 0s of 1 plus KG s, and 0s of G cl is are the same as the 0s of G s. Now, if the number of 0s of G s, this number z on the poles of G cl. Now, when we talk about the 0s, the 0s of G cl are the same as 0s of G s and poles of G cl are the same as the 0s of this thing.

So, of course if I put the qualifier right half plain 0s of G cl, they are the same as right half plain 0s of G s and let me call that z. And similarly, right half poles of G cl is the same as right half poles 0s of 1 plus KG s. And let me call that P, then z minus P enclosures of the point minus 1 by the Nyquist plot of G s. So z minus P are the number of enclosures of minus 1 by the Nyquist plot of G s, where z is right half plain 0s of G cl which is the same as right half plain 0s of G s and P is the right half poles of G cl which is the same as the right half plain 0s of 1 plus KG s.

Now, if you start of with a transfer function which is stable, suppose you have G s is stable, this implies that poles of G s in right half plain is 0. Now, G s is stable therefore, the poles of G s in the right half plain is 0. Therefore, the number of times that the Nyquist plot of G s will encircle the origin will always be clockwise and will correspond to the number of 0s of G s in the right half plane, but the 0s of G s in the right half plane are the same as the 0s of the close loop transfer function in the right half plane.

Now, if instead of looking at the enclosures of the origin, one looks at the enclosure of minus 1 of the transfer function, and it turns out that the enclosure of minus 1 there are no enclosures whereas, the enclosures of origin there is 1. Then, what the input mean is that the there is a right half pole of G cl along with a right half 0 of G cl. And therefore, the number of the enclosures of minus 1 is 0. And this gives us an idea about the close loop transfer function stability by looking at the Nyquist plot of the open loop transfer function.

Now, I am sure most of you have gone through the Nyquist plot way of determining the transfer function for a close loop transfer function from open loop transfer function. And this plays a very important role when you, when it comes to non-linear dynamical systems. Because in the non-linear dynamical system, this particular special point minus 1, that sought of captures the feedback as far as the linear system is concerned and in case of a non-linear system on the feedback loop if you have a non-linear element, then the un certainty of the non-linear element gets manifested in some region around this minus 1. And so you have theorems where you know if you are so in the in the analysis of non-linear system you typically have a situation.

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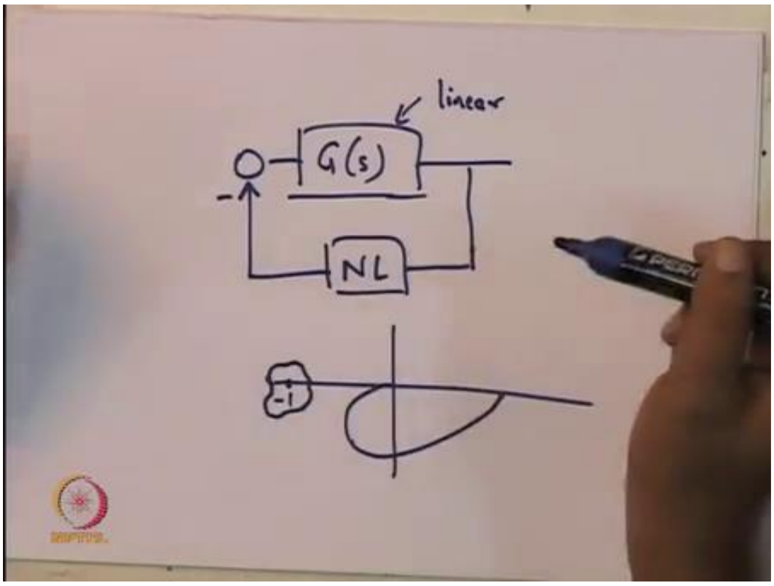

Where you have a linear plant and you have a nonlinearity and you attached this nonlinearity to the linear plant, and you want to talk about the stability of this whole system, but this is a non-linear thing. So this G s, this is linear and as this is linear you can obtain a Nyquist plot of this G s. Now, earlier when you used for feedback, when you used nonlinearity, then it was this point minus 1. And whether that minus 1 gets enclosed by the by the Nyquist plot of G s determined whether the close loop transfer function was stable. When it is a nonlinearity turns out that, this nonlinearity is in sound way approximated by a linearity. Suppose is nonlinearity approximated by a linearity.

Then, it will be just this point minus 1, but when it is, when it is not approximated by a linearity, but you really think of a nonlinearity, you will find that there is some region around minus 1, which accounts for the nonlinearity of this element. And the Nyquist plot of G of s should enclose or not enclose this. In the same way as the Nyquist for the linear system the Nyquist plot should or should not enclose minus 1.

So, instead of having a just one point to consider in the non-linear analysis, it turns out to be a whole area around minus 1 which should be a enclosure should not be enclosed. Of course, what I am talking about now will become clear in the later lectures, when we actually find out this area which talks about, which captures something about this nonlinearity. So, much for now as far as the Nyquist plot is concerned.