Nonlinear Dynamical Systems

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Lecture - 14
1) Norms of Signals
2) Systems (operators)
3) Finite gain L2 stable

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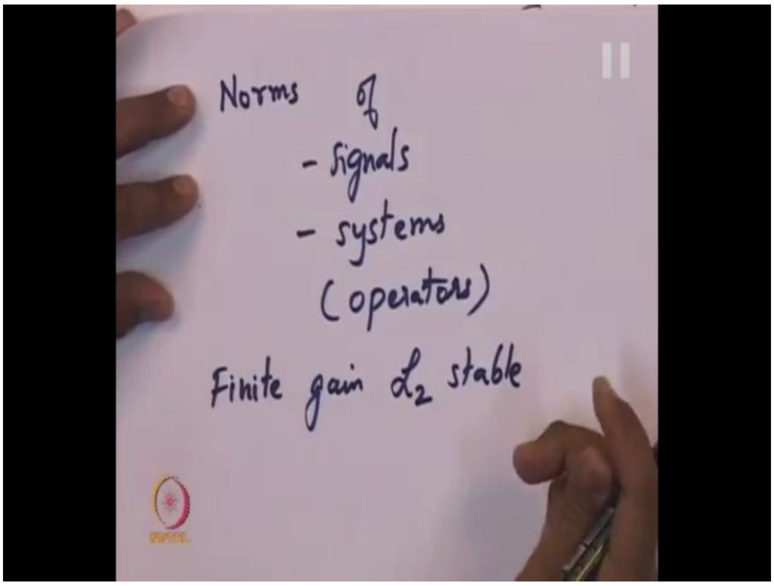

Welcome to this lecture, we are going to continue about norms. Norms of signals we have already seen, we are going to see in more detail about systems, which we are going to consider as operators. So, we have going to see in more detail about this in particular we are going to speak about finite gain L 2 stable, very important class of stable systems, this is what we will see in more detail today, we had just began in the previous lecture.

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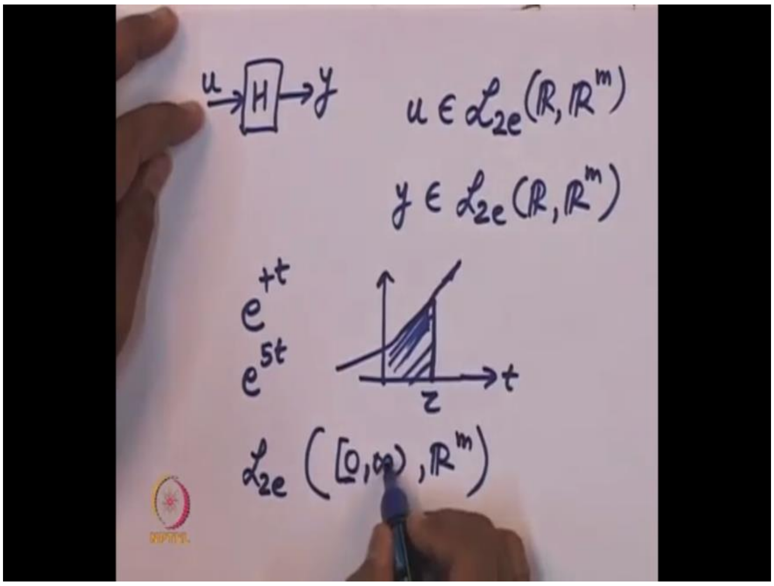

So, we saw that if this is a system - input, output system, there are various notions of stability. And we are speaking of input output stability in which u’s and y’s live in some space of signals. We said a good spaces signals is L 2 e; L 2 e is a rich space of signals, for examples there are e to the power plus t, e to the power any 5 t etcetera. These are all there inside L 2 e R, they are all growing you see these signals are all growing, and then ((Refer Time: 01:51)) L 2 for that for that reason why simply because there all growing, and they are not going to be square integrable, but for each chopped version at whatever tau one chops the chopped version is in L 2.

So, we recall that the definition of L 2 e was you take and put you take a function f and put it into L 2 e, if f tau is in L 2 for every chopped f tau. Now, one can also speak of L 2 e, 0 to infinity to R m. Here we are considering function that I had defined over the domain 0 to plus infinity not from minus infinity to plus infinity like in this, but from 0 to infinity like in this.

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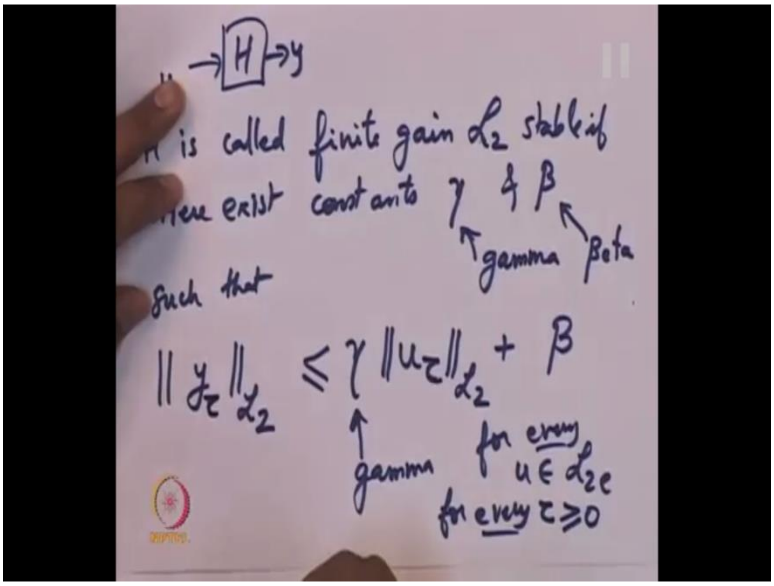

So, we defined a system h with input u, output y to be finite gain L 2 stable, so h is called finite gain L 2 stable L 2 plays a key role in the norm that we are taking. If there exists constants gamma ad beta, so this one is not y, but gamma and this one is beta such that such that what inequality satisfied y notice which one we are putting at various places is lesser or equal to u. The inequality that I am writing now is not yet correct, we will correct it very soon gamma this one is gamma, for every for every u in L 2 e.

So, notice that if u is in L 2 e then it is L 2 norm need not be defined, of course L 2 space of functions is contained inside L 2 e, but you take obituary u in L 2 e is L 2 norm is not defined L 2 norm of which functions that define after chopping by tau. So, notice that you have to chop by tau chop at value tau, then it comes into a L 2 space and then the L 2 norm is defined for a add for every for every tau greater than or equal to 0. So, you chop and then it is in L 2, but then the right hand side, you might chop it different values of tau.

If you chop very late, then the right hand side might become large, if you chop very early the right hand side might become very small. So, notice that you should also be chopping at the same value here, so here also y will be in L 2 after this chop, now is when this equation has lot of meaning, so how much energy comes? So, we are going to chop it tau and how much energy comes in the output y we have measuring the output energy in the output and energy.

So far, this tau you chop by tau and then take the L 2 norm means you look at the energy that has come out. So far, that energy cannot exceed the energy that went in times a constant the energy that went in so far times a constant and perhaps a offset this off set beta is not very important. You should understand this offset beta as suppose you do not give any energy inside, you do not give any energy input u at if 0 is the input, then this particular term becomes 0.

Then, the output can still be non 0, but it as to be bounded for every tau whatever comes out is bounded by some number beta. So, the energy that comes out for every tau is bounded by some by fixed constant beta. So, gamma and beta are some fixed constant that are not allowed to be changed depending on the particular u that you take from L 2 e and the particular tau at which you decide to chop. This is independent of which u and which tau the same gamma and beta has to work for this inequality to hold, that is the significance of this particular result so we will see some consequences of this inequality.

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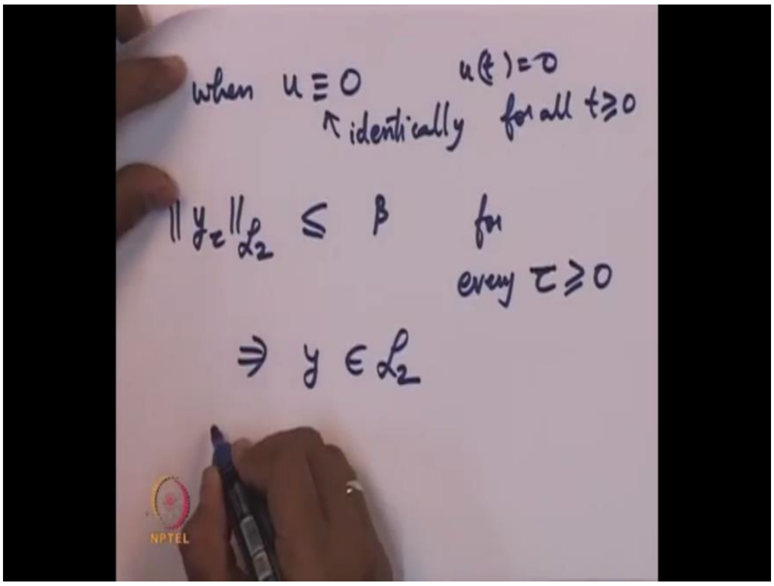

First consequence is when u is equal to 0 u is identically equal to 0. This should be understood as identically not just equal at some time instant, but equal to 0 at every time instant u, u of t equal to 0 for all for all t all over signals are define only for t greater than or equal to 0. Hence, u of t equal to 0 for all t is greater than or equal to 0, then what this says is y tau L 2 lesser or equal to beta for it is not u anyway for every tau.

So, in particular you make this tau very large for every tau for every positive tau, you can make this tau 10 to infinity and still it will be bounded. So, this implies that y itself is in L 2 y because for every tau the chopped version is bounded by some number beta independent beta itself is not allowed to depend on beta itself is not allowed to depend on the tau. Hence, this L 2 norm of the chopped y is independent of the tau and hence y itself has to be in L 2 this is not very hard to prove.

What this says is that when you give 0 input and the energy in the output is finite, the energy in the output is finite not just when you take the energy. So far until now, but also when you integrate up to plus infinity, so what does this say in terms to LTI system that is what we will see in very in much detail very soon, now also let see some more consequence.

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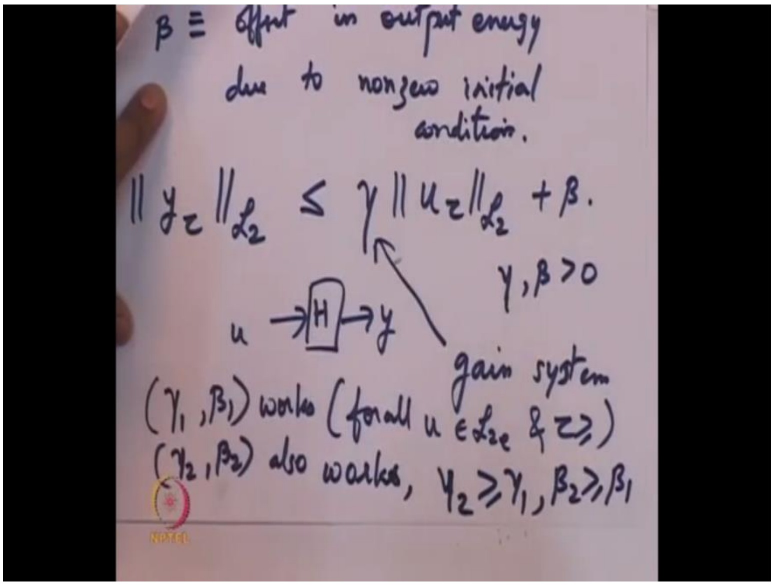

So, beta should be understood as some offset in output energy due to non 0 initial condition, for example there might be some a initial condition inside this system. This might cause the output to be non zero, even though the input is sin integral is 0, so the significance of of beta you have already understood when the input is 0 that time the output can still have some energy. The output energy cannot exceed beta for any tau and hence y itself will be in L 2 for this special case that 0 is the input. Now, let us understand in more detail, let us look at this inequality again, of course plus beta.

So, if you give some finite amount of energy into the system from the input, then the output will also have some energy, but the output's energy is expect to be more if input more energy is given to the through the to the system from the input. So, for linear system, also if you give more energy into the system, of course output should be allowed to give output should be allowed to have more energy in its output in the output signal should be allowed to have more energy.

The output's energy cannot exceed some constant time's the input energy, this kind of quantifies that all the energy that comes in the output has to have gone in through the input plus some constant, which is perhaps because of non 0 initial conditions. This kind of also quantifies that the system h itself is not does not have a source of energy inside why this rules out the source of energy within, because if there is a source of energy within even when you give 0 input the output can have lot of energy outside that is being ruled out by this fact that when you give 0 identity.

When you give input identical is 0, then the output's energy is bounded from above by a value beta, of course what remained to say was that gamma and beta are greater than 0, but that easily follows you see left hand side is some positive number L 2 norm. It is a norm, it is non negative quantity and hence the right hand side is also non negative, both gamma and beta have to be positive numbers because this u itself is also positive, beta is positive follows by taking the fact that u itself might be identically 0. Hence, the left hand side is non negative; hence the right hand side is also non negative, so this is how you should understand this finite gain L 2 stable notion of stability.

So, what is finite gain about it this gamma is like a gain of this system the way we have defined it here called the system h finite gain L 2 stable L 2 played a role because of the particular norm L 2 norm that we took here? Hence, u and y taken from L 2 e and gamma is the gain of the system is it to say the gain, is it unique are gamma and beta unique. We have already noted in the previous lecture that if one gamma and beta pair works anything gamma larger than if gamma one comma beta 1 works.

Then we say works it should say works for all, for all u in L 2 e and for all chopping tau, then gamma 2 comma beta 2 will also works, also works as long as you take gamma 2 greater than or equal to gamma 1 and beta 2 greater than or equal to beta 1.

This pair will also work as so clearly this is not unique one might try to say what is the minimum gamma for which this inequality holds at minimum gamma is the minimum gamma for which inequality holds for all u. For all tau that is resemble to be called the gain of the system the such gamma, but then we do not need that concept in more detail, hence we are going to see the L 2 norm the gain of the system only for the LTI systems SISO.

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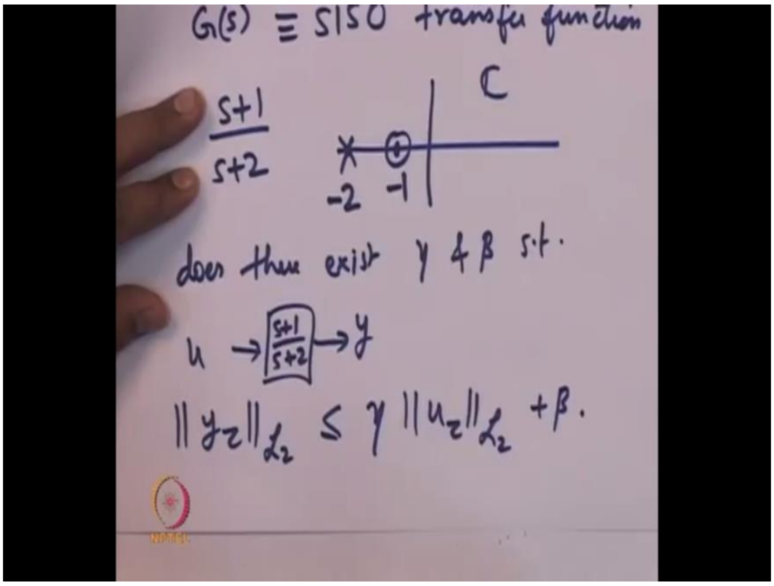

Consider some SISO transfer function for example, take s plus 1 over s plus 2, so this particular system is a stable, it has it is poles in the left half complex plain, this is the 0 the pole minus 2 minus 1. So, in this particular for this transfer function this is stable, so this does there exist gamma and beta such that such that for this system s plus 1 over s plus 2 for this system. This, as I said this beta depends on the initial condition, let us assume that the initial condition is 0, if the initial condition is more, of course more energy will come out in the output even when the input is identically 0.

Hence, this beta is playing a role because the initial condition also could be sitting inside and could be non zero, but we are going to consider the case that the initial condition is equal to 0. Hence, we will not be considering beta the role the beta plays that is the relatively easier issue anyway. So, we want to see what is the minimum gamma that allows us to call that minimum gamma as the, so called L 2 induced norm of the system.

So, this L 2 induce norm has a very important significance, it is also the peak value of Bode magnitude plot, that is what we will see in detail.

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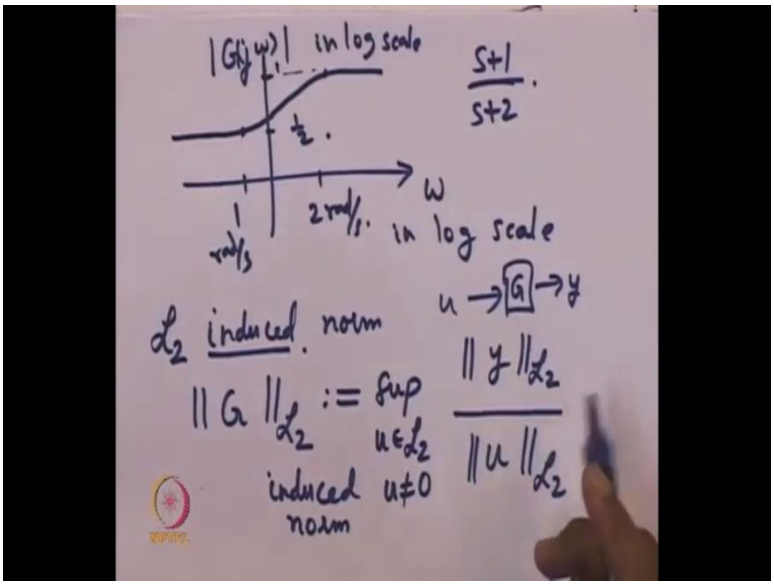

So, it is well known, consider again this transfer function s plus 1 over s plus 2, this is how the Bode plot looks magnitude of g of j omega in the log scale. Also, this is in log scale, it starts increasing at the 0, 1 radiant's per second, it stops increasing at 2 radiant's per second at a pole. So, these are the cut off frequencies, so notice that the peak is equal to one which is 0 dB and this corresponds to 1 by 2 and then s is equal to 0, s equal to 0 which is the d c gain. This norm this gain is 1 by 2 and as s tends to infinity, which is the gain as at very high frequencies this is equal to 1 at s tends to infinity.

You see s this constant terms do not play role because the leading co efficient are equal and hence it becomes tends to one that dealers are also same by proper, the numerator and the denominator decrease a equal to each other. So, this tends to on max, so you can ask what the question the L 2 induced norm what is induced about it. So, you take the system g is defined as induced norm this is g is not a signal, we define L 2 norm of signals so far, but now we are going to define so called induced norm. This is not the L 2 norm that you saw so far, we are defining norm of a operator of a system that takes input u output y and has a transfer function like we had so far. Now, we have a LTI system with a transfer function g, hence we had defined it is L 2 norm with g here as output L 2 norm divided by input L 2 norm.

This ratio is resembled call this as a gain of the system, but this ratio might be different for different inputs that you give even though it is being scaled by the inputs, it being divided by the energy in the input. Still, this ratio might depend on what frequency you have concentrated your energy in we want to look at this supremum over all u in L 2 and of course u should not be equal to u should not be equal to 0. Otherwise, you will have 0 in the denominator, so this supremum only means that is like the maximum. We are not looking at the maximum value over a finite set L 2 is a very large set not a compact set and hence we are replacing max by sup, sup is a supremum.

So, this ratio the this ratio indicates the maximum amplification that the system g can cause to an input u and this amplification is being measured in terms of the L 2 norms. We notice that these L 2 this L 2 and this L 2 here stand for signals for systems for signal that you have already seen. Hence, this is what you have already understood unlike here where this is a L 2 norm being used for a system g. So, by using the ratio of L 2 norms of output divided by input and then the taking of supremum over all such signals u, we have used that to induce a notion of norm for an operator G.

So, these induced norms have also have those properties that we have indicated so far like triangle inequality and the norm being at least greater than or equal to 0, and norm is equal to 0 only when the operator itself is a 0. Further, if we scale the operator by some number and the L 2 norm also get scaled by the actual value of that number. All this property are satisfied for norm in addition to some more properties called sub multiplicative property that did not need in much detail in this course, when we need we will see.

So, L the induced norm is defined like this it turns out to this norm will meet the purpose of the gamma written there. So, let gamma be defined as the induced norm L 2 induced norm since we have used L 2 for a operator it is understood that this L 2 is not of a system not of a signal. The induced norm that we use an induced refers to taking the ratio of the output sent to norm divide by the input’s L 2 norm. So, since this one was defined as supremum over all u in L 2 of output’s L 2 norm divided by input’s L 2 norm, since we took this as a ratio, we are now going to quickly see that this induced norm will serve the purpose of the gamma.

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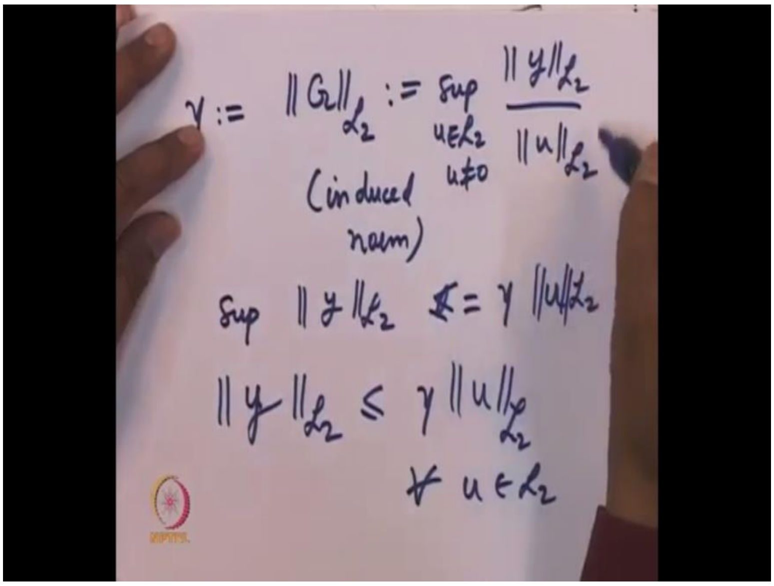

So, define gamma as this L 2 induced norm when this L 2 as being used in the context of a system of a operator, then we are going to understand it as obviously being the induced norm from the context. It is clear because G is not a signal it is a system, hence this L 2 here refers to the supremum that we just now defined of the output's L 2 norm divided by the input's L 2 norm. Of course, we have not yet seen whether this is finite, whether this exists or not, but you will take the supremum of this ratio supremum over what supremum over all u in L 2.

This means you vary this u for different values in different signals in L 2, and look at this ratio and ensure that denominator is not equal to 0. That is ensured by taking u unequal to 0 unequal to the signal that is identically 0 and this we will define as gamma. What this means is that the supremum of y L 2 is lesser than or equal to is equal to gamma times u L 2, L 2 norm of the input if the maximum the supremum is as good as the maximum.

As I said, it almost reaches this that is not attained that is only concerned see if the supremum is equal to this. Then, any other is less than or equal to gamma times u L 2 norm of u, why this inequality has come because this gamma was defined as a supremum. What all we did is, we took this denominator from this side to that side and we wrote this in equal this equality gamma equal to supremum of y L 2 norm divided by u L 2 norm.

That is what we have multiplied and written here because we know that the denominator is unequal to 0 and if this supremum of the left hand side is equal to this. You if you get rid of the supremum, then this inequality ends up coming, now this is true for all u in L 2. That is how we have obtained back that inequality, we started with except that one might say that what happened to the e the extended that extended has not come because we have already assumed that this ratio exist.

This ratio could easily have been unbounded, it might not have existed, it might be very large and infinity infinite. So, this ratio this is finite if and only g is stable that is what we are going to see in detail now, so when g has no poles in the right of plan, nor does it have poles in the imaginary axis, that is when this gamma will exist.

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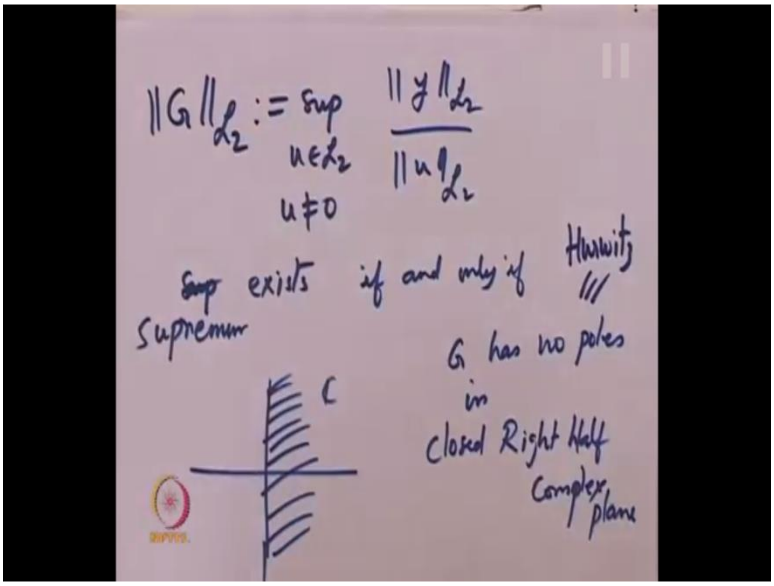

So, g L 2 is defines as supremum over all u in L 2 of y L 2 norm L 2 norm and u unequal to 0 this sup sup exists sup meaning supremum, sup is the short form for supremum. Supremum exists if and only if input result G has no poles in closed right half plain right half complex plain.

If G has no poles in a closed right half complex plain, what is right half, of course that is clear what is closed about it no poles on the imaginary axis. Either all the poles are in the open left half complex plain, what is open about it the boundary of the left half complex plain is imaginary axis the boundary has not been included.

That is what makes it open left half complex plain, so to say all the poles are in the open left half complex plain is same as saying there are no poles in the closed right half complex plain. The right half complex plain is where I put my fingers to say closed means it includes its boundary the imaginary axis is, it is boundary if it has no poles in the closed right half complex plain. That is when we will call g also as Hurwitz and this property G has no poles in the right half complex plain is also called Hurwitz.

Hurwitz also depends on the context and in this context, it means when G is rational, when G is ratio of two polynomials G having no poles in the closed right half of complex plain is also called as Hurwitz. That is precised, the case when this supremum that we have defined here is a finite value, this supremum could easily become infinity, because the numerator might become unbounded even though the denominator is finite. That is when this ratio need not exist, it might become very large and infinite, that is when we will say L 2 norm does not exist.

When will the L 2 norm exist when is supremum finite, when it is bounded, it is bounded precisely when G has no poles in the right half complex plain nor does have poles on the imaginary axis. This is precisely the case that the L 2 norm exist in that case you give a u in L 2, now output will also be in L 2 and then this ratio will exist and that justifies. Let us go back to previous slide that justifies for that case this gamma is finite and hence this inequality holds for all u in L 2 the L 2 e, we do not have to write the e because this holds for all u in L 2 itself.

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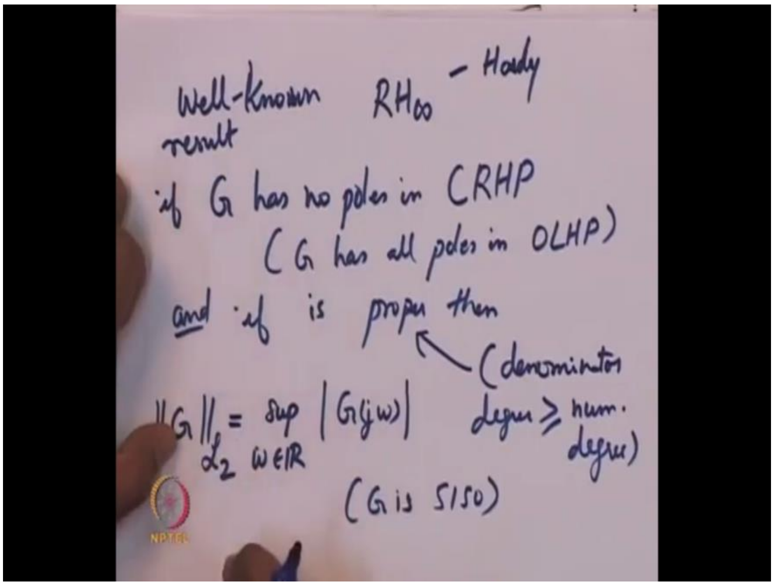

So, now let us look at a notion of gain for some more systems what is the meaning of this maximum, so well known. There are many CRHP is also called R H infinity well known result if g has no poles in closed right half plain which means G has all poles in open left half complex plain and if G is proper. This proper is another condition I forgot in the previous slide that is properness is also required if you do not want. If you want G to have a finite L 2 norm, if you want, the induce L 2 norm of g to be finite, then G has to have all its poles in the left half complex plain in the open left half open plain.

Further, g also has to be proper is same as saying denominator degree greater than or equal to numerator degree. This means that G is proper which means as s tends to infinity G s exits, G infinity is a finite number. What is well known is that the l L 2 induce norm that we defined is equal to is attained is in fact attained on the imaginary axis this supremum value, supremum over omega in R of g of j omega. Consider further case that g is SISO and G is single input and single output that time this max L 2 induce norm is nothing but supremum over all omega in R of g of j omega. This is a very important well known result and what is R h infinity that I have written here R stands for real rational. Those transfer function is g which is a ratio of two real polynomials, polynomials whose co efficient are all real that is what R stands for h infinity stands for those transfer function which are proper.

They have all their poles in the open left half complex plain h h is in memory of person called hardy who worked a lot in such spaces and many others and is a close associate of well known Ramanujan. So, this class of transfer functions G, which have real rational and which have all the poles in the open left half complex plain, it is well known that the L 2 induce norm that we defined is equal to just the supremum of this particular value. Here, take the absolute value of g at different points on the imaginary axis and look at this supremum over all these points on the imaginary axis, so that is what brings us to the Bode plot the Bode magnitude plot to be precise.

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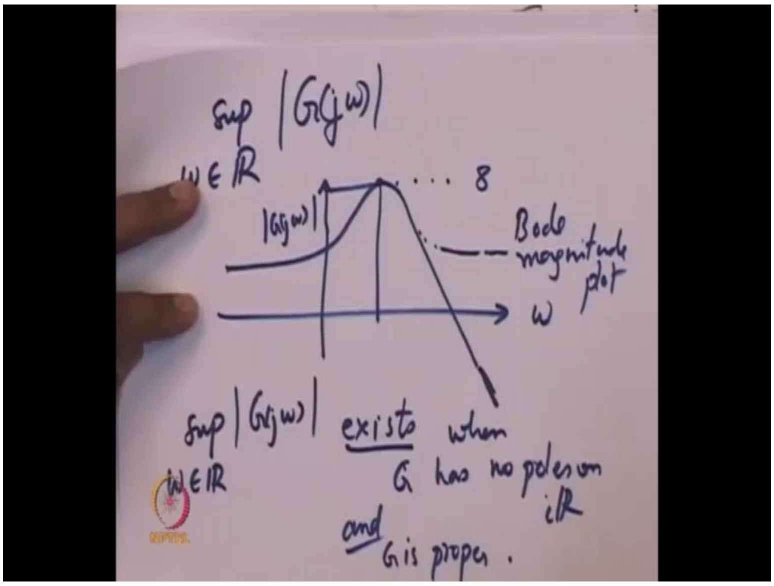

So, supremum over all omega in R of g of j omega for this is a complex quantity, we take the absolute value this to do this is nothing but suppose this is a Bode magnitude plot, then this supremum this value here suppose this value is equal to 8. Then, we will say this is a Bode magnitude plot, why will this need finite because g has no poles on the imaginary axis. Because G has no poles on the imaginary axis, nor is g proper nor is g improper because g is proper as s tends as for as omega tends to infinity, it either slows down, which is a case, when g is strictly proper.

The denominator dB is strictly more than the numerator degree, it saturates it can saturate to a value different from a d c gained if g is by proper to say. It saturates means as omega tends to infinity it is non 0 finite values and to say it is finite means G is proper the nominator degree does not exceed the denominator degree.

To say that it is non zero means that two degrees are equal, so it will either come down like this with some roll of roll of that depends on the difference in the degrees of the numerator and denominator. It will saturate to some value for the case that the degrees are equal, so this peak value is finite as soon as these two cases are satisfied g as no poles on the imaginary axis and G is proper what the value of the peak that is equal to 8. This peak will indeed be equal to the L 2 induce norm precisely when g has no poles in the right half complex plain.

Also, supremum of g of j omega supremum over all omega in R exists, when g has no poles on the imaginary axis and two conditions that required and g is proper. So, this supremum when will it exist, it could easily become infinity, it could become infinity for it it will become infinity. For example, if g has poles on the imaginary axis in which case this goes off this become very large, it becomes unbounded the Bode magnitude plot.

So, if g has no poles on the imaginary axis at no finite value omega does not become unbounded and if g is proper then as omega tends to infinity. Also, it does not become unbounded, so when these two conditions are satisfied that time, this supremum exists. We will say if exists means it is a finite value, but just because it exists does not mean make it does not make the supremum equal to the L 2 induce norm when is it equal to the L 2 induce norm.

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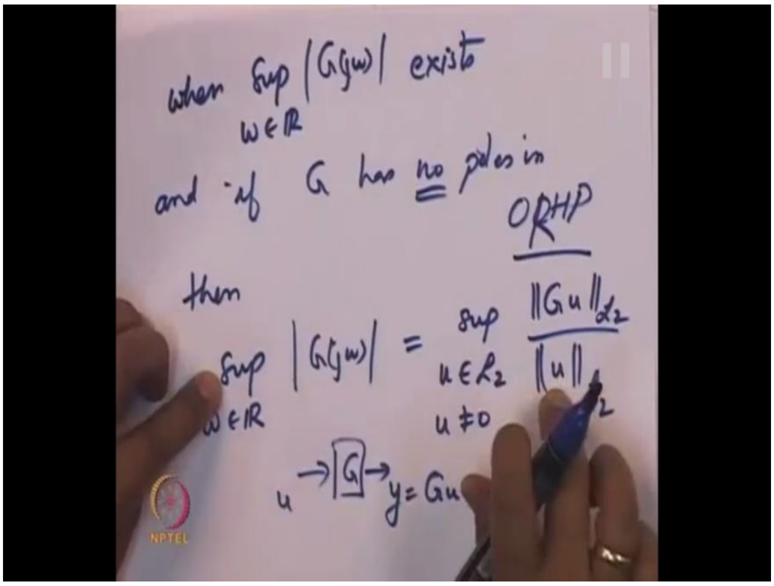

When Supremum g j omega exists and if G has no poles in the open left half complex plain, open left half complex plain has poles have not got, oh sorry no open, right half no poles no poles in the open right half complex plain this has got ruled out. Imaginary axis already got ruled out because supremum exists, then this supremum is equal to the L 2 induced norm, this is the frequency domain condition and what I write here is a time domain. So, you look at this system u this system g with input u, output y equal to we are going to say just G u g acting on u.

For this system, you look at the output's L 2 norm divide by the input L 2 norm, this L 2 norm is what we define in the time domain for the signal. So, this right hand side quantity is a supermom over all L 2 signals and u not equal to 0, that ratio we have taken in the time domain that is equal t this frequency domain condition the peak value of the Bode magnitude plot.

For these two to be equal you want g to have no poles in the open right half complex plain, the left hand side exists as soon as G has no poles on the imaginary axis nor no G has no poles on the imaginary axis and G is proper in that case the left hand side exists. It will become equal to the right hand side when G has no poles in the open right half plain also, here are some exercises.

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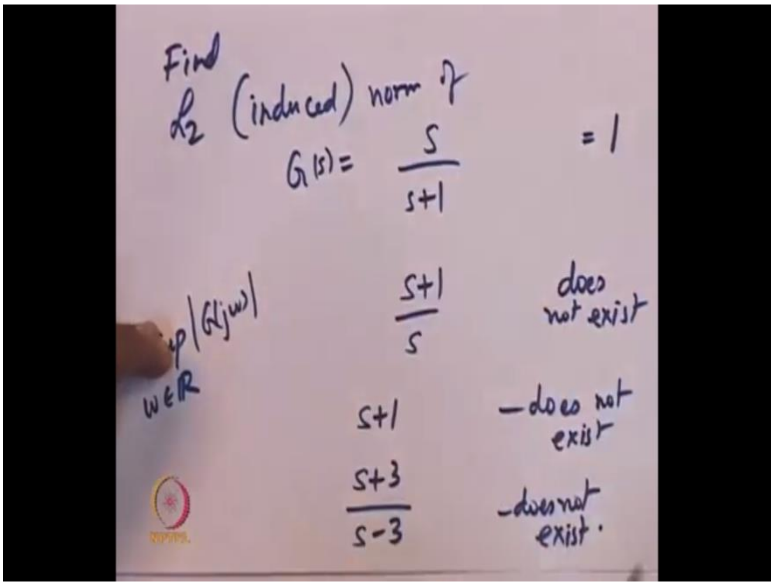

So, find L 2, L 2 norm of a system means it is a induced norm of g equal to s over s plus 1 s plus 1 over s, s plus 1 and s plus 3 over s minus 3, what are these values equal to please take your time and calculate to get for this the value is equal to 1. G of s equal to this, these L 2 induced norm of this equal to 1, this does not exists here, also does not exists, please take your time to find out why they do not exists, these do not exists, these all three do not exist. This is because there is a pole on the imaginary axis, this is because is improper here, because the there is a pole in the right half complex plain, but for each of this four cases also please check for which case does supremum of g of j omega.

As I said, this supremum can exist under a milder conditions, under what condition does this exist this supremum need not be equal to the L 2 induced norm of g, it will be equal only under certain conditions and for which these four cases does this exist. Of course, it will be equal to this for this case, but for other three cases what requires thorough investigation. So, we have we have seen some examples of transfer functions and when they have a finite L 2 induced norm.

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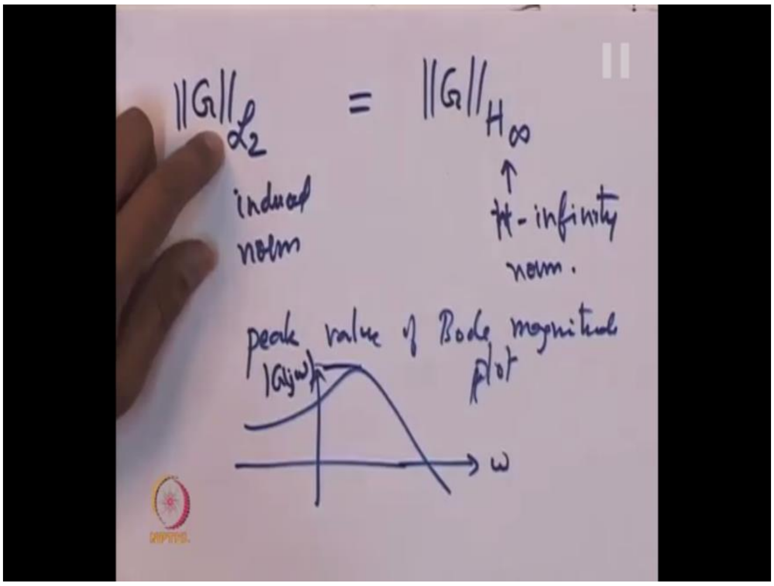

What remained to be told is a very important word L 2 induce norm has another name, just it is only a different name that is also called as H infinity norm, H infinity norm, what is a H infinity norm? It is nothing but the L 2 induced norm for single input single output systems, it is nothing but peak value of Bode magnitude plot for single input single output systems.

It is very easy to calculate, it is the equal to the peak value of the Bode magnitude plot this peak value exists if G has no poles on the imaginary axis, it is equal to the L 2 induced norm. When g has no poles on the imaginary axis, when G is proper and G has no poles in the right half complex plain under that condition is when the H infinity norm equivalent to the L 2 induced norm exists and is finite that time it will be equal to the peak value. The peak is what is equal to the H infinity norm, precisely when G is stable when G has all its poles in the left half complex plain and also when g is proper. So, what is this peak value it has significance for the case that G is stable G is stable, one can also look at its Nyquist plot.

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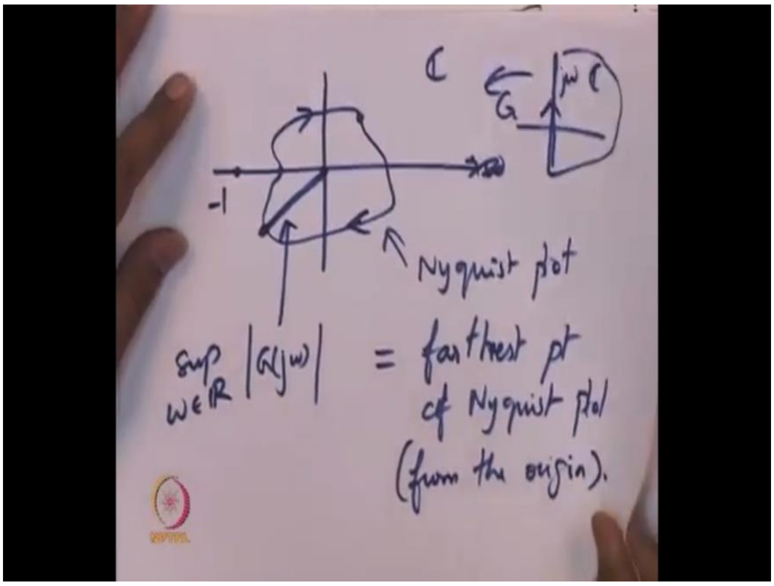

So, let suppose this is the Nyquist plot, of course it is symmetric above the imaginary axis with some orientation. Let us say this itself a complex plain Nyquist plot at any omega, this is where G of j omega is for the Nyquist plot is nothing but a map of the imaginary axis under the action of G. This is the G omega, the imaginary axis, it gets mapped under G to this particular contour with a orientation.

Of course, if closed contour gets mapped to this closed contour in some orientation important orientation and there is this point minus 1. We can ask at each point what is the distance of that G of j omega from the origin the peak value is nothing but the maximum distance. This is absolute value of g of j omega just a complex number at any omega g of j omega is a complex number and absolute value is its distance from the origin. So, supremum is nothing but the maximum distance as you go as you travel along the Nyquist plot and you have never gone to infinity. This point has not gone to infinity precisely because G has no poles on the imaginary axis and also because G is proper in that case.

This contour will be a closed contour and that time one can look at the peak distance from the origin. So, this supremum over all omega in R is nothing but farthest point of Nyquist plot farthest from what farthest from the origin from the origin. This is a significance in the Nyquist plot, of course we will see this in little more detail very soon. So, we are going to after this development of L 2 induced norm, we are closed to seeing an example called the small gain theorem.

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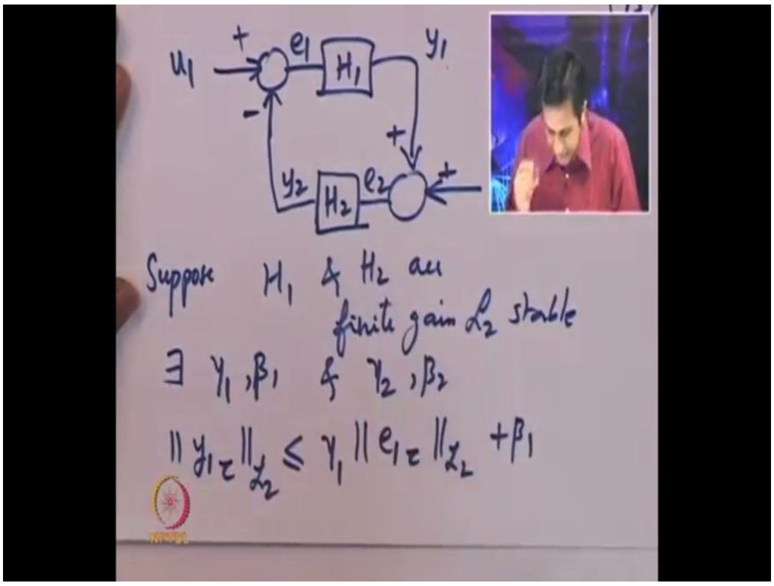

So, for that purpose we need to see the notion of feedback interconnection being stable, let us think of one them with a negative sign for the feedback for the small gain theorem. It does not matter whether this is plus or minus suppose H 1 and H 2 are finite gain L 2 stable finite gain L 2 stable, means that exist comma this particular symbol with e reversed. This means there exists gamma 1 beta 1 for this is for H 1 and gamma 2 comma beta 2 for H 2 such that what inequality satisfied this inequalities. What you have seen several times today already y 1 chopped L 2 norm is less than or equal to gamma 1 times e 1 chopped L 2 norm plus beta 1. Similarly, the same inequality for H 2 which puts an inequality between the L 2 norm of y 2 chopped and L 2 norm of e 2 chopped related by gamma 2 instead of gamma 1.

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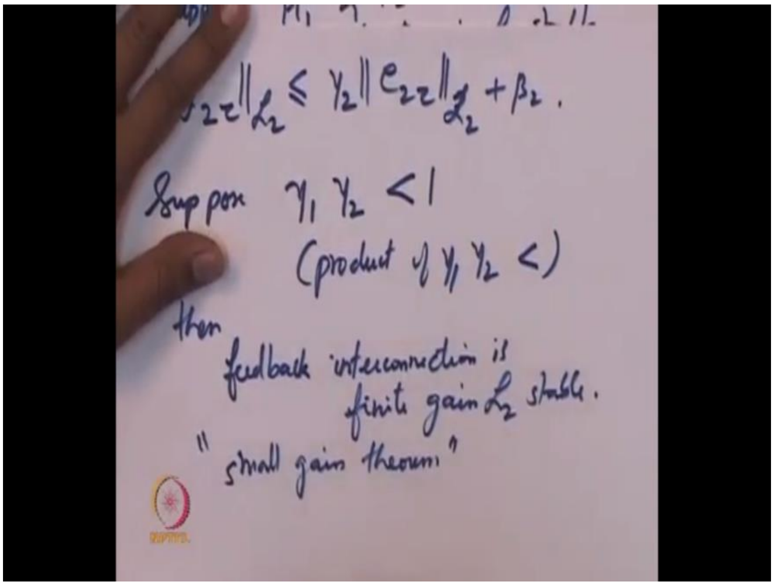

So, we can ask the question is a feedback interconnection stable that is the question that we are trying to answer. So, before we come to that question, look at this feedback interconnection we have already assumed that H 1 is L 2 gain finite gain, L 2 stable. We have assumed that H 1 is finite gain L 2 stable, we have assumed that H 2 is also finite gain L 2 stable and we want to ask is the interconnection also finite gain L 2 stable. Now, the question arises, what does it mean for the interconnection to be finite gain L 2 stable to say that the interconnection is finite gain L 2 stable.

This means that the map with u 1 u 2 as a input to y 1 y 2 as the output we can think of this interconnected system as a map as a system which takes u 1 u 2 as a input and gives a output y 1 y 2. Of course, you might say should the output be y 1 y 2 or should the output be e 1 e 2, both are and you can also consider u 1 u 2 as the input and all four signals as the output. If this map from u 1 to u 2 to all these signals is also finite gain L 2 stable, then we will say this feedback interconnection is finite gain L 2 stable, this is a important concept, so let me just recap this concept.

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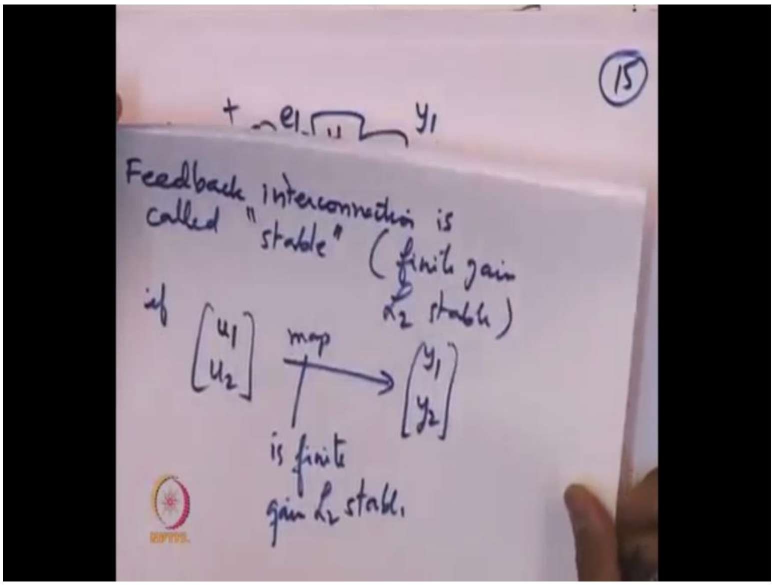

Feedback interconnection, interconnection is called stable here by this word stable here, we mean finite gain L 2 stable if what is satisfied u 1 u 2 y 1 y 2 map is finite gain stable finite gain L 2 stable. This particular map from the input u 1, u 2 to the outputs y 1, y 2 if this map is also finite gain L 2 stable, then we will say that this feedback interconnection is stable is finite gain L 2 stable, which feedback interconnection this finite feedback. This feedback interconnection, you might ask y take outputs y 1 y 2, it can be shown that if u if the map from u 1, u 2 to y 1, y 2 is finite gain L 2 stable.

Then, the map from u 1 u 2 to e 1 e 2 is also finite gain L 2 stable, these are equivalent, so between y 1 y 2 and even e 2, you can take any one pair as the output. If you want to take all four of them both pairs that is that is also, but it is not to take this e 1, e 1, y 1, one should take cross y 1, y 2 or e 1 e 2. So, this a meaning of feedback interconnection stable, now we want to ask the question under what conditions on h 1, h 2 will we have stability of the feedback interconnection stability of the feedback interconnection. As I said here is finite gain L 2 stable, from this these inputs u 1, u 2, these are like being injected at the interconnection points.

There is also notion of well posedness of interconnection, well posedness of interconnection means that u 1, u 2 are genuinely are inputs and as soon as u 1, u 2 inputs are given, rest all outputs are determined rest all variables are outputs. This means that they all get determined as soon as u 1, u 2 get determined.

So, to say that this, if you invite interconnection is well defined to say it is well posed means for every u 1, u 2 input, there is a unique output e 1, y 1, e 2, y 2, all these four are unique. So, we have already assumed that this particular feedback interconnection is well defined, after it is well defined we have introduce a notion of feedback interconnection being stable feedback interconnection. This means stable means that this interconnection finite gain L 2 stable the map from u 1, u 2 to y 1, y 2 is feedback is finite gain L 2 stable.

So, for that for this feedback interconnection to be finite gain L 2 stable, we are going to see a sufficient condition. So, for that sufficient condition you have assume that both H 1 H 2 are finite gain L 2 stable to say that H 1, H 2 are finite gain L 2 stable is that there exists gamma one beta 1 gamma 2 beta 2 such that these two inequalities are satisfied. We are seeing an example one sufficient condition for the feedback interconnection to be stable, of course the feedback interconnection can be stable in when several other conditions are also satisfied not necessarily.

This is just an example of one sufficient condition, what is the sufficient condition suppose gamma 1, gamma 2 product is strictly less than one product of gamma 1, gamma 2 is strictly less than 1. Then, if this is satisfied, then feedback interconnection is finite gain L 2 stable is this is called the small gain theorem, what is a small gain theorem, it is an example, it is a sufficient condition. The feedback interconnection to be stable that sufficient condition is as follows that we consider this feedback interconnection, suppose it turns out that H 1, H 2 are themselves finite gain L 2 stable to say that they themselves finite gain L 2 stable.

This means that these two inequalities are true, of course for all for all e in L 2 e and for all e one for all chopping greater than or equal to 0 and here for all e 2 in L 2 e, this is particular symbol. For all, it is a currently commonly used to say, for all for every under the condition that H 1, H 2 are both finite gain in L 2 stable and if the product is strictly less than 1, then one can also guarantee that the feedback interconnection is also finite gain L 2 stable.

This is a sufficient condition for the finite gain for the feedback interconnection to be finite gain L 2 stable and this theorem is called the small gain theorem. One can understand the small gain theorem like this, consider this feedback interconnection.

Again, remove u 1, u 2 start from e 1, go to y 1, the maximum gain that can happen is gamma 1, then you go from here to here the maximum gain is gamma 2. So, when you forget the minus sign multiplication by minus sign does not really change the gain, gain of the minus 1 is nothing but the plus 1. When you round this loop, how much amplification have occurred from here to here by gamma 1, there is this plus u 2, there is no u 2 from here to here it is gamma 2, but the product of gamma 1 gamma 2 strictly less than 1.

So, when one travels the travels the loop completely and comes back here, the net magnification is strictly less than 1, the extremely important point to note that u travels the loop and come back to the same point. The maximum amplification that can occur is strictly less than 1 if somebody assures that the maximum amplification, they can occur is strictly less than 1. Then, the small gain theorem assures us that the feedback interconnection is finite gain L 2 stable and for that maximum amplification strictly less than 1, notice that this plus sign or minus sign does not matter.

It is because multiplication may minus 1 does not change the maximum amplification at all because the gain on the operator minus 1 is again equal to plus 1 due to that reason. So, this complete this topic about norms and various signals norms of the operator, we have seen what the meaning of the finite gain L 2 stable in the context of LTS systems. For that case, we have seen that it is nothing but all poles in the left half complex plain in the open left half complex plain and G is proper.

For that case, we have seen that the L 2 induced norm, which is also called the H infinity norm is nothing but the peak value of the Bode magnitude plot. So, this and this peak value of the Bode magnitude plot is nothing but the distance of the farthest point of the Nyquist plot, farthest from the origin the distance from the origin. So, this completes this important topic and this is required for various other topics, we will continue with the next topic in the next lecture.

Thank you.