Nonlinear Dynamical Systems

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Lecture - 10
Bendixson Criterion and Poincare-Bendixson Criterion
Example: Lotka-Volterra Predator Prey Model

Welcome everyone to lecture number 10 on non-linear dynamical systems. We will continue with the Bendixson criteria and the Poincare Bendixson criteria. In particular we will see important examples. One is Lotka Volterra predator prey models and also the van der pol oscillator. Let us start with the Lotka Volterra predator prey model.

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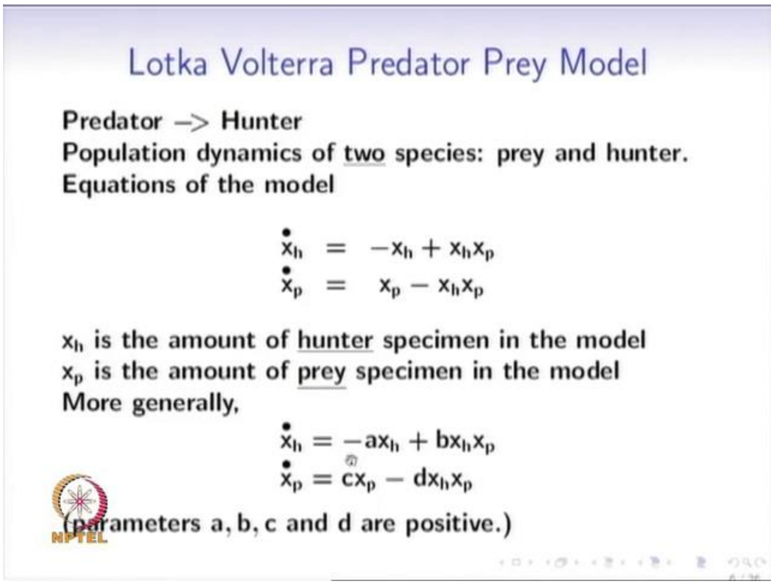

This is studying how the population of two species vary as a function of time. These two species are classified into prey and hunter. So, there one specie that is a prey and other species that is the hunter and we will study the model of this prey and hunter species. Of course, we are studying a simplified model. Let x h be the hunter specimen in the model and let x p be the prey specimen in the model. So, what does this equation say, x dot is equal to x dot h equal to minus x of h plus some quantity that depends on both x h and x p and x dot p equal to x p minus x h x p.

So, the first term in each equation is how the particular specie would evolve, if there were no other species. So, the first term first equation says that, if there were no prey that is if x p were equal to 0 then x h would just decrease a as function of time, it would decrease exponentially because there is no food. So, left to itself the hunter specie would just decrease, but for each interaction between x h and x p, the hunter eats the prey. Hence this extra, the next term the second term in this right hand side is causing an increase in the hunter population.

So, the hunter population decreases because of its own population and it increases because of its interaction with x p. So, the rate of increase is proportional to both x p and x h population. It is bi linear to it is equal to the product that is the increase causing term. On the other hand the prey itself is just going to multiply, it is going to increase exponentially when left to itself. If there had been no hunter specie and interaction with the hunter specie causes x p to decrease.

So, quantities x h and x p are all positive and whether the increase or decrease depends on its own population and also population of the other species. This is the reasonable model for how dynamics of 2 species that interact with each other evolves as the function of time. Of course, we have simplified most importantly in the sense that, more generally there will be some constants x dot h would be equal to minus a times x h plus b times x h x p and x dot p is equal to c times x p the rate of increases proportional to some c times x p in general. Decrease the interaction causes a decrease with this multiplication with d.

So, this is how one could study a general model, but 1 can consider that, we are choosing a different unit for x h and x p. So, that this constant will become will equal to one and also, there is some normalization that has been done, so that we are studying this model. Of course this itself a simplification, this model is also a simplification because there might be some higher order derivatives. We have seen already how the population of just one species can vary with resource availability with ability to reproduce depending on the interaction between species. All that have been ignored, assume this with first order dynamics with respect to itself and just the product the interaction is just the product of the two species population.

So, the questions that we can ask for this particular model is, what are the equilibrium points what is the nature of equilibrium points of the linearized system are they periodic orbits.

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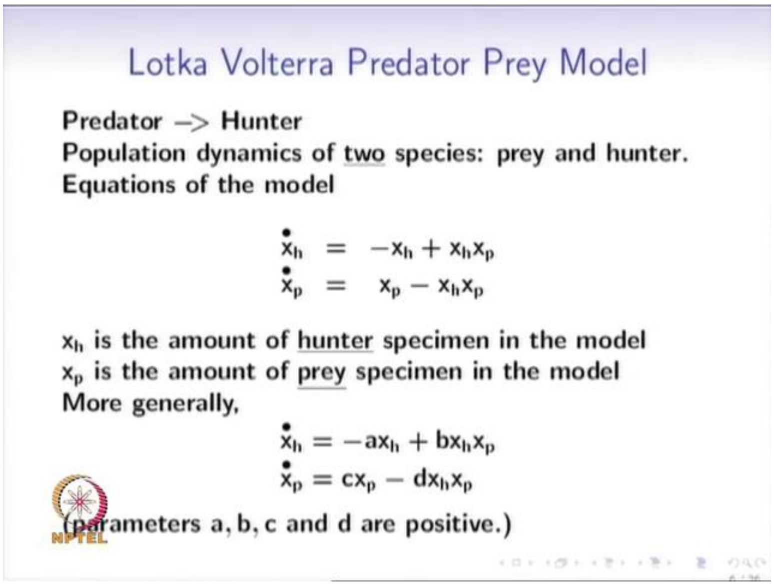

These are the questions we will ask. So, let us go back to this particular model and we will find the equilibrium points for this system.

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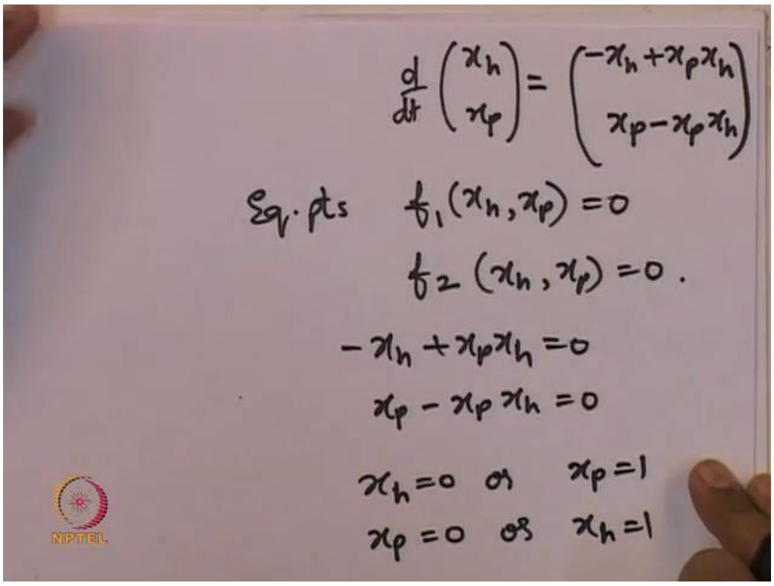

d by d t of x h and x p is equal to minus x h plus x p times x h and this is x p minus x p times x h. So, this is our f, so equilibrium points. Points are those values of x h and x p where f 1 of x h, x p equal to 0 and also f 2 of x h, x p equal to 0. So, what do we get by equating x h minus x h plus x p x h equal to 0 and x p minus x p x h equal to 0 for a particular value of x h and x p to be an equilibrium point. These two equations have to be satisfied. Let us see what are values for which these equations are satisfied first equation says h equal to 0 or x p equal to 1. Second equation says x p equal to 0 or x h equal to 1. So, this gives us, how many pairs of equilibrium point. The equilibrium point has an x p and x h coordinate. So, let us see what all possibilities are there for equilibrium point.

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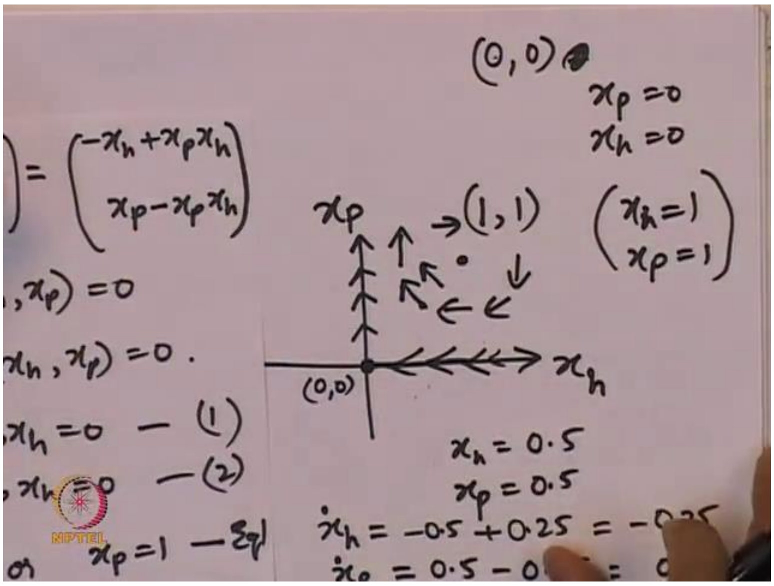

See both equations have to be satisfied then one can have 0, 0. This is nothing but x p equal to 0 and x h equal to 0. The first component in this is x h specie value, second is the x p population value. So, both equal to 0 is 1 equilibrium point, that is what we get from here and both equal to 1, 1. So, which means x h equal to 1 and x p equal to 1. This is another value for the equilibrium point. You see notice that other this if x h is equal to 0, you cannot have x p equal to 1 equal to 1 right because for the other for both equations this is equation 1 and this is equation 2. This is 1 this is 2, equation 1 says that, any 1 of these 2 possibilities, equation 2 says any 1 of these 2 possibilities.

When we combine them we get that these 2 equilibrium points. These 2 points these 2 values of x p and x h are situations where the population species does not change as a function of time. So, this is where x h this x p 1 equilibrium point is here and other equilibrium point is here. This is the equilibrium point 1, 1 this is equilibrium point 0, 0 as I said the first component denotes the x h value. Let us see what happens if x p is always equal to 0 x p equal to 0 means this is the hunter population.

So, our dynamical equation system says that if x p is equal to 0 which means that the second term is always equal to 0. If you put x p equal to 0 here and x h is just decreasing that is why we have drawn these arrows and if x h were equal to 0. So, this is sitting on the x p axis, then x p just goes on increasing, this is how the arrows look, but more generally it is the combination of the 2. For example, let us take what happens at 0.5, 0.5 that is draw the arrow at this particular point which corresponds to point 0.5 and 0.5. So, at x h equal to half and x p equal to half we get x dot h equal to.

So, this is just substituting 0.5 in place of these 2. So, we get minus 0.5 plus 0.25 which is equal to minus 0.25 and x p dot is just 0.5 minus 0.25, which is equal to 0.25. So, this is the vector whose x h component is negative, but x p component is positive. So, this is an arrow that looks like this. So, that its x h component the horizontal component is x h, it is decreasing, but x p component is increasing. So, like this we can draw arrows for all the points. One can check that this is how we get.

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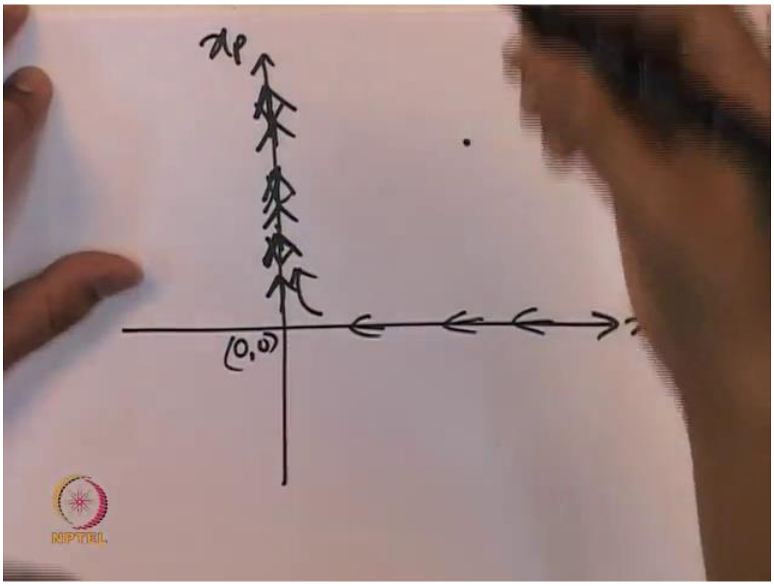

Let me draw a bigger figure. Only the first quadrant is visible because the population do not become negative. So, this is the point 1, 1 this is 0, 0. As I said x h population is going to decrease if x p is equal to 0 x p equal to 0 corresponds to this x h axis and x p axis corresponds to x h equal to 0. So, I am sorry.

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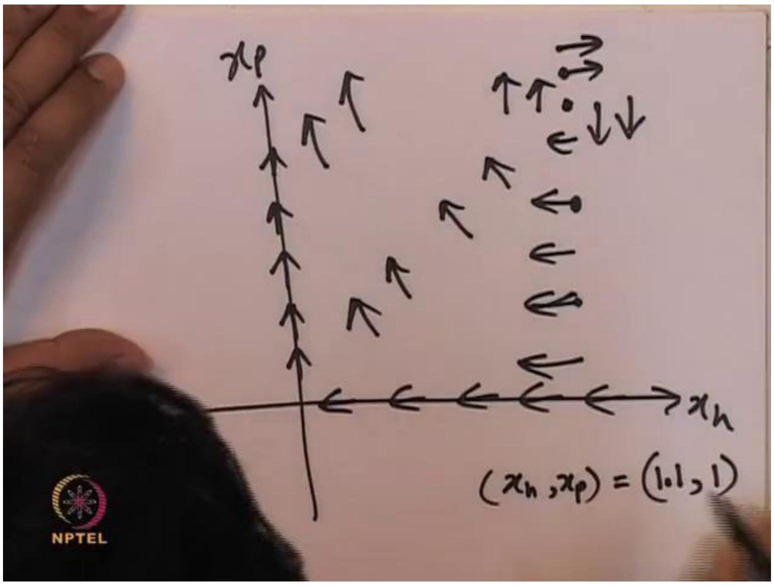

So, x h x p when left to itself the prey population is going to increase. That is why the arrows should all be in the direction of increasing x p. So, the correct figure should be and this is the point 1, 1 and you already checked that intermediate points at this point, it is like this the one. If it is a little higher let us verify this that this is how it looks. So, this itself an equilibrium point if it happens to be at point 1, 1.

If the hunter population is equal to one unit and the prey population also equal to one unit then it remains constant, but for small participations above that point the arrows I have drawn like this, but this requires verification. So, let us take a sample point, this particular point has x p coordinate equal to 1, but x h coordinate slightly more than 1. So, for example, let us think of consider the point x h, x p equal to 1.1, 1

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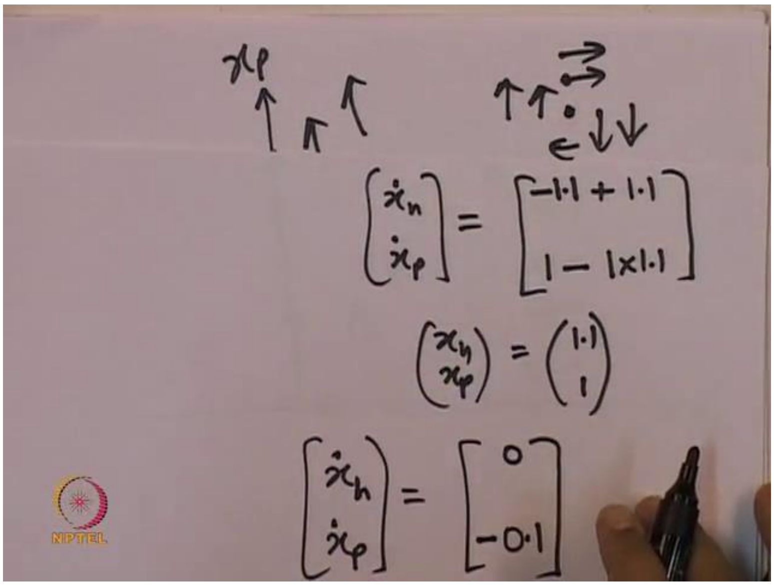

Let us see what happens for this particular point. For this particular point we have drawn the arrow like this, but let us check whether it indeed is like this. So, x h dot x p dot equal to we are evaluating at point x h, x p equal to equal to 1.1 and 1. So, this is minus 1.1 plus 1 times 1.1. So, this is equal to 1.1 and x p population rate of change of the prey population is equal to 1 minus 1 into 1.1. So, this turns on to be is equal to x h dot x p dot is equal to the top component is 0 and lower value is minus 0.1.

This is what happens when x h is slightly more than 1, slightly more than equilibrium point, but x p is equal to the equilibrium point value that is equal to 1. So, when we do this then we are speaking on this point here. For this point we are getting that x h rate of change is equal to 0. So, the horizontal component is equal to 0 and the vertical component is equal to minus 0.1 that is why it is vertically downwards. So, similarly, one can check for each of these 4 points what is the property of this point its x p population the prey population is slightly more than 1, but x h population the hunter population is equal to 1. For each of these 4 points one can verify and see that the arrows are indeed like this, suggesting that there is a periodic orbit around this point. So, there are periodic orbits close to this, but this point on the other hand looks like a saddle point.

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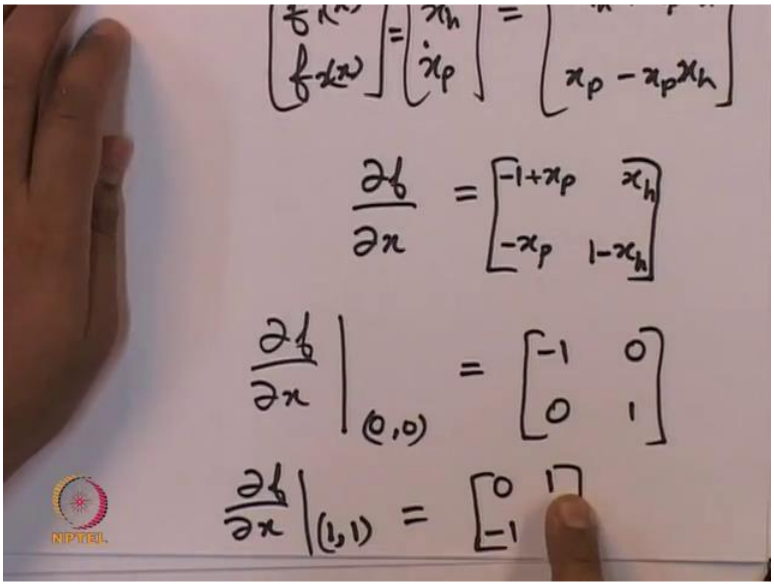

So, let us verify this by linearizing the system at each of these 2 equilibrium points. Let us go back to the dynamical system x h dot x p dot minus x h plus x p x h x p minus x p x h. So, del f by del x this is equal to f 1 of x f 2 of x equals this. So, the first row the first function here is called f 1 of x, second function here is f 2 of x del f by del x is equal to a 2 by 2 matrix. The entry here is derivative of this with respect to x h that is equal to minus 1 plus x p. The entry here is derivative of this with respect to second component of x, that is x p, this is equal to x h.

The entry that comes here is derivative of f 2 with respect to x h here we get minus x p and the entry that comes here is the derivative of this with respect to x p. The second component of the state for that we get 1 minus x h. So, as expected this is a matrix, this is a 2 by 2 matrix which depends on x p and x h. So, we are going to evaluate this matrix at the equilibrium point. So, del f and del x evaluated at the equilibrium point 0, 0 is one of the equilibrium point. For this particular equilibrium point we get minus 1 0 0 1 and del f by del x evaluated at the other equilibrium point 1,1 we get equal to by putting x p and x h both equal to 1, we get 0 1 minus 1 0.

So, we have these 2 a matrices 1 a matrix for the equilibrium point 0, 0 and the other a matrix for the equilibrium point 1, 1. It is not difficult to see the Eigen values of these 2 matrices. So, equilibrium point 0, 0 has Eigen values for a diagonal matrix. The Eigen values are nothing, but diagonal entries.

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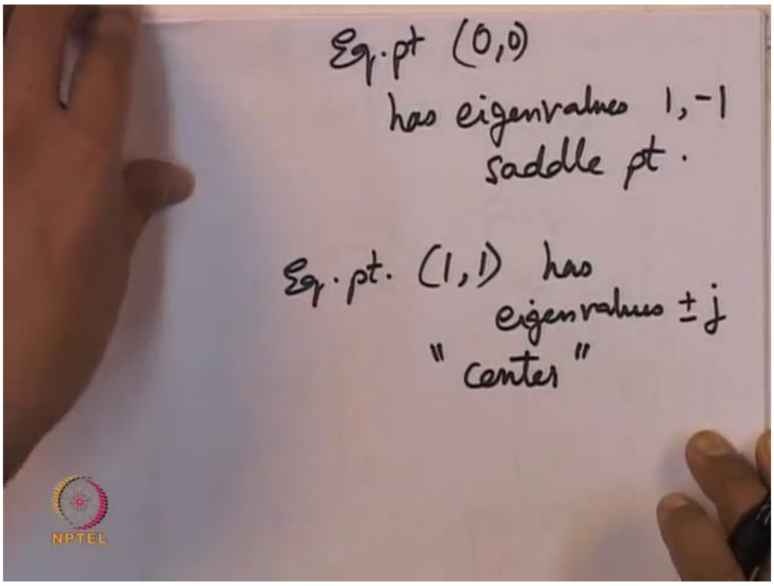

The equilibrium point 0, 0 has Eigen values 1 and minus 1. So, we already saw that, this is an example of a saddle point. The equilibrium point 1, 1 has Eigen values, what are the Eigen values of the matrix, which matrix of this matrix. Eigen values of this matrix are plus minus j.

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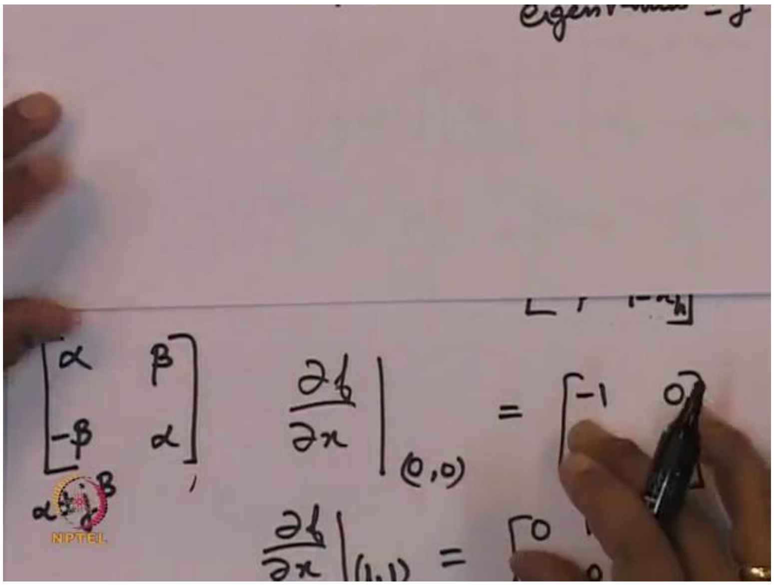

As we noted in one of the first few lectures that, the Eigen value of such a matrix, if beta is not equal to 0, then the Eigen value of this matrix are equal to alpha plus minus j beta. The Eigen values of such a matrix are complex precisely what complex values are the

Eigen values. Alpha plus minus j beta the diagonal entries are the real elements real part and the off diagonal entries with opposite signs correspond to the imaginary part of Eigen value.

These are the Eigen values even then beta is equal to 0. So, for this particular equilibrium point we have this special case. So, the Eigen values are plus minus j, which we know corresponds to a center.

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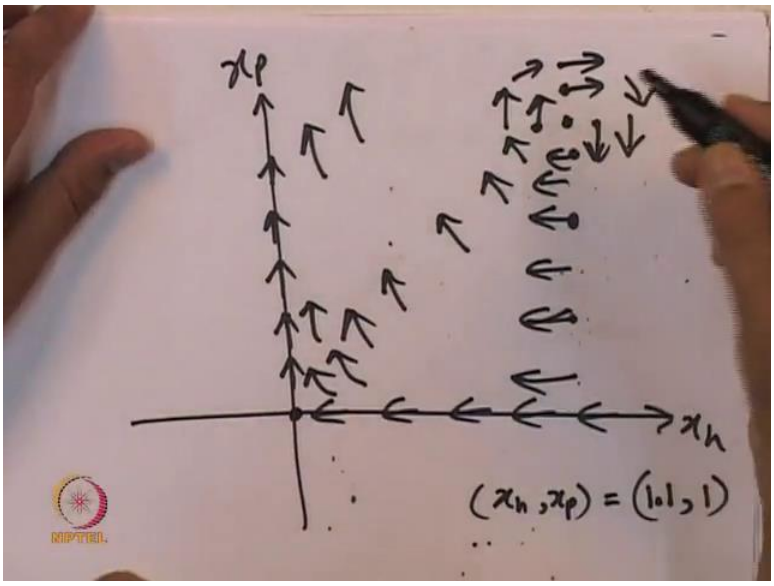

The equilibrium point is what we call a center. So, our center is one that has periodic orbits. We always saw that, for this particular plot indeed this particular equilibrium point has periodic orbits and this is a saddle point. So, the linearized system is a center, which is nothing, but a continuum of periodic orbits. Very close by different initial conditions correspond to different periodic orbits. They all correspond to periodic orbits and different periodic orbits is that the same for the non-linear system also this is the topic that we will see in detail today. So, please note that we have investigated the Lotka Volterra predator prey model. For convenience the predator we have called as hunter. So, that we can use a subscript h.

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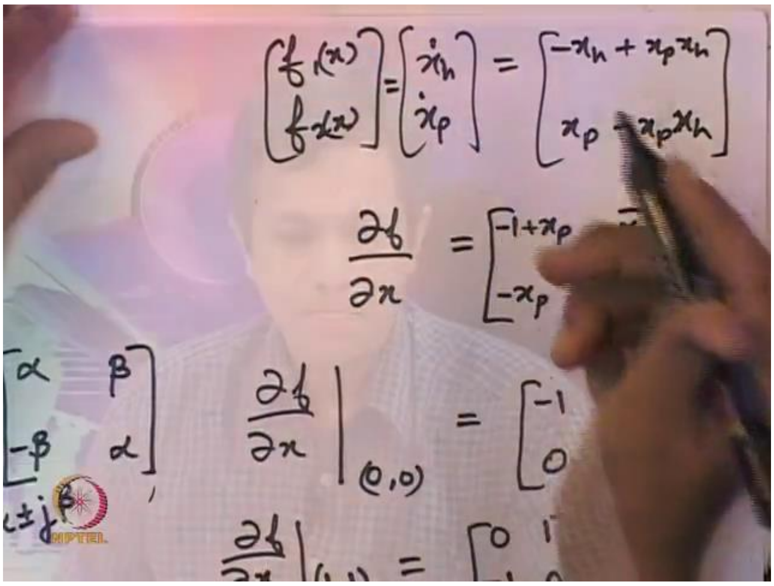

The prey we continued to call p x p. The simplified model shows 2 equilibrium points 1 equilibrium point the linearized system is a saddle point. The other equilibrium of the Lotka Volterra predator prey model corresponding to 1, 1 corresponds to a center, after linearizing. So, what is important is, that this particular equilibrium point is a center. We already saw that these arrows are suggesting like this.

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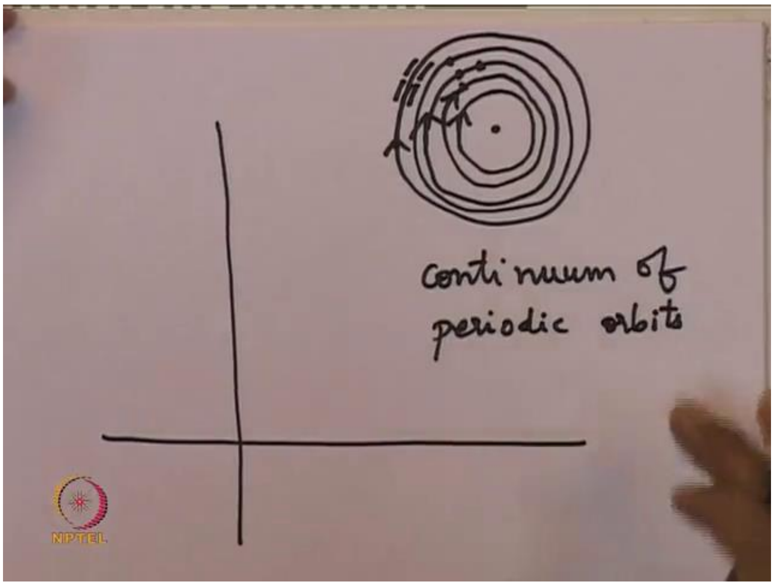

If it were a linear system, then when we go close this is the periodic orbit, when we go close to this and another initial condition it may or may not be different periodic orbit the linearized system says so, but it need not mean for the original non-linear system also. For example, these are two different initial conditions they correspond to the same periodic orbit, but different initial conditions like this may correspond to different periodic orbits or might converge to the same periodic orbit.

This is the subject that we will see in detail today. So, if all these initial conditions correspond to different periodic orbits then we will like to say that there is a continuum, continuum of periodic orbits. These periodic orbits are not isolated, but very close to each periodic orbit there is another periodic orbit, in a very close vicinity. Suppose this is a periodic orbit, the initial conditions starting form here correspond to periodic orbits also. In that sense there is continuum of periodic orbit.

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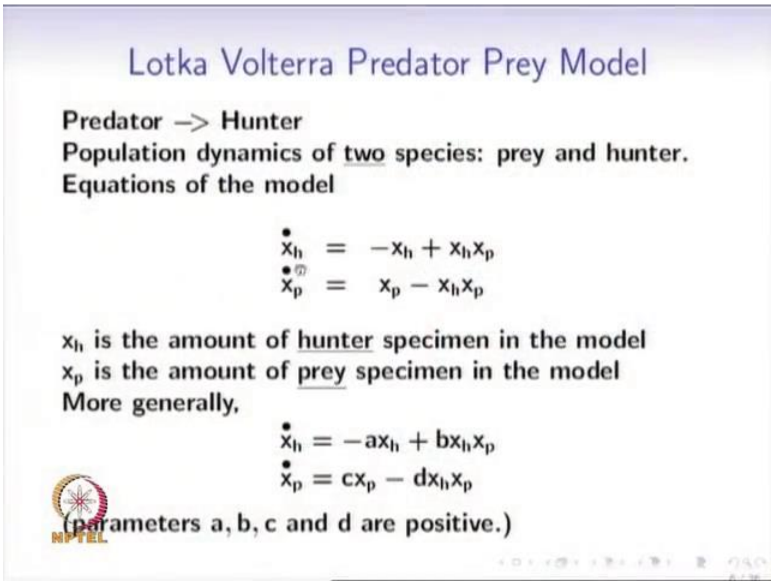

So, it is a very well known important fact that for the particular Lotka Volterra model that we have taken for, let us go back here. For this particular Lotka Volterra predator prey model for constants a b c d we have 2 equilibrium points 0, 0 and 1, 1. When you assume a b c d equal to 1, but for a different point when a b c d are some positive constants possibly not equal to 1, there are 2 equilibrium points while the 0, 0 is a saddle point, the other equilibrium point is a center and more over for the non-linear system for this Lotka Volterra predator prey model, there is a continuum of periodic orbits. This particular fact for this particular model I knew these 2 models is a very important fact.

One can modify this model suitably. So, that we have isolated periodic orbits. So, today we are going to see a different example where there are indeed isolate periodic orbits. So, let us use Poincare Bendixson criteria and the Bendixson criteria.

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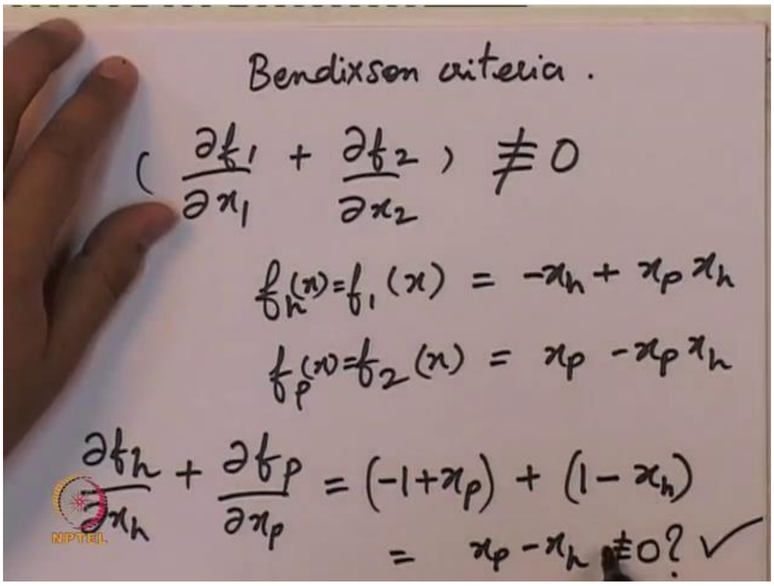

To check if there are periodic orbits, let us see the Bendixson criteria. What does this Bendixson criteria say, we will evaluate this particular quantity and check whether this is identically equal to 0 or not. If it is not identically equal to 0 only then we can go ahead and apply the Bendixson criteria. So, let us evaluate this particular quantity for our example. For our example f 1 of x was equal to minus x h plus x p times x h and f 2 of x is equal to x p minus x p times x h.

So, this f 2 we could also call as f p and this is equal to f h f h denotes the rate of change of x h and f p denotes the rate of change of x p. So, let us evaluate del f h by del x h plus del f p by del x p, when we evaluate this we get derivative of this with respect to x h is equal to minus 1 plus x p plus derivative of this with respect to x p. We get this equal to 1 minus x h. So, this is equal to x p minus x h. So, is this identically equal to 0, no it is not identically equal to 0. So, it is that is why we can go ahead and apply the Bendixson criteria. Let us now apply it and see x p minus x h sin of this quantity.

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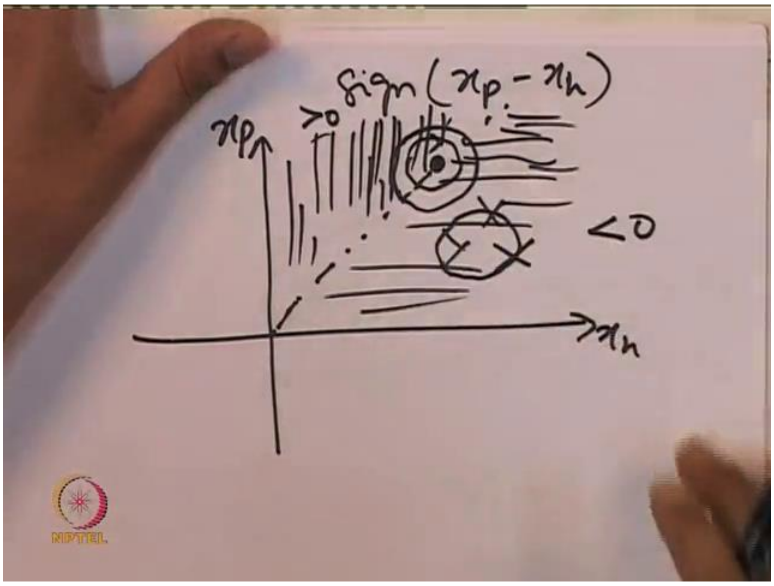

If the sign does not change over a region then Bendixson criteria says that if the sign of this particular quantity does not change on a region, then there are no periodic orbits contained inside that region. So, when is x p minus x h equal to 0 it is along this line. So, everywhere to the right of this line. This is x h this is x p to the right of this line this quantity is negative and above this line or to the left of this line this quantity is positive.

So, the Bendixson criteria says that there cannot be a periodic orbit contained to the right of this line nor there can be a periodic orbit to the top of this line. It does not, this is the equilibrium point 1, 1. The Bendixson criteria does not rule out such a periodic orbit, that does not lie entirely in this region nor does it lie in this region. So, this is an important property to note, that the Bendixson criteria only says that can such a periodic orbit exists inside this region, no this is not possible. Can a periodic orbit lie entirely in this region where the sign of this is all positive, that is also not possible. However this particular periodic orbit could exist. So, Bendixson criteria is only a sufficient condition for non-existence of a periodic orbit lying entirely inside a region. Let us now check what the Bendixson criteria says for a linear system x dot is equal to A x forms the equilibrium point is 0, 0.

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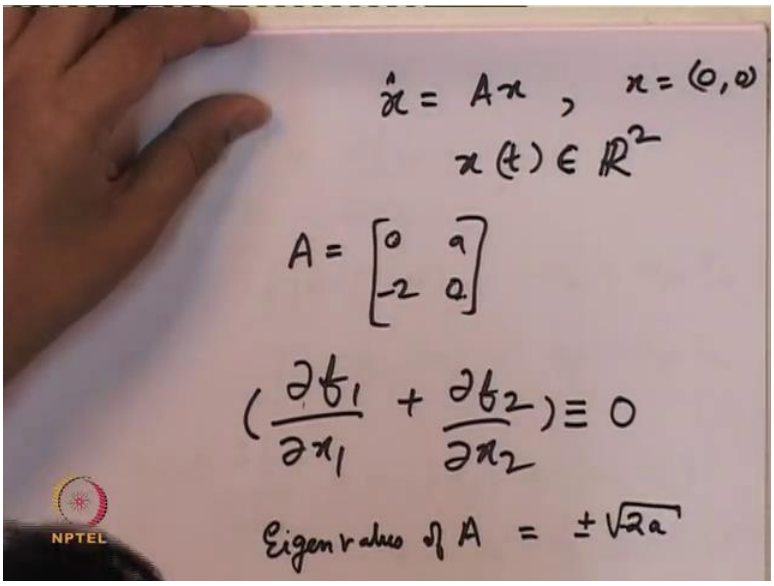

So, Bendixson criteria is applicable when for the planar case, that is when x has 2 components at each time instant x of t has 2 components x 1 and x 2. So, suppose A was equal to made be we see a slide about this. So, for the Lotka Volterra predator prey model, we have already seen this before. We see another example let us see this particular case periodic orbit for A, that looks that is of this form. So, our A we have already assumed it is of this form.

Now, we will do del f 1 by del x 1 plus del f 2 by del x 2. Notice that these 2 terms are nothing, but the diagonal entries of this matrix A. So, for this particular A, the diagonal entry is are both zeros. So, they add up to 0 also, they are identically equal to 0, no matter which x 1 x 2 you check this is going to be equal to 0. This particular quantity is expected to be independent of x 1 x 2 for linear systems. Why for linear systems, for linear time invariant systems these 4 entries are all independent of x.

Hence you differentiate f 1 and f 2 f 1 with respect to x 1 f 2 with respect to x 2 which is nothing but just picking up these entries, picking the values at these 2 positions and they are going to be independent of x. So, for this particular A, we get this identically equal to 0. So, do we say that Bendixson criteria is not applicable or do we say that there are no periodic orbits. Of course, we know that for this particular A, the Eigen values of A are equal to plus minus square root of 2 times A. So, if A is positive then the Eigen values are plus minus 2 minus of 2 times A. We will just verify this.

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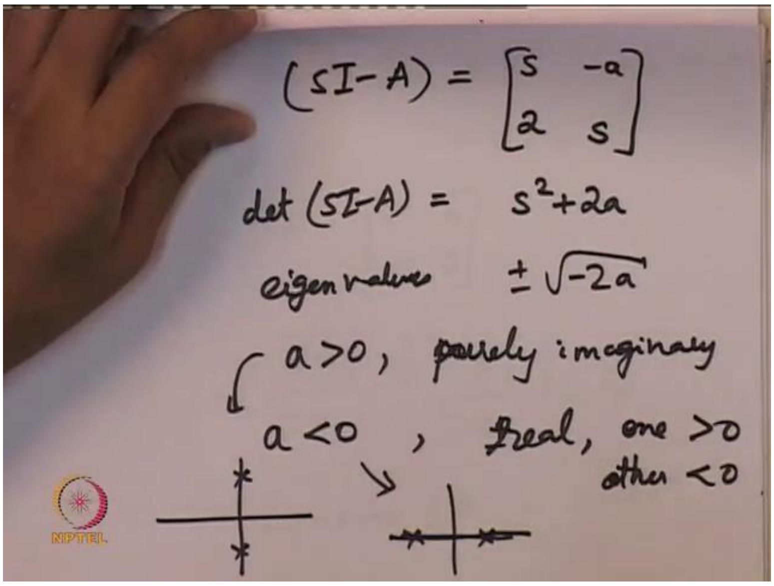

So, what is S I minus A S I minus A is equal to. So, determinant of S I minus A is equal to S square plus 2 A. So, Eigen values of the A matrix are nothing but the roots of the determinant. The roots are square root of minus 2 A plus minus. So, if A is positive, A greater than 0, then complex purely imaginary in fact. If the Eigen values are purely imaginary, then we know for a linear system there are periodic orbits. If A is less than 0 then Eigen values are plus are real. One of them is greater than 0 other is less than 0, why because this if A is negative this quantity itself under the square root sign is positive.

So, we can take the square root and one is positive and one is negative. So, for this case the Eigen values are here and for this case the Eigen values are here. Here how far from the origin depends on the value of A of course, but whether they are depending whether A is positive or negative affects whether the roots are purely imaginary or real. So, we know that for this case the equilibrium point is a center and there are periodic orbits.

While for this case the equilibrium point is a saddle and there are no periodic orbits. So, the important case, when this is identically equal to 0 that particular case could correspond to either there are periodic orbits or there are no periodic orbits. This is just to see that the Bendixson criteria is unable to say anything when this is identically equal to 0. That is precisely the reason that Bendixson criteria assume that this is not identically equal to 0 and then we start looking at whether the sign changes or not.

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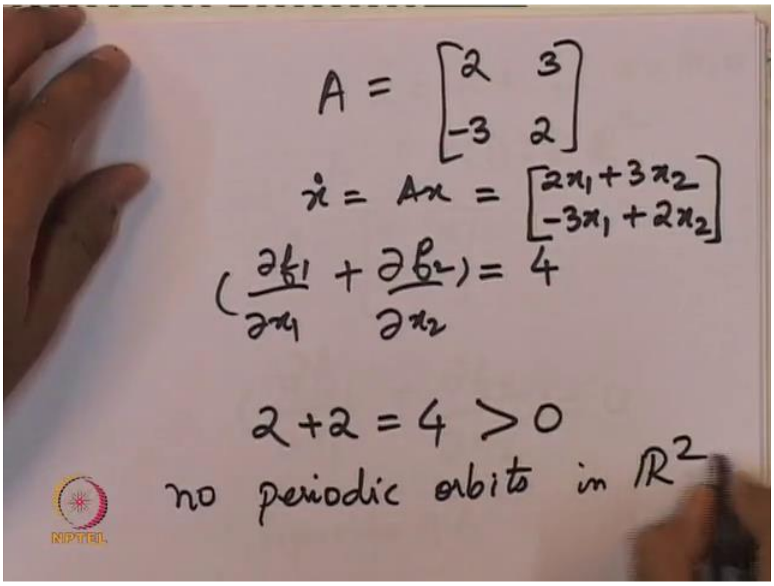

Let us take a case where for x dot is equal to A x. Let us check what is del f 1 by del x 1 plus del f 2 and del x 2, what is the value of this. We will check that this is equal to 4 by calculation explicitly. So, x dot is equal to A x means 2 x 1 plus 3 x 2, that is a meaning of A acting on x and the second row of A will be used to multiply with x to get minus 3 x 1 plus 2 x 2. When we do this, then we differentiate the first component of x dot with respect to x 1 and we get this equal to 2. We are picking at just this entry and the second component of x dot it is f 2 of x with respect to x 2 that doing this.

If you notice that derivative of this with respect to x is just this component this first one by one entry and the derivative of this with respect to x 2 is just this entry. That is the reason that I said that doing this particular to evaluate this quantity is nothing but to add the diagonal entries for a linear time invariant system. So, to get this equal to 4, this is greater than 0 and it is independent of x 1 x 2. For linear systems we expect that this will not depend on x 1 x 2 and it is indeed independent of x 1 x 2. Since, it is greater than 0 for all x 1 x 2 we get that no periodic orbits, no periodic orbits in R 2 in the entire state space in the entire plane there are no periodic conflicts.

So, for linear systems we can check that, as long as the diagonal entries do not add up to 0 this quantity will not be identically 0. Then we can see that periodic orbits are ruled out. When would periodic orbits be possible, if the diagonal entries add up to 0. If the diagonal entries add up to 0 we cannot say that the periodic orbits exists because of

Bendixson criteria is silent for that case. It does not say anything when the diagonal entries add up to 0 identically. We only saw that it is possible that there are periodic orbits, it is also possible that periodic orbits do not exists when the diagonal entries add up to 0.

So, this is already the complication for linear system. So, for the Lotka Volterra predator prey model to show that there are periodic orbits is a difficult thing and it is an important research topic after which it has been concluded that, there is a continuum of periodic orbits for the particular model that we studied. Now, let us take some other examples of dynamical systems and check whether there are periodic orbits or not.

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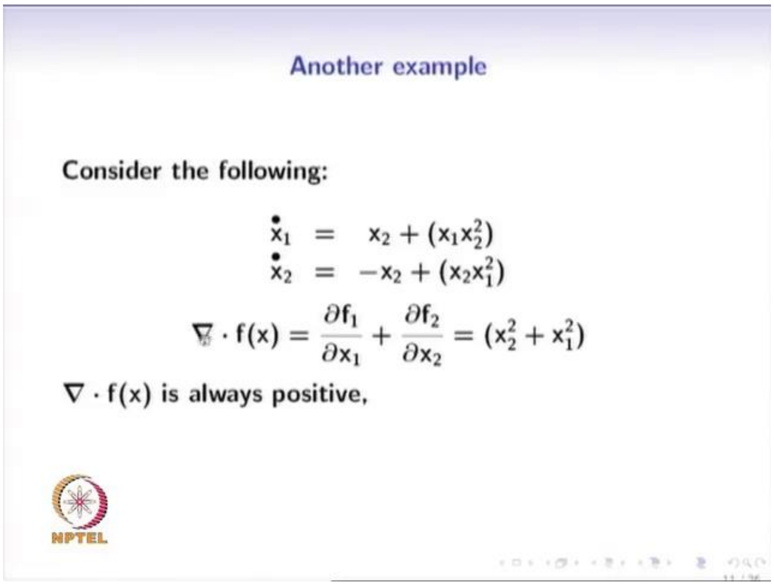

So, consider this example. So, in which x 1 dot equal to x 2 plus x 1 times x 2 square x 2 dot is equal to minus x 2 plus x 2 times x 1 square. So, we differentiate the first in order to use Bendixson criteria, this particular quantity that we were to evaluate is nothing but divergence of f, divergence of f is also denoted as dot product of this operator with f. So, when we evaluate this we get x 2 square here. So, there is something wrong here, this x 2 should have been x 1. So, please note then there is a small mistake here.

If we have x 1 here then, we get this. Now, we have that this is always positive after you substitute x 1 here you get that this is always positive. So, if the Lotka this is very similar to the Lotka Volterra predator prey model after we have x 1 here this is very similar to the Lotka Volterra predator prey model.

If it had depended on not just the product of x 1 and x 2, but some higher order power of 1 of the species then it is possible to show that the Bendixson criteria says that there are no periodic orbits.

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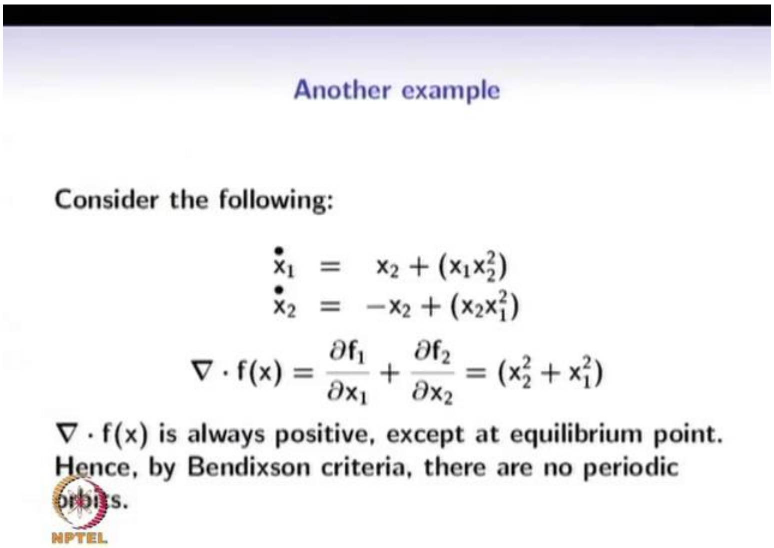

Why because divergence of f is always positive except at the equilibrium point. So, hence by Bendixson criteria there are no periodic orbits. Now, consider this example also has we should have a modification here we should have x 1 here in place of in second equation x dot equal to minus x 2 minus x 2 times x 1 square.

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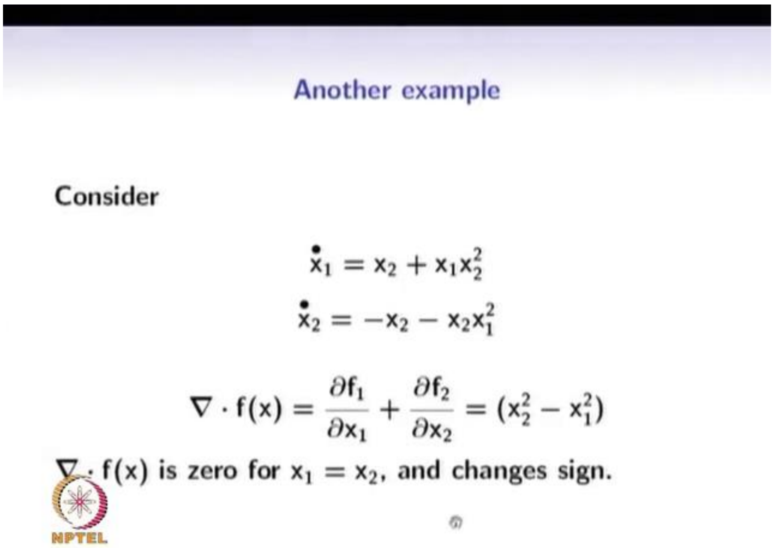

So, here we see that this changes sign. This only means that inside the region where it has the same sign, there the rest of the periodic orbit contain inside the region. This is an example that we have already seen.

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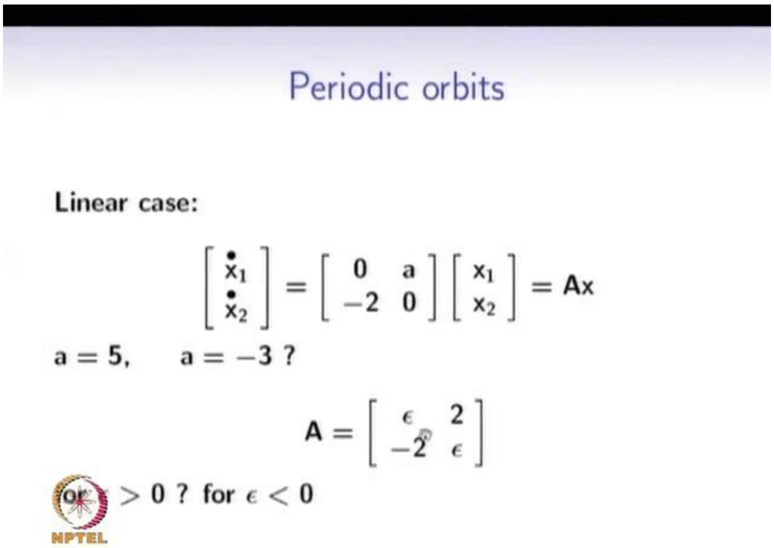

Now, we will study an important case where we have an epsilon here along the diagonal entries. So, for this particular A in which we have epsilon along the diagonal, what can we say about the equilibrium point. So, when epsilon is greater than 0, then the equilibrium point 0, 0 is unstable focus. This is something we have already seen, what happens when epsilon is less than 0.

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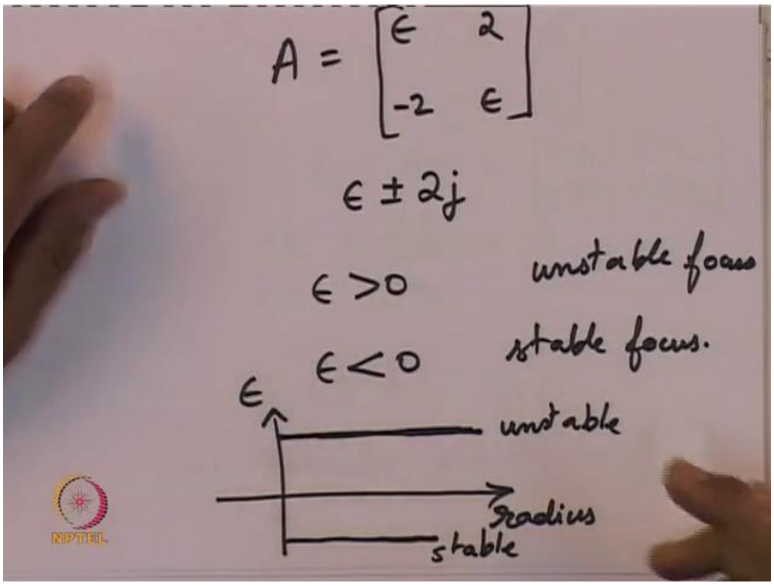

When epsilon is less than 0 Eigen values of this matrix are epsilon plus minus 2 j. So, epsilon greater than 0 implies unstable. Unstable mode or focus it is unstable focus because the measuring part is non 0 and for epsilon less than 0 we have a stable focus. So, this is what we will say that as A, if this epsilon this is our epsilon value. This is let us say radius distance of the point from the origin. So, if epsilon is some positive quantity epsilon it is not dependent on x 1 x 2.

So, it is just the same positive number, positive means unstable focus. If it is some same fixed negative number then it is stable. So, this is unstable and this is a stable focus. So, how about modifying this epsilon as a function of radius. So, that we have trajectories that converge to a periodic orbit. So, this is what we will see in detail. So, what if epsilon changes it sign depends on the distance from the origin and changes it sign.

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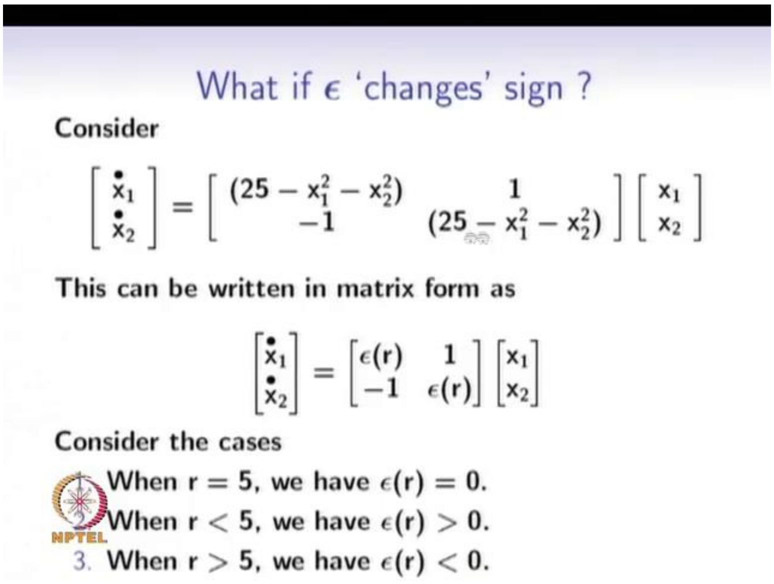

So, consider this differential equation in which along the diagonal we have put 25 minus x 1 square minus x 2 square along the diagonal and off diagonal term we keep constant does not depend with change with radius. This is nothing but writing it in this form in which along the diagonal we have some function that depends on the radius. It depends on the distance of the origin. Now, we consider the case when r is equal to 5. For that case we have epsilon of r is equal to 0.

We can consider the case when r is less than 5. For r less than 5 the diagonal entries are positive and when r is greater than 5 the diagonal entry is r negative or both negative. We cannot speak of Eigen values of this the Eigen of this matrix itself depends on x 1 and x 2. We speak of Eigen values of only constant matrices. So, it appears like if we make this radius, if we make this diagonal entry depend on radius. Then we will have trajectory either coming towards origin or going away from the origin depending on whether we are inside a particular circle. Whether we are inside the circle of radius 5 or outside or on the circle itself we are going to be remaining on the circle.

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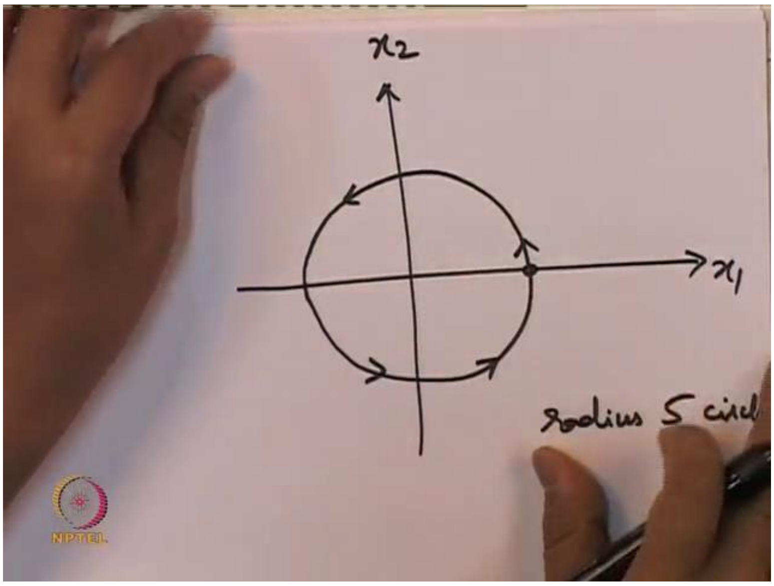

So, let us check this is a circle of radius 5. So, when x 1 is equal to 0. So, this is radius 5 circles when x 1 is equal to 0 and x 2 is equal to 5 the time x 1 dot x 1 equal to 0.

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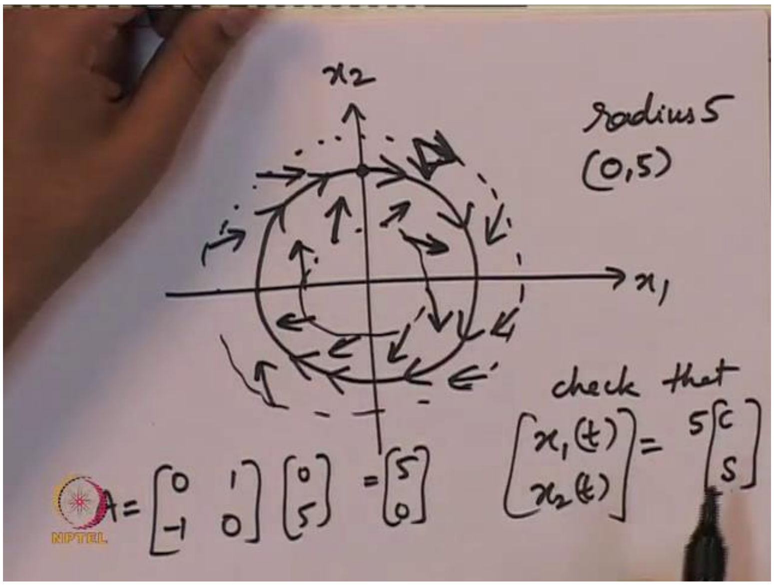

The orientations, we start again. How do we get this orientation we expect that for radius equal to 5 we have periodic orbit, why is it that we have a periodic orbit. You put r equal to 5 and you see that the matrix A then it looks just as if so, check that x 1 of t x 2 of t equal to 5 times along the circle x 1 x 2 are like cos and sin or function of what frequency t of the cos t and sin t why because omega is equal to 1.

For this particular A the solutions are x 1 x 2 are equal to cos and sin, to the sin of the quantities and of radius 5 and why cos t sin t. In general it would have been cos omega t sin omega t and the omega is equal to 1 because of diagonal entries are equal to 1. Now, we are going to decide why this clock wise and not anti-clock wise that we can check by taking some sample points. So, consider this point, this is x 1 component is equal to 0 and x 2 equal to 5. So, consider the point 0, 5 and 0, 5 when this acts on 0, 5 when the A acts on this matrix then we get that this is equal to 5, 0.

So, the x 1 component is increasing at this particular point. That is why it is in this direction. So, by using the same argument we can decide where cos of t should come where sin of t should come and whether there should be a negative sin to one of these. Now, the focus of this particular topic is, to see what happens to the radius larger than 5, for radius larger than 5 and for radius smaller than 5. So, for radius larger than 5 there is the off diagonal term indeed cause some rotation, but the diagonal entries cause a decrease in the radius that is why it is coming inwards.

So, we have these arrows coming inwards. For the circles inside the circle of radius 5 that is for circles of radius less than 5 there is some rotation caused because of the off diagonal terms, but the diagonal entries themselves are positive which is causing this radius to grow as a function of time.

This is an important property that we will exploit to see that, all initial conditions except the equilibrium point 0, 0 are all converging to this special periodic orbit, which periodic orbit the periodic orbit of radius equal to 5. So, let us check, let us use Poincare Bendixson criteria and check that there indeed exists a periodic orbit.

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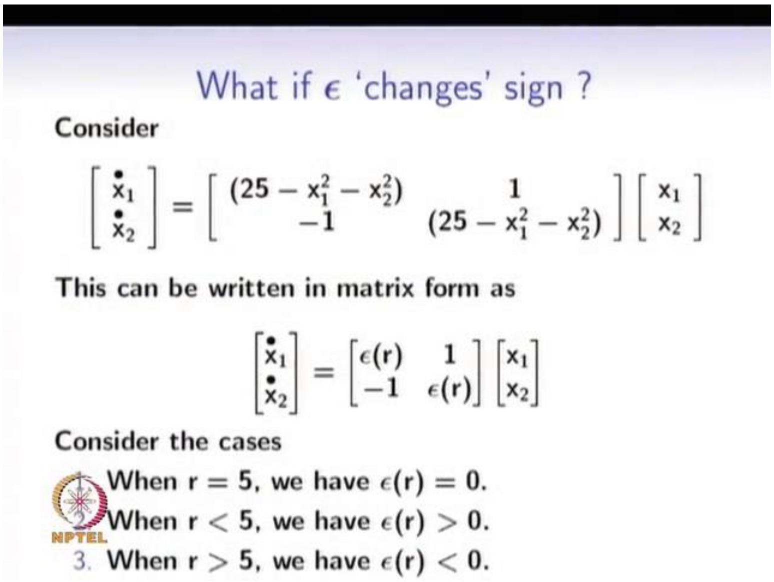

We are not able to say that this is a center kind of arguments because we can use that only for the linear system by linearizing at an equilibrium point.

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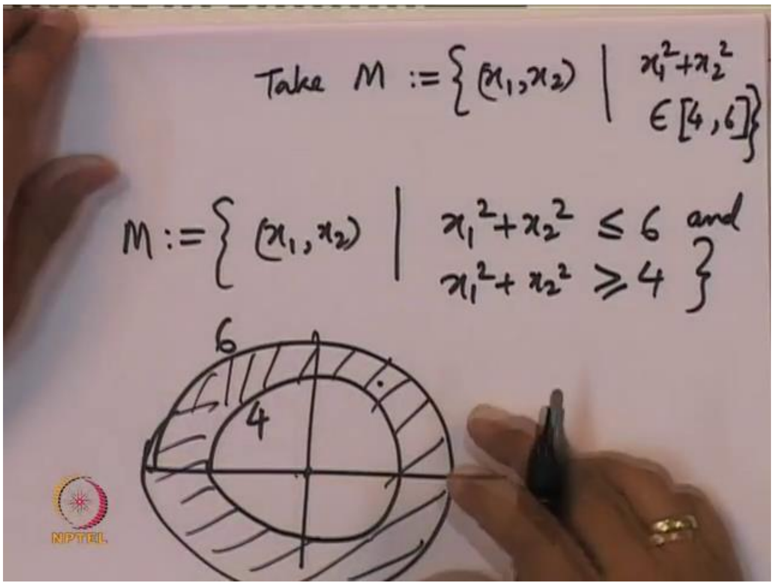

So, take M to be equal to the set of all x 1, x 2, such that x 1 square plus x 2 square lies in the interval 4 to 6, closed interval. What is our set M, let me write again set M is a set of all x 1, x 2 points such that x 1 square plus x 2 square is less than or equal to 6 and x 1 square plus x 2 square equal to 4. In other words this is a circle, supposed to be circle. This is another circle this is a circle of radius 4, this is a circle of radius 6. All the points in this ring these are all the points whose distance from the origin is greater than or equal to 4 and less than or equal to 6 also, this and this.

So, we will now check that this particular set M is positively invariant and has no equilibrium points inside it and it is compact. The compactness is satisfied because this is the compact set and it is the close set because these inequalities are not strict inequalities they are not strict in equalities, but non strict inequalities because of that fact this is the close set and it is a bounded set because we see that all the points are at most distance fixed away from the origin hence it is a bounded set.

So, in order to use Poincare Bendixson criteria we are going to check that. This set M is a positively invariant set also, when would Poincare Bendixson criteria be applicable, the set time should be a close and bounded set should be positively invariant and there should be no equilibrium points inside it or at most 1 equilibrium point, which is either an unstable node or an unstable focus. So, let us check what is the property of this particular M?

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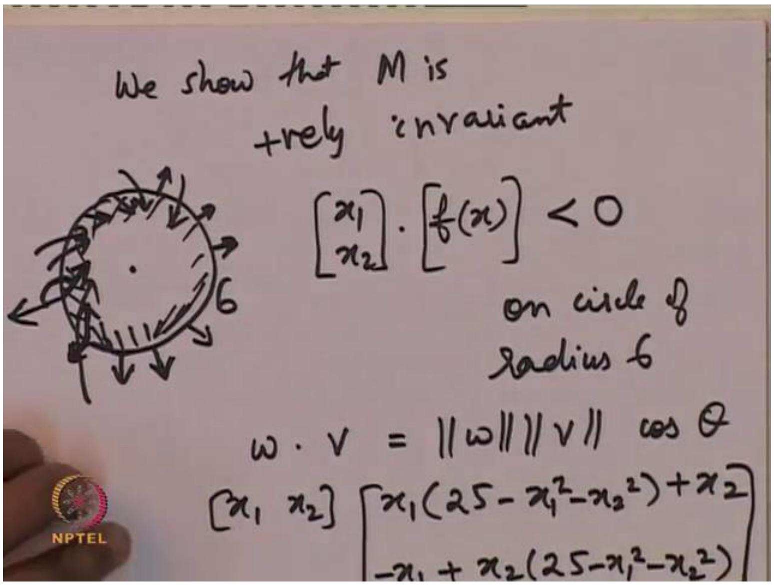

We show that M is positively invariant. So, how can we show that this set M is positively invariant. So, what we will do is, we will take the circle of radius 6 and we will check that this circle of radius 6 the outward, the vector that is perpendicular to this boundary is outward like this. This is a vector we will check what is the inner product of this vector with the vector field. If at every point along the boundary this vector field is directed inwards. It means that all the points are trajectories are coming inwards. What are these vectors, this is the unit vector perpendicular to the boundary and directed outside the region. The region is inside, this as far as the boundary 6 is concerned as well as this boundary of radius 6 circle of radius 6 is concerned. This is a vector that is directed outwards. So, let us check what is this vector it is nothing but x 1 x 2 vector, it is inner product with f of x at that point, f of x at each point again a vector of dimension 2. We will check whether this inner product is positive or negative. If this inner product is negative on circle of radius 6, it means that along the circle all trajectories are going inwards, why is it inwards because this is a vector outward and this is the f of x.

If this particular angle is this dot product being negative mean that the angle between 2 vectors is an obtuse angle, why because what is the dot product of w dot product with v. This is equal to w norm times v norm times cos of the angel between w and v. So, if this quantity is negative, it means these 2 quantities can be negative. So, this cos theta is negative and cos theta is negative only for theta beyond 90 degrees which means that this angle between these 2 vectors is greater than 90 degrees.

Since, this vector is a vector that is directed outside the boundary this angel being obtuse which means that f is directed inwards. So, let us check whether this quantity is negative. So, x 1 x 2 times f of x, f of x is what we can see from this particular thing f 1 of x is first rho times x 1 x 2. So, this is nothing, but x 1 times twenty 5 minus x 1 square minus x 2 square plus x 2. This is f 1 of x and f 2 of x is minus x 1 plus x 2 times 25 times x 1 square minus x 2 square. So, when we do the dot product of this that is nothing but this row vector times this column vector.

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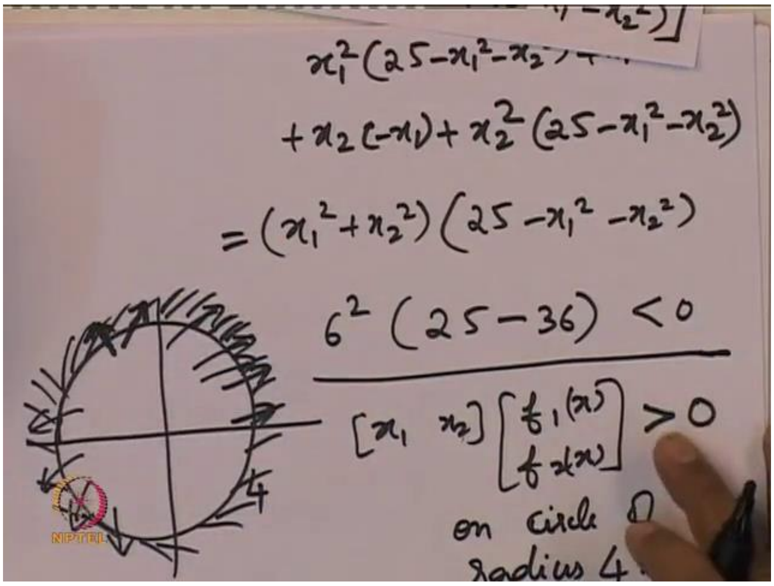

When we evaluate this let us see what we get. So, x 1 square times 25 minus x 1 square minus x 2 square plus x 1 x 2. This is just this quantity here that I have written is just x 1 times the first component here plus x 2 times minus x 1 plus x 2 square times 25 minus x 1 square minus x 2 square. So, x 1 x 2 minus x 1 x 2 these both cancel. So, we get x 1 square plus x 2 square in common 25 minus x 1 square plus x 2 square.

Now, note that we are going to evaluate this along the circle of radius 6. So, we get 6 square times 25 minus 36. So, this is some quantity that is less than 0, that is all we needed. So, this proves that along the circle of radius 6 the arrow is directed in wards. Now, let us check what happens along the inner circle. This is a circle of radius 4 and along this boundary this is a unit normal and the f itself, there are 2 vectors at each point along the boundary.

One vector is the direction of f of x at that point and another vector is the direction of the unit normal and in this case, this vector says that it is directed in wards, why because M lies to the outside of this region, outside of this circle. M was the set of all points of radius greater than or equal to 4 and less than or equal to 6. Since, we are taking a circle of radius 4 the circle region is to the inside. So, this vector is directed inwards of the region. So, at each point what is this vector it is this x 1 x 2 again. The direction of f at that point is f 1 x f 2 of x. Now, because this vector is a vector directed towards inside the region along the boundary, this particular quantity being greater than 0 means that the angle is acute angle.

The angel between the 2 vectors is less than 90 degrees and then it would mean that the trajectories are all coming inwards into the region. So, that are all these trajectories that are coming into the region as well as the boundary is concerned. As well as the boundary is concerned the circle of radius 4. So, by the same argument all we have to do is written a substitute x 1 square plus x 2 square is equal to 4 square and not 6 square. This quantity becomes 4 square when we are looking at the circle of radius 4 and this quantity becomes 25 minus 16 which is now positive. So, this is greater than zero on circle of radius 4. So, this proves that the set M is positively invariant. How have we shown that this set is positively invariant.

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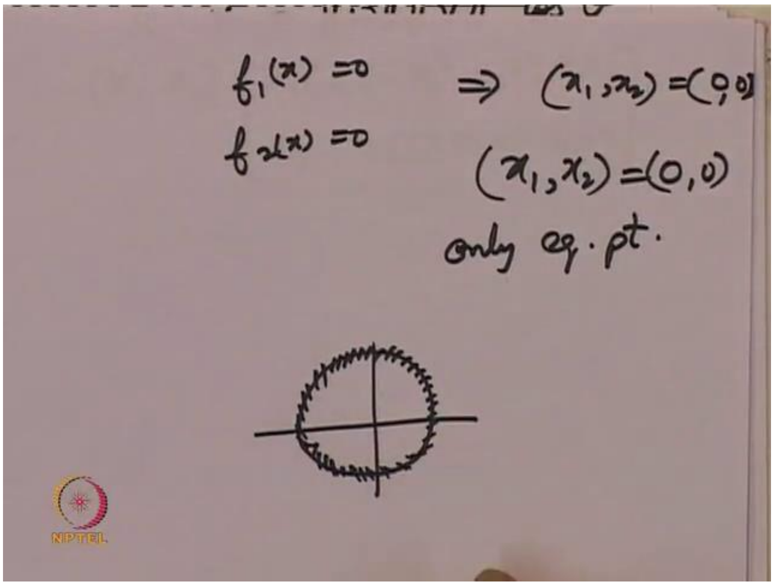

We have said that this set M has boundary consisting of 2 circles. So, this is one boundary circle 4, the other boundary is circle of radius 6 and the region is like this. This is a region M all along the outer circle the trajectories are coming inwards into the region. This is what we checked because the angle was obtuse there. All along the inner boundary also the trajectories are coming inwards. So, check that as long as the region M is defined as set of all points where the radius of the x 1 square plus x 2 square is greater than or equal to say 4.9 and less than or equal to 5.1.

It will still be positively invariant, that is only property that we used. Such an M will be positively invariant. Hence it will contain a periodic orbit. Is there an equilibrium point inside this region, that is another thing we are supposed to check before we use the Poincare Bendixson criteria this is the last thing we will check before today's lecture ends.

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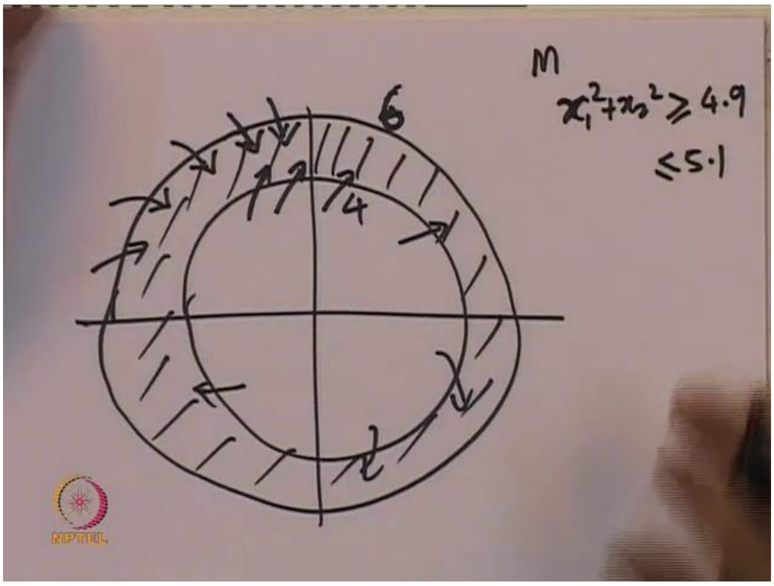

So, let us put f 1 of x equal to 0 and f 2 of x equal to 0. So, f 1 of x we had evaluated and we had got that equal to what is shown here is f 1 of x and this is f 2 of x. We have to substitute both equal to 0 and find the values of x 1 x 2 such that both these functions are equal to 0 those x 1 x 2 values will comprise the equilibrium points.

So, one can check that the only equilibrium point for this is x 1 x 2 equal to 0, 0. In other words if M is a set of all points of distance greater than or equal to 0. If the radius is strictly greater than 0 then M will not have any equilibrium points that is why we can. So, this implies that x 1, x 2 equal to 0, 0 is a only equilibrium point. The equilibrium point itself is stable or unstable one can check, let this be as a home work that this equilibrium point is unstable why because at this equilibrium point we ensure that the diagonal entries of this particular matrix, of which matrix. Let us go back to the slide, of this particular matrix for x one equal to 0 and x 2 equal to 0.

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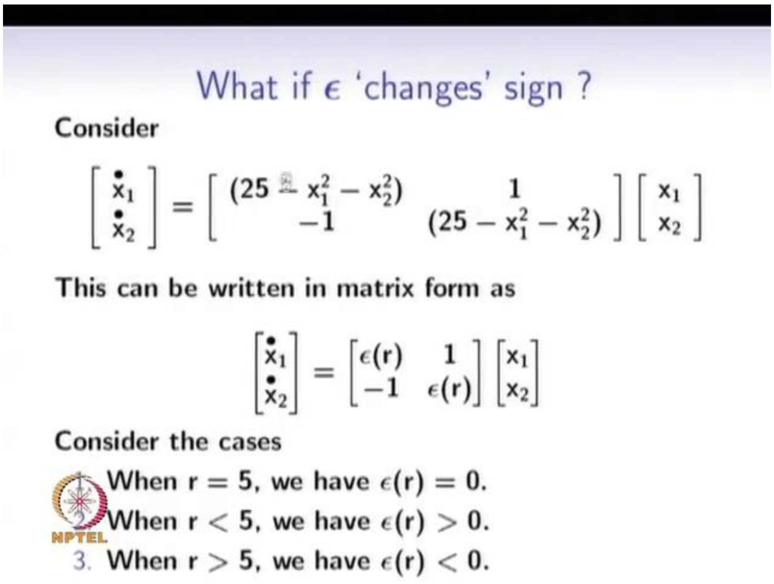

This particular matrix has diagonal entries positive. Hence the equilibrium point is unstable focus. So, this allows us to use Poincare Bendixson criteria. Since the region that we have considered set of all points of radius greater than 4 and less than 6 greater than or equal to 4 or less than or equal to 6 is compact, is positively invariant and has no equilibrium point. Hence it is a periodic point, by making this set M smaller and smaller such that it just contains the circle of radius 5.

So, we can take the region of M to be set of all points of distance slightly less than 5 and slightly more than 5. It will still the same argument will hold and there will be a periodic orbit. This shows that, there is there is no continuum of periodic orbits here that is an isolated periodic orbit for this particular example. We will consider modifying this example obtain the Vander Paul oscillator as a special case.

Thank you.