Introduction to Robotics
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Lecture - 2.1
Kinematics - Coordinate Transformations
Manipulator Kinematics

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Very good morning. Welcome back to this course on Introduction Robotics. So we will start the topic Manipulator Kinematics today. In the last few classes I mentioned about the development of robotics in general and the applications of robotic technology in various fields, and I mentioned that industrial robotics is one of the major areas of robotics application.

And when we discuss about the important topics in robotics, I mentioned that manipulator kinematics is one of the most important part which any robotics engineer should need to understand and then this will actually form the basis for learning robotics in many other applications also like field and service robotics. You need to have a basic understanding of manipulator kinematics.

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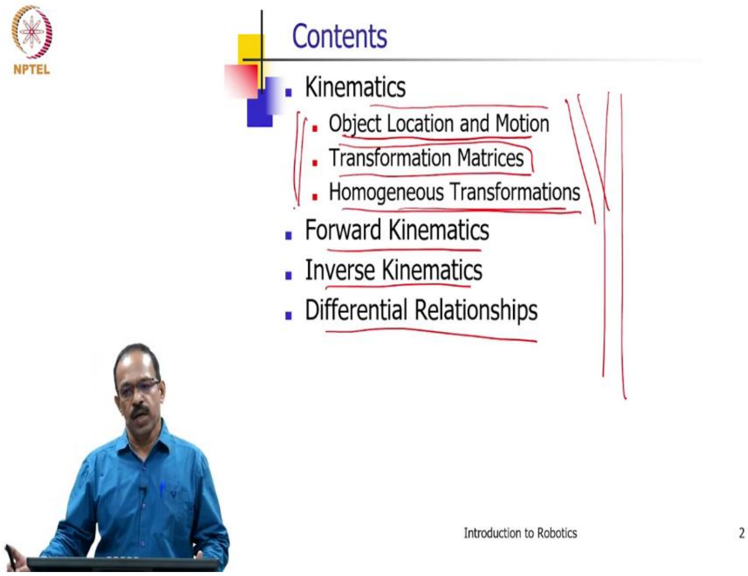

So, we will be starting this discussion today and as part of this kinematics, we will be talking about the object location and motion as the first step. So, kinematics basically talks about the position and velocity relationships in robotics. So, in order to understand that first we will talk about the Object Location and Motion. How do we actually represent an object in 3D space? And how do we actually represent the motion of an object in 3D space will be discussed.

And then we will talk something about the Transformation Matrices. So if an object is moving in the 3D space and when the movement includes the translation rotation then how do we actually represent these transformations can be explained using Transformation Matrices. So we talk about Transformation Matrices and then we talk some, about something called Homogeneous Transformations. So, to represent the object transformation we need to use something called Homogeneous Transformations

And once we learn this, then we will talk about the Forward Kinematics of the industrial manipulators. And then we go to the Inverse Kinematics and Differential Relationships. So, these are the topics that will be covered in the in this part of kinematics. So first we will start with the basic relationships of Object Location and Motion.

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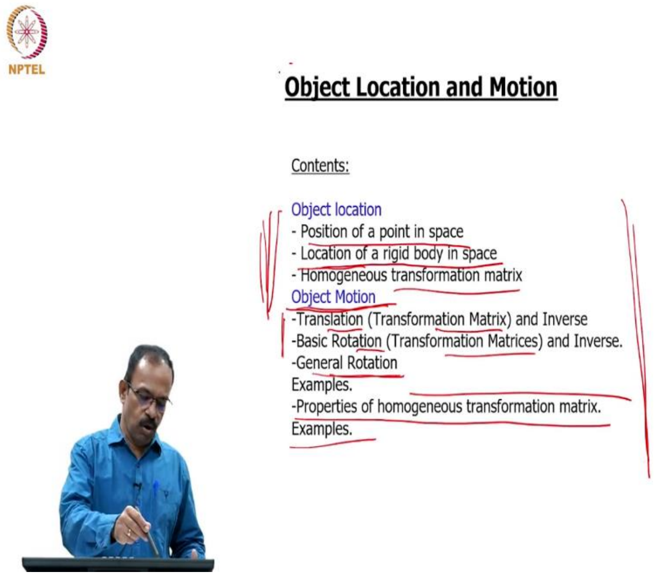

So, now if you look at the Object Location and Motion as the first topic we can actually see that there are a few things we need to understand before we talk about the motion of an object in 3D space. Since you are from different backgrounds, some of you are from civil engineering, some from electrical, some from mechanical, some from chemical, I want to go through some basic mathematical relationships which some of you may be already familiar, but this is to make sure that all of you understand the basic concepts.

So, that when we go to the forward kinematics and inverse kinematics, you may you will find it easy to follow. So, the first thing that we will be discussing is the Position of a point in space. How do we represent a position of point in space very basic thing which most of you already know, but then we will start with that and then see, how do we represent the location of a rigid body in space? Because the position of a point in space can be represented, but an object when you have an object, it is not only the position, you need to had, need its orientation also because an object it is a three-dimensional object.

So, position alone is not sufficient to represent an object. So we need to have the location of an object where location says means the position and orientation of an object and then we talk about the Homogeneous transformation matrix. And in object motion, we need to see I mean there are two ways an object can actually move.

So one is the translation. You can actually move linearly from one point to the other point translation motion. The other one is a rotation, rotary motion. So you can have a translation and rotation for an object. So how do we actually represent this translation and rotation using matrices that is basically the Transformation matrices, translation matrix and rotation matrix.

And then we talk about a General Rotation principle also, and finally, we will have a Homogeneous transformation matrix and then we take few examples to explain how the homogeneous transformation matrix can be represented, can be used to represent the motion of an object in 3D space. So that is what we are going to discuss in the, the first part of object, location and motion, before we get into the robot kinematics part we first talk about the general object location and motion.

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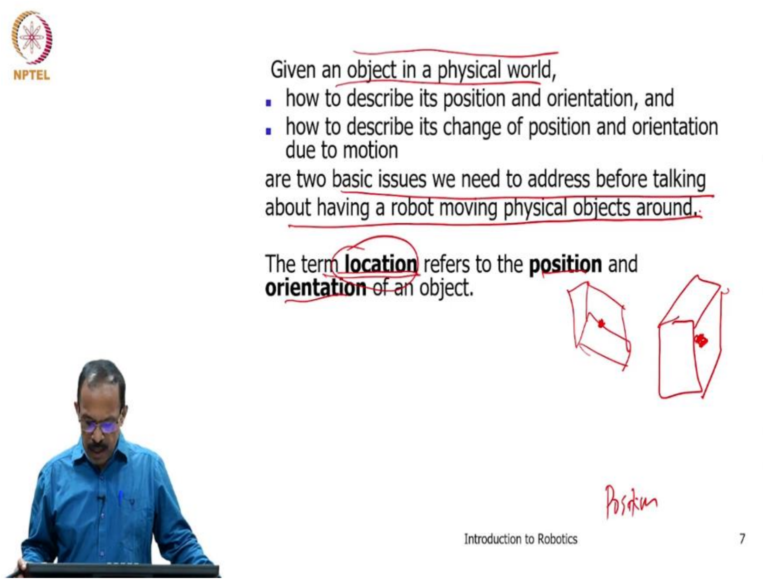

So, let us see this, so you have an object in physical world. Suppose you have an object in physical world. It is a three-dimensional object and then it has got a sense EG and we will just put that point in there also. Now if I want to say that, what is the position of this object or how do you actually represent this object in a physical world?

So, we need to have two things to represent it; one is that we need to saw, what is the position? What is the position of this object? And then we need to know, what is the orientation? Because the object can actually be in this like this way also you can actually put this object in this way both are the same object, but the position if it is the same position if you actually place it in this orientation, it becomes a different way of representation.

So, how do we actually represent the position and orientation of an object? And that is basically the location of an object and how to describes it's change of position also? Now this object has moved from here this is object has moved from here. So this was the initial position. It has moved to here. Now, how do we actually represent this change of position and change of orientation? What way we can actually represent it mathematically is the question that is where we try to see how do we actually represent this object location in 3D space

And when it is moving from one location to another location, how do you represent the, this transformation of location also? So that is the, these are the two basic issues we need to address before talking about having a robot moving a physical object. Yes, finally a robot is being used to move physical object. So I have an object here. I want this to be moved to here in a different orientation. So, how do I represent this using in the case of robot that can be understood only if we know how to represent 3D object in 3Ds, I mean object in space.

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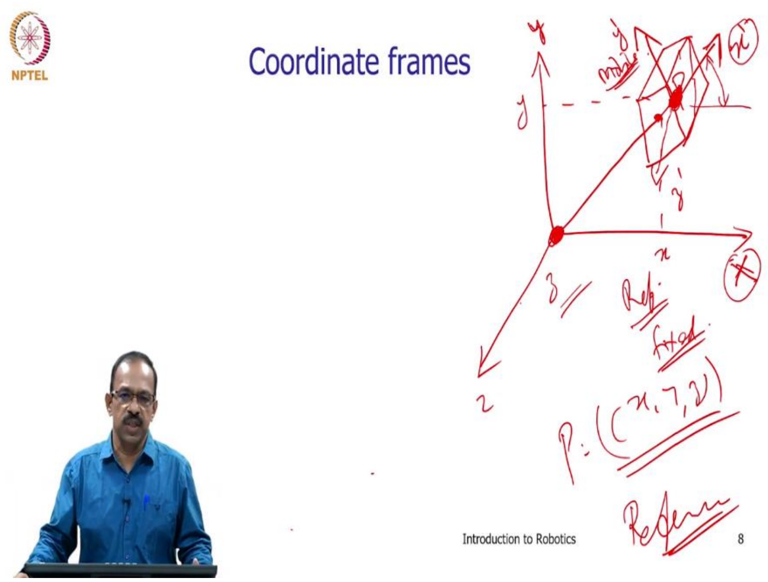

So what we do is to we will try to look at something called Coordinate frames using the coordinates of the point. Suppose you have a three dimensional coordinate frame like this where we write this as X, Y and Z. Now we know that if you have a point in space if you have point in space I can actually represent this point using its coordinates.

So, I can say that the coordinates are X, Y and Z and then we will say, this is the X coordinate, this is the Y coordinate and this may be the Z coordinates. So X, Y and Z. So, we will say that at this point P can be represented as X, Y, Z that is the way how you represent a point in 3D space. This is a vector P the using the coordinate frame you will be able to represent the vector P. Now if is the same as an object in space. Suppose this is an object a 3D object in space then we can actually say that the position of the centre of this object can be represented again using the coordinate points.

So, you have these coordinates to represent the, the centre of this object, but that is not sufficient in the case of a 3D object because that actually gives you only the position of the object but we need to know how this object is oriented with respect to a frame. So, we need to always have a reference frame. So we called this is the reference frame with respect to which we can represent a position. So whether it is a point or an object, we can use a reference frame to represent the position of a object or a point or an object and that is what we called as the coordinate frame.

So this coordinate frame becomes the reference frame to represent the position of an object or a point in 3D in space. So it can be 3D space or any dimensional space, we can use a coordinate frame to represent position of object or point in the space. Now, if you want to represent the orientation of the object in space then we need to have something called then we need to have another reference frame. We will say that this also has got a reference frame. So we have a frame attached to this which is like this. So I will say that, there is a reference there is a frame attached to this, so I will say this is x', y' and z'.

So, I will say, this is the frame which is attached to the body. So, now we have a reference frame and a frame attached to the body also. Now the position of the body or this object can be represented as the position of the coordinate frame, this frame. So we if we know the origin of this frame, then we can represent the origin of this frame using the coordinates as the position of the object and the orientation of the object can be represented as a, as reference to this frame.

So, what is the orientation of this frame with respect to this frame? So I call this as the reference frame and I call this as the mobile frame. So I call this as a mobile frame or an object frame whatever you name you want to called. So if I call this as a mobile frame and this as a reference frame, then I can represent the orientation of the object by looking at the orientation of this coordinate frame with respect to this coordinate frame.

So how these two coordinate frames are aligned that actually represents the orientation of the object. So in order to represent the position and orientation of object in 3D space, we need to refer, we need to represent them in terms of coordinate frames. So we need to identify the reference frames and with respect to that frame we can actually represent the position and we can have an frame attached to the object and the orientation of that frame with respect to the reference frame gives the orientation of the object with respect to the reference frame. So this way we can define the position and orientation of objects with respect to coordinate frames.

Now the frame which is used as a reference is normally referred to as a fixed frame and the frame which is attached to the body is known as a mobile frame. So, we have a fixed coordinate frame with respect to which the position can be defined and we have a mobile frame with respect to and the orientation of the mobile frame with respect to the fixed frame can be used to represent the orientation of the object. So, that is the way how we can represent the position and orientation of objects in space. Now, let us see let us go to the the basic definition for the coordinate coordinates and the coordinate frames and then see how we can represent the orientation using the coordinate frames.

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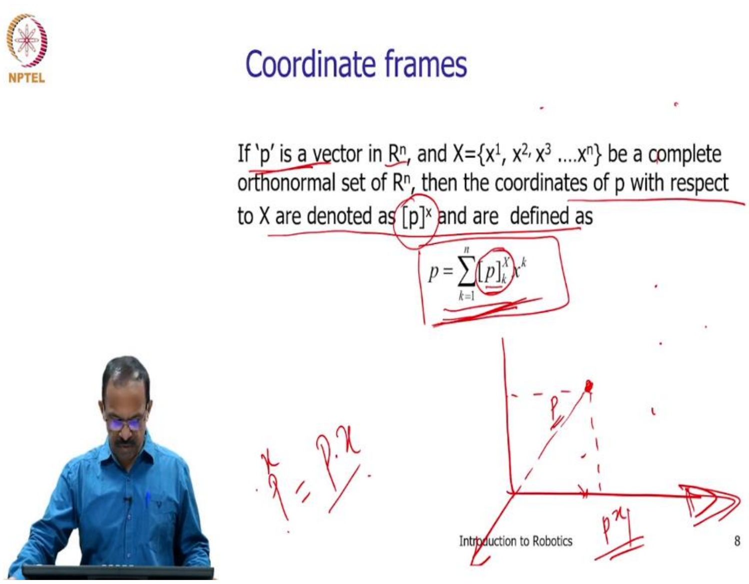

So, now if you have a vector p suppose p is a vector. In R, dimensional space, I mean n dimensional space where R is; R^n is n dimensional space, then the coordinates of p with respect to X are denoted as $[P]^x$. So, the coordinate of p with respect to X are denoted as X and the vector p can be defined as $P = \sum_{k-1}^{n} [P]_k^X x^k$

So this is the way how we can actually define the vector where this is the coordinate of p with respect to the kth axis. So, you have this x1, x2, x3, xn as the axis of the coordinate frame or the n dimensional coordinate frame then we will get this as the coordinates of p with respect to X and pX, xk is the complete vector, how you define the vector in n-dimensional space. Now we say three-dimensional space you will be able to see that this p1, p2, p3 will be the coordinates and we will be able to get the vector. Now, how do we actually define these coordinates? Suppose you have a coordinate frame like this and this is your p then this is the your px that is the coordinates of p in the x axis.

So that is the px. Now this px can actually be obtained as by taking this vector p. So if you take the dot product of p with respect to x, then we get this as px that is the dot product of this vector with the unit vector of x axis you will be getting this as px. So, px or the coordinate of p with respect to x is nothing but the dot product of the vector p with respect to the x axis.

$$[P]^x = P.x$$

Similarly, py will be the dot product with respect to y and pz will be dot product with respect to z. So, that is the way how we actually define the coordinates of p in this coordinate frame.

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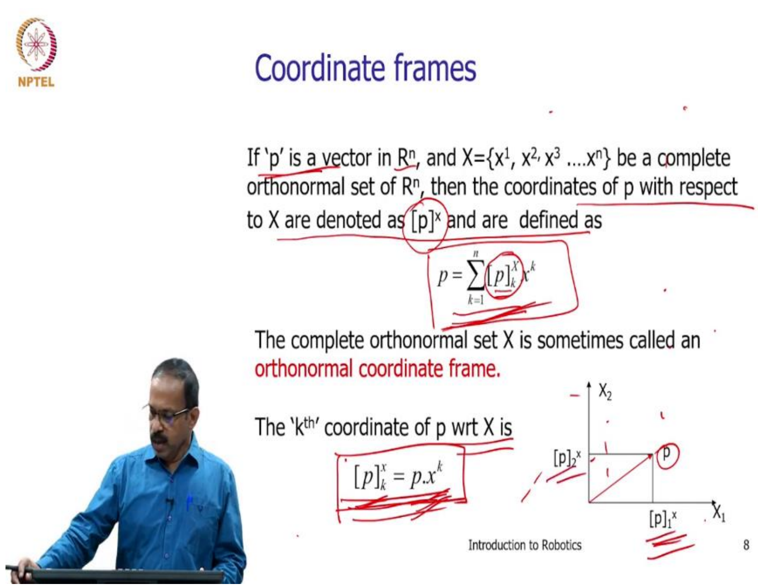

So, the kth coordinates of p with respect to X is defined as

$[P]_k^x = P.x^k$

So we can see that the kth coordinate of p with respect to x or with respect to the coordinate frame x is x1, x2, x3 depending on the number of the dimension of the coordinate frame. So, here if it in three dimensional coordinate frame then you have k is equal to 3, maximum.

So ,this is $[P]_k^x = P.x^k$

So if it is p with respect to X, then we have this coordinate $[P]_k^x = P.x^k$. So you take the dot product of the vector p with respect to the kth axis you will get the coordinate of p. So this is the way how we define the coordinate of a point in space. Now if you have, that is the explanation here given here so you have this as $[P]_1^x$, $[P]_2^x$ etcetera. So, we have this as this is the p that is the vector p and $[P]_1^x$ is nothing but p.x^1 and $[P]_2^x$ is nothing but p. x^2. So that is the way how we get the coordinates of p.

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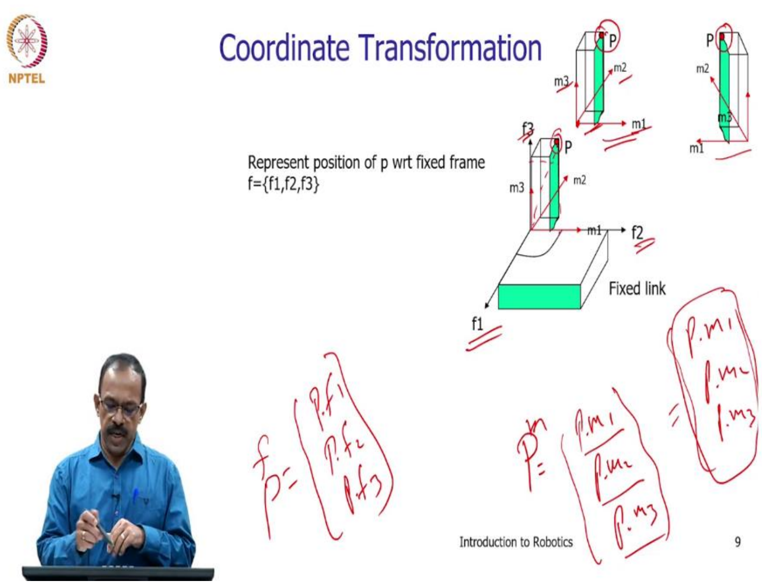

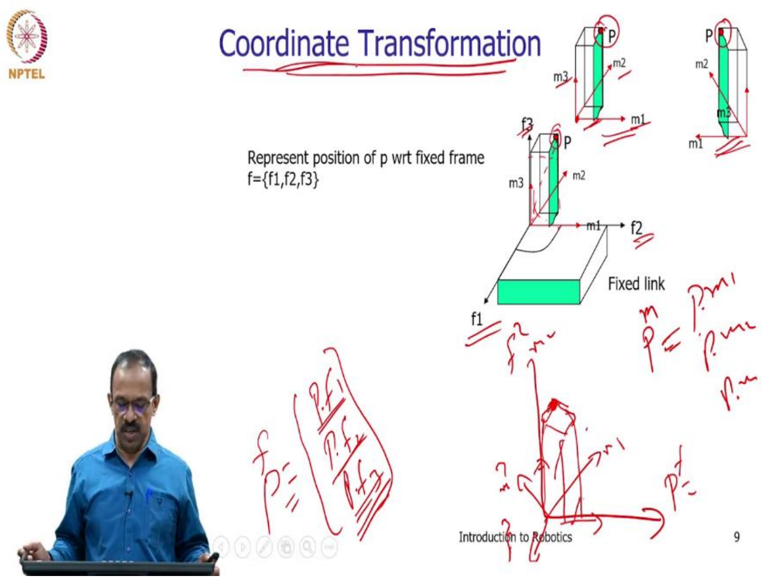

Now, having seen this now we need to look at, how do we actually represent this in different situations. That is how do we actually represent the transformation of a object. That was about the point p. Now, suppose we have an object like this suppose we have an object three dimensional object in the three-dimensional object we define the coordinate frame as m1, m2 m3. So we define a coordinate frame m1, m2, m3.

Suppose this is I take this as a object. So I have a object three-dimensional object and I define a coordinate frame like this m1, m2 and m3 three three-dimensional coordinate frame. And as you can see I define a point P at one corner. So, I can see I can define a point P like this in this coordinate frame. So, as you can see, I am using this point and this is the origin of the coordinate frame and m1, m2, m3 are the axis of this coordinate frame.

Now, this point P to the coordinates of P can be actually be obtained by taking $P = \begin{bmatrix} P.m_1 \\ P.m_2 \\ P.m_3 \end{bmatrix}$. So that that will be the coordinates of P with respect to the frame m1, m2, m3. So, I attach this to the object and therefore I call this as mobile frame because I assume that it actually moves along with the object the frame moves along with the object. So I called it as mobile frame and I will be defining this I mean the point P it here and I will get the coordinates as $P = \begin{bmatrix} P.m_1 \\ P.m_2 \\ P.m_3 \end{bmatrix}$.

Now, I assume that this object is rotating by 90 degree or 180 degree. So, I am having this object and the here this point is P, I am assuming that it actually rotates by 180 degree. So, it actually move like this and the point is P now here. So point P as I mean that is rotated and I want to know, what is the position of P after the after rotation.

So, the object is rotated 180 degree and I want to know, what is the position of P with respect to the frame now? So, now the P is this and you have this frame also rotated like this I mean along with the object the frame rotated and you want to know the position of P. So, what will be the position of P now will it be the same as previous one or the P will change now the position of P will change with respect to the frame?

It is not going change because now $P = \begin{bmatrix} P.m_1 \\ P.m_2 \\ P.m_3 \end{bmatrix}$. So that is the coordinate of P and since m1, m2, m3 also rotated along with the object there will not be any change in this coordinates of P. So, P will remain same, so any point P in this three dimensional object will remain as the same as the object moves object rotates or translate whatever it is because the coordinate frame also moves along with that therefore the coordinates of P remains same as it has the object moves. So there is no change in the position of P with respect to the mobile frame

Now, we assume that this object we actually move this object to or we actually place the objects in this frame, I have a fixed frame now. So I have a fixed frame. I will say that the fixed frame is this one. I will assume that there is a fixed frame with respect to my body or what the room, I will say that I am initially aligning this object with the fixed frame such that the, the axis of mobile frame and the fixed frame are aligned.

That is, I am placing it like this, I am placing this object in this frame with a fixed frame there is a fixed link or I have a call this as fixed frame f1, f2, f3. So f1, f2, f3 is a fixed reference frame and initially I place this object p in the in this frame aligning m1 with it is slightly different in this case, but I assume that it is m1, m2, m3 are aligned in this case.

Now if I align this and then get the point P with respect to m will remain the same. So P with respect to m will remain the same here. Now, I will find out what is the point of P with respect to f. So I am interested in knowing now, what is P^f? So this was P^m mobile frame I am interested in knowing, what is P^f? What is the position of P with respect to the fixed frame? I can actually get it again by using the same principle. I take $P^f = \begin{bmatrix} P.f_1 \\ P.f_2 \\ P.f_3 \end{bmatrix}$ I can take the dot product of this vector P with respect to f1, f2, f3 I will get the position of this.

And the position of $P^m = \begin{bmatrix} P.m_1 \\ P.m_2 \\ P.m_3 \end{bmatrix}$. Now if I rotate this object 90 degree now at this location if I rotate it by 90 degree, so initially we have this I will put this f1, f2 and f3 and we had this objects and we had this m1, m2 and m3. So, this was point P. Now, if I have rotated this objects by 90 degree and this P, actually I have turned this and the object actually came like this and the P actually reached here this position. So, this is P now. So m1, m2 m3 also rotated. So, now m1, m2 and m3 is going like this, this will be m1, this will be m2 and this will be m3. So you have this object initially this object was in this position and it rotated by 90 degree or 180 degree as like this and the object has actually moved this position the P has actually reached here.

Now if I want to get the position of P after rotation, so if I get the position of P after rotation. So P with respect to mobile frame after rotation will again remain same as $P^m = \begin{bmatrix} P.m_1 \\ P.m_2 \\ P.m_3 \end{bmatrix}$. So there is now change in the position of P with respect to mobile frame. But since this is rotated by 90 degree its position with respect to f can be still represented using the P dot f1, P dot f2 can be represented using $P^f = \begin{bmatrix} P.f_1 \\ P.f_2 \\ P.f_3 \end{bmatrix}$, but since this object has change and its mobile frame has moved therefore the values of $P^f = \begin{bmatrix} P.f_1 \\ P.f_2 \\ P.f_3 \end{bmatrix}$ will not remain same because the object has rotated.

So, what actually it says that when the object is rotating the position of P remains same with respect to a mobile frame but its position changes with respect to the fixed frame and therefore we need to if we need to represent the position of P with respect to the fixed frame when the it is moving we will need to find a method to find out what is P^f when the robot, when the object is moving with respect to the fixed frame or with respect to its own frame. So that is basically the,

the transformation of coordinates. How do we represent the position of a point in 3D space or position of a point in that in an object when the object is moving with respect to a reference frame? How do you represent the position is the question of coordinate transformation?

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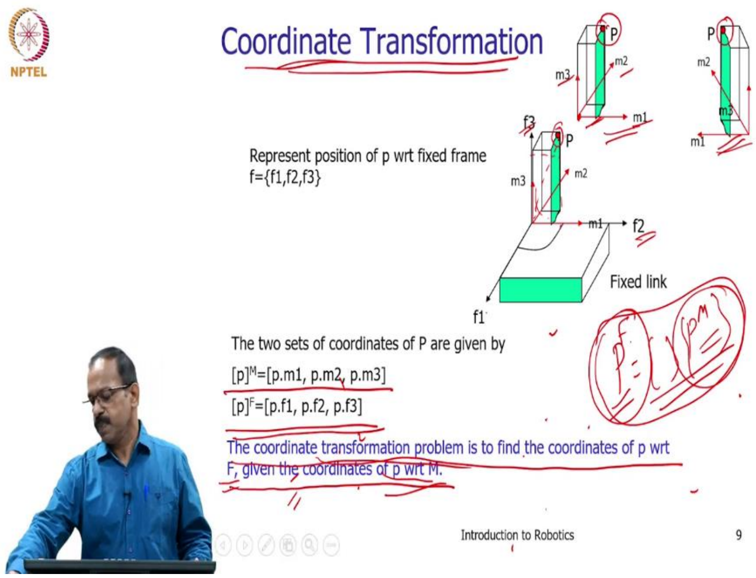

So, let us look at this in a slightly detailed way. So, assume that the P the two coordinate frames P^f P^m is basically $P^m = \begin{bmatrix} P.m_1 \\ P.m_2 \\ P.m_3 \end{bmatrix}$ as you can see. So this is $\begin{bmatrix} P.m_1 \\ P.m_2 \\ P.m_3 \end{bmatrix}$. Now this is $P^f = \begin{bmatrix} P.f_1 \\ P.f_2 \\ P.f_3 \end{bmatrix}$.

That is the point P with respect to fixed frame can be represented using $\begin{bmatrix} P.f_1 \\ P.f_2 \\ P.f_3 \end{bmatrix}$. Suppose there is a rotation so the coordinate transformation problem is to find the coordinates of P with respect to f given the coordinate coordinates of P with respect to m, suppose we know the coordinates of P with respect to m. How do we find the coordinates of P with respect to f is the coordinate transformation problem?

So we have this P now with respect to m. And we want to know, what is P with respect to f when this P is moving when this object is moving. How do we actually represent P with respect to f, if we know P^m and when it is moving when the object is moving, how do we represent P^f, if we know P^m. Because P^m remains same with respect to the mobile frame it remains same but with respect to the fixed frame it changes because of the motion of the object. So, how do we represent P^f, if we know P^m is basically known as the coordinate transformation problem.

So, we know the coordinates of P with respect to M. But we want to know coordinates of P with respect to F. So, we want to know this P^f, when this P is moving it is moving in different ways still we want to know, what is the P with respect to F. So that we can represent the position and orientation of the object with respect to a reference frame whatever happens to the object whether the object is moving rotating translating, we still want to know what is the position and orientation of the object.

So if there is a car or a mobile robot in the room and the mobile robot is moving around. So, we want to know its position, so we need to have a reference frame and we want to know it is orientation also, so if you want to know the position orientation of the robot in the room, we represent the position with respect to a reference frame and the orientation will find out the mobile frame and then see how much it has rotated and we try to find out the orientation also. So the question here is how do we get this P^f? That is the fixed frame coordinates when there is a movement for the object and for the point P which is when it is moving how do we represent is basically the coordinate transformation problem.

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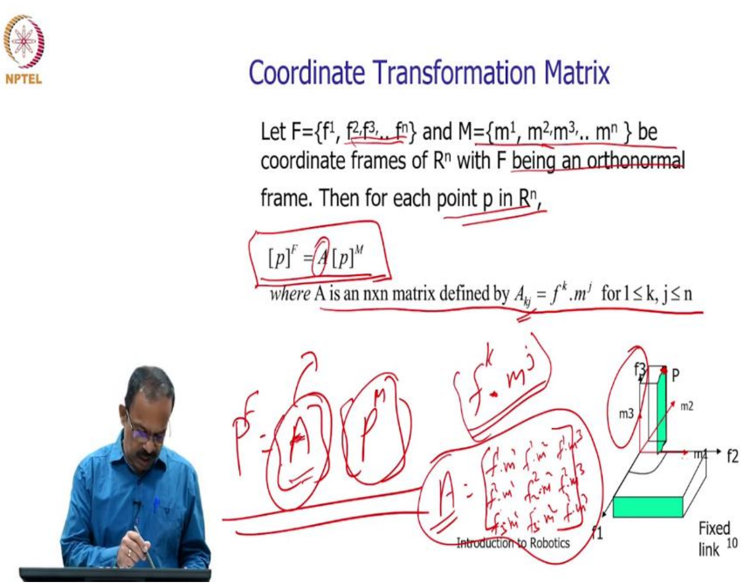

Now if we look at this in detail. We can see that so the $F = \{f^1, f^2, f^3 \ldots f^n\}$ and $M = \{m^1, m^2, m^3 \ldots m^n\}$ coordinate frames and F being an orthonormal frame. Then for each point P in Rn that is the point P in R^n as you can see, this is the point P in Rn we can say that $[P]^F = A[P]^M$. So we can represent P^f as matrix A multiplied by P^m and this A is an n x n matrix defined by $f^k.m^j$.

So, this A is known as the coordinate transformation matrix. So, basically we are telling that we can write P^f as a matrix A multiplied by P^m that is if we know P^m. If we know this P^m if P represent to mobile frame then the point P with respect to the fixed frame can always be represented using this relationship where A multiplied by P^m where A is known as the coordinate transformation matrix, and the elements of this transformation matrix can be obtained as $f^k.m^j$.

So, F is the fixed frame M is the mobile frame F is fixed frame, M is the mobile frame. So $f^k.m^j$ gives you the elements of $A_{kj} = f^k.m^j$. So if the three dimension frame, so if have f1, f2, f3 and m1, m2, m3. So, we can see this initially they are aligned. So f^k, F and M are aligned and then when it is rotating you will be having a different point to be represented and that point P can be represented with respect to F by using this A.

So, now if you write A for a three dimensional, for a three-dimensional space f1, f2, f3 and m1, m2, m3 then we can say that $A = \begin{bmatrix} f^1.m^1 & f^1.m^2 & f^1.m^3 \\ f^2.m^1 & f^2.m^2 & f^2.m^3 \\ f^3.m^1 & f^3.m^2 & f^3.m^3 \end{bmatrix}$

Basically, it says that if you have f1, m1, f2, m2, f3, m3 and if there is a rotation, now there aligned so you can see in this case f1 dot m1. So this, so here not aligned.

So $f^1.m^1$ and this is m1 this is f1, so $f^1.m^1$will be 0. So there will be they are not aligned so it will be getting it as 0. So whenever there is an alignment you will see $f^3.m^3$ you will see $f^3.m^3$ is 1. And same way you can get, $f^2.m^2$ etc and we will be able to get this matrix A there is a transformation matrix A. So, the transformation between two coordinate frames. So these are two coordinate frame one is mobile and other one is fixed.

So, the coordinate transformation between these two frames can always represented using a matrix A and the elements of A can be obtained by taking the dot product of $f^k.m^j$. So, we will be getting all the elements of the rotation matrix by taking the dot product. So that is basically the coordinate transformation matrix between two coordinate frames one is a fixed frame another one is a mobile frame in this case.

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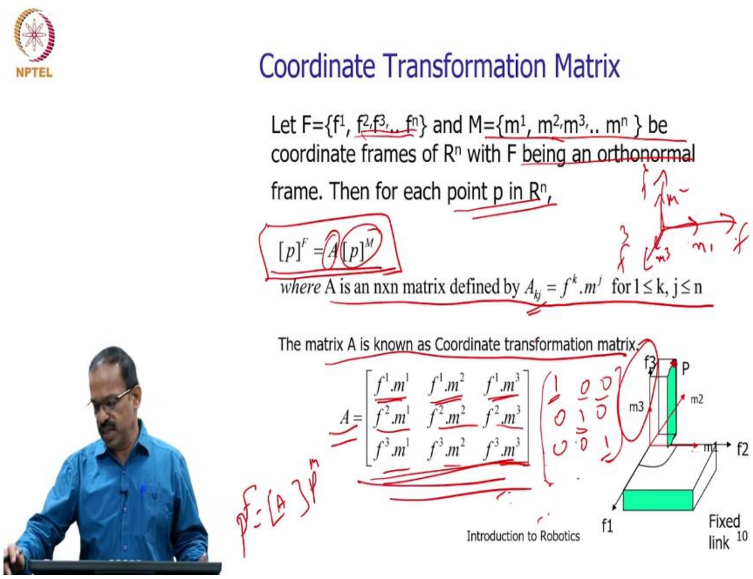

So, the matrix A is known as the coordinate transformation matrix and A is given as $f^1.m^1$, $f^1.m^2$, $f^1.m^3$ and early this one also. So this is known as the coordinate transformation matrix. So now if can if you know the point P in mobile frame. And you want to find out P^f. So what you need to know is what is A and this is A depends on how it is actually rotating with respect to the fixed frame how the coordinate frames are aligned.

The coordinate frames are aligned both f1 m1, f2 m2 and f3 m3 they are aligned then this will be 1, 0, 0; 0, 1, 0; 0, 0, 1. If F and M are aligned, for example, if this is f1, f2 and f3 and this is m1 and this is m2 and this is m3 then you will see that if f1 is aligned with the m1 then $f^1.m^1$ will be 1, $f^1.m^2$ will be 0, $f^1.m^3$ will be 0. Similarly, $f^2.m^2$ will be 1, $f^3.m^3$ will be 1. If they are aligned, then the dot the matrix will be identity matrix and that will says that P^f and P^m are the same. So, the point P in fixed frame and mobile frame will be having the same coordinate.

If they are not aligned, then there will be a different coordinate in the fixed frame and that is obtained by using this relationship. So, that is what actually the coordinate transformation matrix. Hope you have understood the coordinate transformation matrix. As I mentioned some of you must be knowing all these things. But for those who are not familiar, I just wanted to ensure that you understand the basic principle of coordinate transformation matrix. This transformation matrix is used in many other field also.

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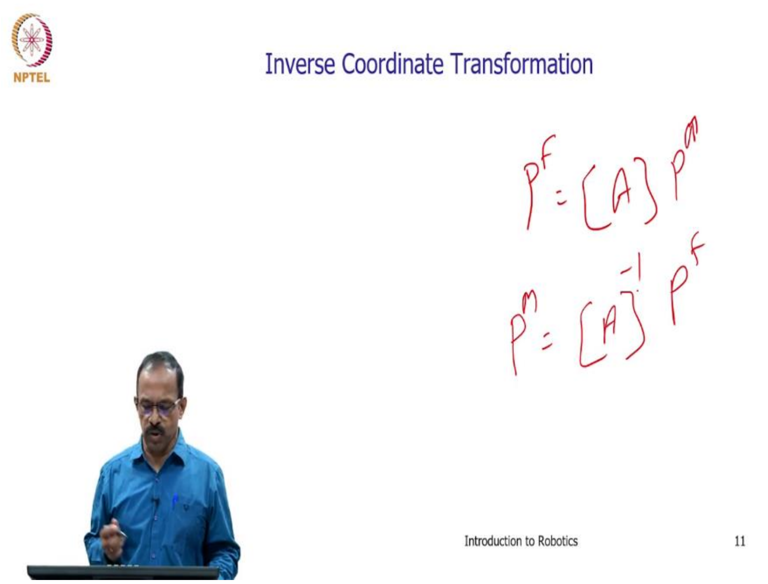

Now, if we know this coordinate transformation matrix, so that is basically we have P^f is equal to A P^m but we can do the inverse also, if we know P^f then we can actually find out P^m from P^f. So what we need to do is to use the A inverse. So we can actually take the inverse of the coordinate transformation provided the origin are the same that is the only condition here.

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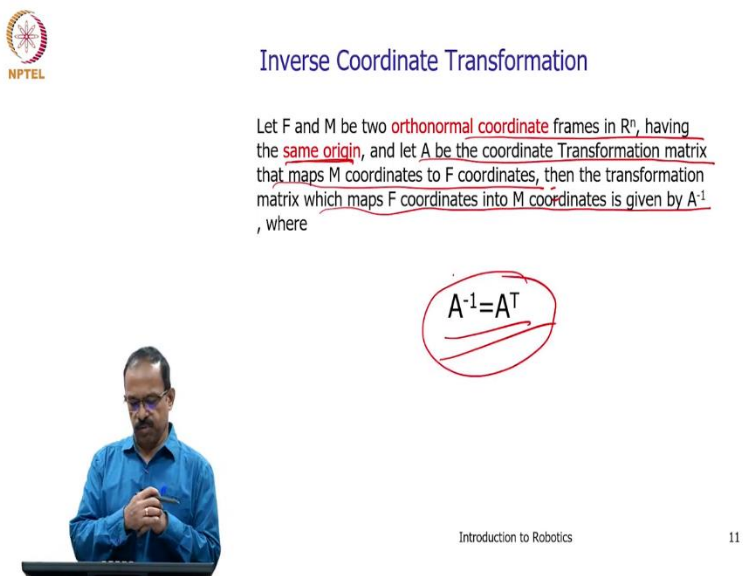

So, we can actually get the inverse I mean inverse can be obtained as a transpose in this case. So if F and M the orthogonal coordinates Rn having the same origin and let A be the coordinate transformation matrix that maps M coordinates to F coordinates. Then the transformation matrix, which maps F coordinates to M coordinates is given by A inverse, where $A^{-1} = A^T$. So, if you know A, you can use the A transposed to get the inverse coordinate transformation provided both the frames are at the same origin. So both having the same origin, then we will be able to use A inverse equal to A transpose. So that is the inverse coordinate transformation matrix.

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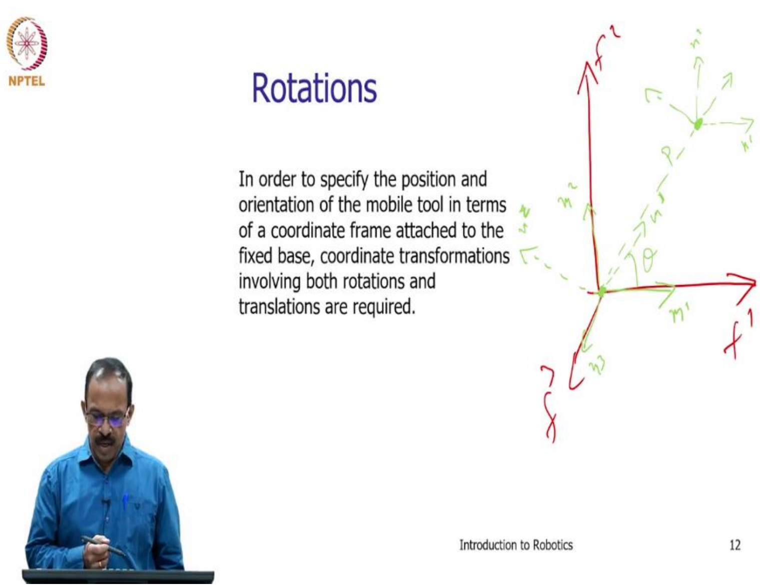

So, that actually talks about the basic principle of coordinate transformation that is you have two coordinate frames and you want to represent a point with respect to one coordinate frame from the other frame or you know, one the point with respect to one coordinate frame the coordinates are known you want to know the coordinates with respect to the other frame. So, one we called a fixed and one we called as mobile. If the frames are moving related to each other. Then we will be able to get the coordinates by using the coordinate transformation matrix. And what we need to do is to take the dot product of the axis and get the matrix, you will get the coordinate points.

Now, let us consider some of the transformations because we know there is a movement of the frame mobile frame. So let us consider some of the movements. So for example if you have a frame like this a fix frame f1, f2 and f3. Now I will define m1, m2 and m3. So these two frames initially they are aligned, they are having the same origin. So there can be two ways it can actually transform or can move two ways; one is that you can actually rotate the mobile frame can actually rotate m1, m2 it can actually rotate assume that it is rotating with respect to m3.

So, you will be getting this as m1 the new m1 will be this and this will be the m2. So, that is one. So basically we are saying that there are coordinate frame is rotated by an angle theta with respect to f3 or m3. So that is one way of rotation one way of moving movement. And another one is that the transformed one it can actually move somewhere here. It can actually translate. It can translate and then assume that it is initially translated like this m1, m2, and then it can actually rotate also I mean you can have a translation rotation or rotation translation anything like that is possible.

So, there are multiple ways in which the coordinate frame can move or an object can move. So when you say coordinate frame is moving you are saying an object is moving and the coordinate frame attached to the object is basically the mobile frame. So, the coordinate frame can actually have a rotation or it can have a translation of P, I can say this coordinate frame has moved from here to here and then rotated also. So you can have a translation rotation or you can have a rotation and translation. So there are different ways of moving.

So we first consider the rotation only. We consider that the coordinate frame is rotating and when it is rotating we are interested to know, what is the transformation matrix when there is a rotation or there is a pure rotation. So we want to specify the position and orientation of mobile tool in terms of coordinate frame attached to this. So normally the transformations involve both rotation and translation. So first we consider the rotation alone that is this mobile frame has rotated by an angle theta with respect to the fix frame and we want to find out what is the transformation matrix.

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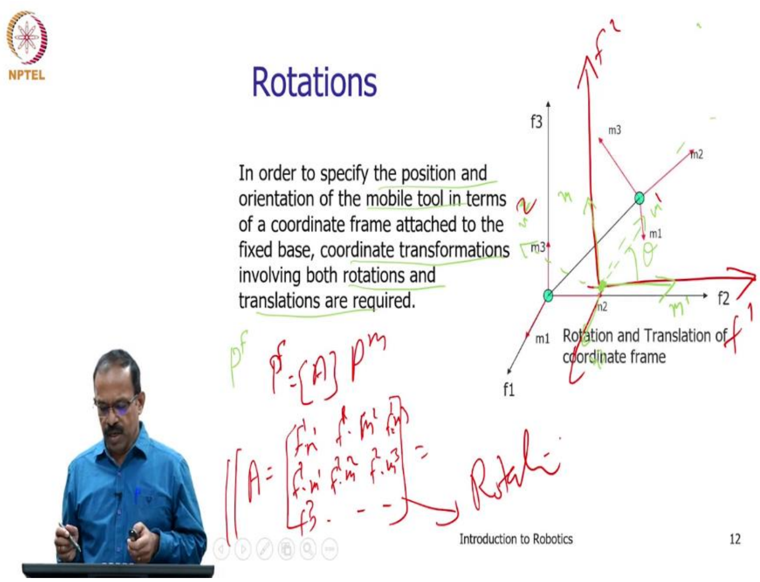

So, we know that P^f is, we know P^f is [A] P^m and A is the transformation matrix. Now we want to find out what is this transformation matrix when there is a rotation. That is, you want to find out. What is this A when there is a rotation? And we know that this rotation matrix, sorry, the transformation matrix is defined $A = \begin{bmatrix} f^1.m^1 & f^1.m^2 & f^1.m^3 \\ f^2.m^1 & f^2.m^2 & f^2.m^3 \\ f^3.m^1 & f^3.m^2 & f^3.m^3 \end{bmatrix}$. So this is the way how the rotation matrix is this matrix is defined. So first if it is only rotation then we call this as a rotation matrix. So we called this as a rotation matrix.

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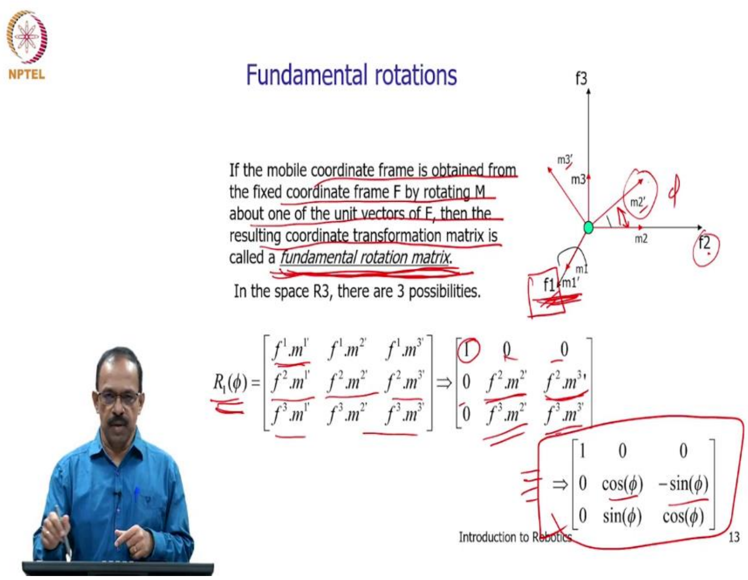

So, we will look into this as a rotation fundamental rotation. So we will say that there is only one rotation if the mobile coordinate frame is obtained from fixed coordinate frame by rotating M about one of the unit vectors of F then the resulting coordinate transformation matrix is known as fundamental rotation matrix.

So, if you have a rotation of the mobile frame with respect to one of the fixed frames then we call this rotation as a, the transformation as a fundamental rotation and the transformation matrix is known as fundamental rotation matrix. So as I showed you, you can actually have a rotation like this. So you have f1 m1, f2 m2, f3 m3 that is the coordinate frames can be F and F and M. So, f1, f2, f3 represents the frame and m1, m2, m3 represents the mobile frame and then assume that the mobile frame has rotated by an angle theta.

Now, $R_1(\emptyset) = \begin{bmatrix} f^1.m^{1\prime} & f^1.m^{2\prime} & f^1.m^{3\prime} \\ f^2.m^{1\prime} & f^2.m^{2\prime} & f^2.m^{3\prime} \\ f^3.m^{1\prime} & f^3.m^{2\prime} & f^3.m^{3\prime} \end{bmatrix}$.So we are actually taking the dot product of the rotated frame, rotated axis with respect to the fixed axis and trying to find out what is the dot product.

So, this R1 phi or the fundamental rotation matrix is f1 dot m1 dash. Now we can see that $f^1.m^{1\prime}$, so this f1 and m1 dash are aligned there is no rotation and there is actually the rotation was with respect to this fixed frame f1 and therefore m1 did not move and therefore you get this as 1. So, $f^1.m^{1\prime}$is 1 in this case fundamental rotation. And this will be 0, 0 and this is $f^2.m^{2\prime}$, so f2 and m2 dash they are not aligned. So there will be a dot product $f^2.m^{2'}$, $f^2.m^{3'}$, $f^3.m^{2'}$, $f^3.m^{3'}$ and so this is 1,0,0; cos phi, minus sin phi, this is by an angle phi. So, you will get this cos phi minus sin phi sin phi cos phi. So this is the fundamental rotation matrix and the rotation with respect to f1.

$$R_1(\emptyset) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & f^2.m^{2'} & f^2.m^{3'} \\ 0 & f^3.m^{2'} & f^3.m^{3'} \end{bmatrix} .= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}.$$

So, when the rotation with respect to the first axis then the rotation matrix is 1, 0, 0; 1, 0, 0; cos phi, minus sin phi, sin phi cos phi. So, this is how you get the fundamental rotation matrix. So whenever there is a rotation of the mobile frame with respect to the first axis of the fixed frame your transformation matrix will be like this and that is known as the fundamental rotation matrix. So, same way we can actually find out if the rotation with respect to f2 or the rotation with respect to f3 you will be able to get the rotation matrix in the same way, the difference will be if it is rotating with respect to f2 then $f^2.m^{2'}$ will be 1. If it is rotating with respect f3 then this will be 1. Otherwise the rotation the matrix formulation will be the same.

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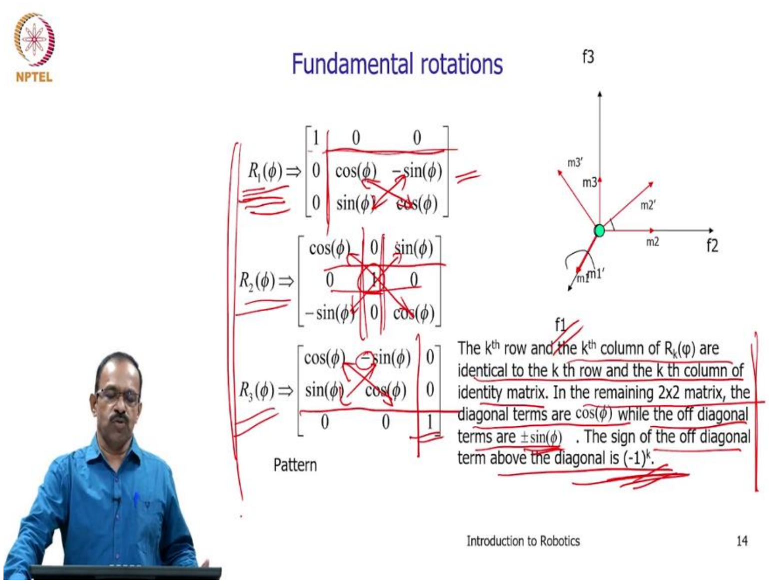

So you can get the fundamental rotations as like this. So R1 it is rotation with respect to the first frame first axis, then this will be the rotation matrix. $R_1(\emptyset) .= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$

And if the rotation with respect to the second axis, $R_2(\emptyset).= \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi & 0 & \cos(\phi) \end{bmatrix}$ f2 then this will be the matrix so we can see this will be 1 because f2 and m2 will be aligned if the rotation is respect to f2. And the third one will be like this $R_3(\emptyset).= \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. So the rotation is with respect to the third axis. So this is the fundamental rotation matrix that you can identify whenever there is a rotation of the mobile frame with respect to the fixed frame and the rotation is with respect to a particular axis.

Then you will be able to easily get that rotation matrix by using this rule. So first axis means the first row and first column you will be able to see like this 1, 0, 0; 1, 0, 0 then this will be the second axis with respect to second axis this one, third axis this one. So these are known as the fundamental rotation matrices. The transformation matrix which represents only rotation then we call it fundamental rotation matrix. So you do not need to really I mean if this will be we will be using this many times in the kinematic analysis, but you do not need to remember it by heart.

So there is a simple way to remember it. So, there is a pattern in this one. So if the rotation is with respect to first axis, you can see the first row and first column, first row and first column will be part of identity matrix 1, 0, 0; 1, 0, 0. If the rotation is with respect to second then the second row and the second column will be part of identity matrix 0, 1, 0; 0, 1, 0. Similarly if the rotation with respect to third, third row and third column will be 0, 0, 1; 0, 0, 1. So that is the pattern in this and then, the diagonal will be always cos phi, see diagonal element will be always cos phi.

So, we can see always it will be cos, off diagonal elements always will be plus or minus sin phi. And the sin of the off diagonal above the diagonal is minus 1 to the power of k. So if you have a sin phi here and the sin of the off diagonal term above the diagonal is minus 1 to the power of k. So minus 1 to the power of k. So this is the above the diagonal is minus this one. So it is minus 1 to the power of 1, this is minus 1 to the power of 2, this is minus 1 to the power of 3, so you will be getting it as minus, plus, minus.

So, we will always see that the kth row and kth column of Rk phi are identical to the kth rank column of identity matrix. And the remaining 2 by 2 matrix the diagonal terms are cos phi and the off diagonal always plus or minus sin phi. So it will be either plus or minus sin phi. And the sin of the off diagonal term above the diagonal is always minus 1 to the power of k. So, if you can remember this you will always able to get this fundamental rotation matrix without any difficulty. So that is the fundamental rotation matrices when the mobile frame is rotating with respect to the fixed frame.

Now, suppose you want to get the composite rotations suppose you have multiple rotations taking place because first this suppose this is an object and you have point P here. Now, this red one is the so this is the black one is the fixed frame and the red one is the mobile frame assume that and if you rotate it, you can actually see that this will be it you can be rotated in with respect to this axis, it can be rotated with respect to this axis, then this will be moving to this side. So, it can actually get the p somewhere here and you can actually rotate again with respect to this you can actually rotate with respect to the vertical axis you will be getting this position as P.

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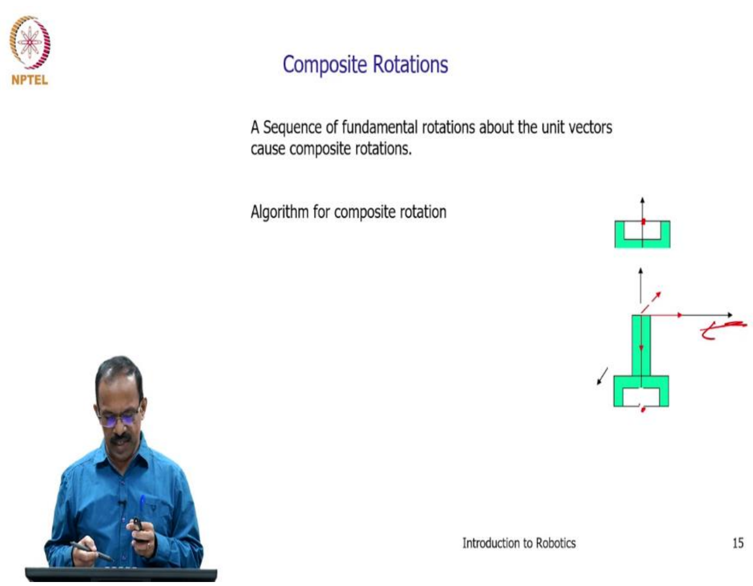

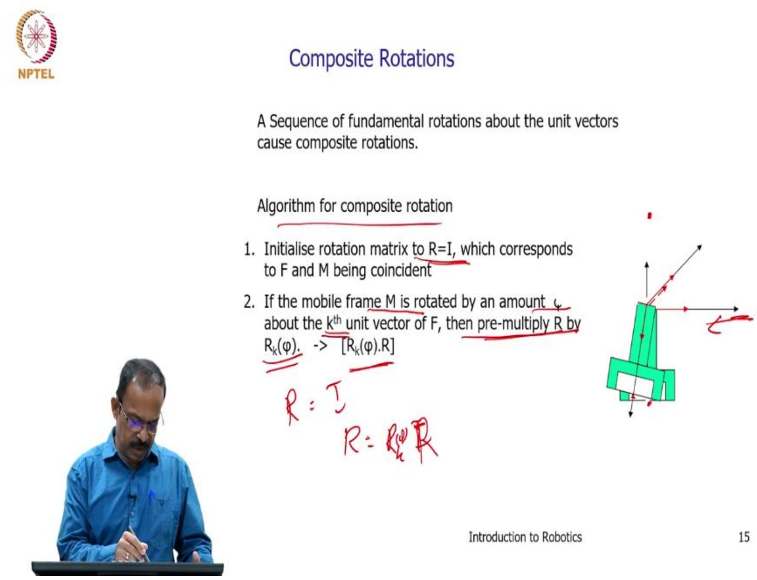

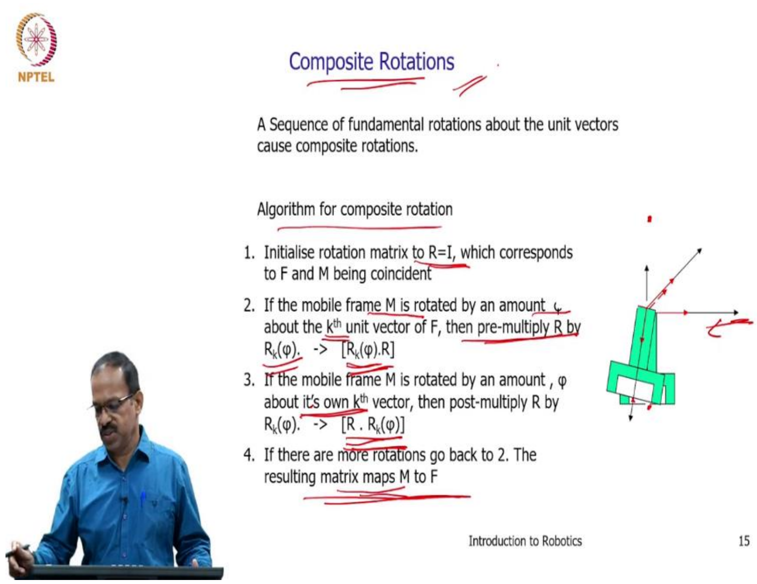

So like the suppose you have multiple rotations taking place. You want to know, what is the point of P? What is the point P with respect to the fixed frame? And this can be done by using a simple algorithm. So we need just need to follow a simple algorithm to get a composite rotation matrix that is initialize the rotation matrix R to I an identity matrix, which corresponds to F and M being coincidence. So assume that F and M are coincident at initially and assume that your rotation matrix R is identity matrix.

And then look at the rotation, if the mobile frame M is rotated by an amount phi about the kth unit vector of F then pre multiply R by Rk phi. So, initially R is I. Now you look at what is the rotation happening if the mobile frame is rotated with respect to the kth unit frame of fixed frame then do your pre multiplication then R is equal to I Rk , k is equal to 1, 2 or 3. So you can get this as a pre multiplication Rk multiplied by R that is R. So R is phi now. So it is Rk multiplied by R so new one.

But if the rotation is with respect to mobile frame or its own frame because the object can actually rotate with respect to its own its own frame also, not necessary that it should always rotate with respect to fixed frame it can rotate with respect to its own frame also, and that is that is the case then what you need to do is to do a post multiplication. So here if the rotation with respect to the kth vector of its own frame its own kth vector then you do here a post multiplication. So then R is multiplied with Rk. So the post multiplication and pre multiplication makes difference and therefore the rotation can actually be represented using either a post or pre multiplication.

So this is the algorithm that we need to use and if you have more multiplication, more rotations keep on doing this, keep on doing this till you complete all the rotations. So that is how you get the composite rotation matrix by looking at whether it is rotating with respect to its own frame or with respect to the mobile, the fixed frame and it continue this till all the rotations are completed and the resulting matrix maps M to F. So you can get the M to F mapping by the resulting composite rotation matrix. So that is the way how you get the composite rotation.

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So, I will stop here. So we will talk about this Yaw Pitch Roll transformation matrix as a composite rotation, and then we will see how to get the transformation matrix using the algorithm that we discussed. So we will discuss this in the next class. Thank you.