Fundamentals of Automotive Systems
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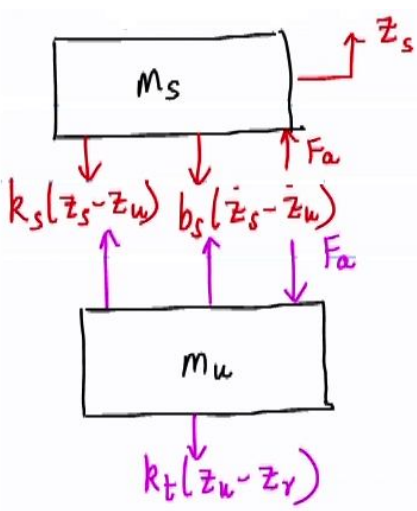
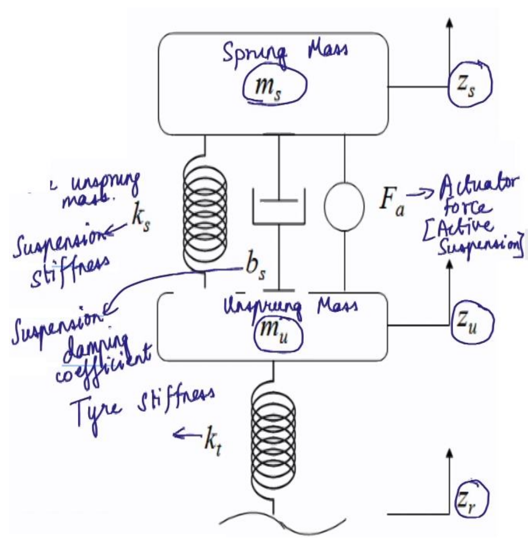
Module No # 12
Lecture No # 70
Dependent Suspension and Suspension Analysis – Part 02





QUARTER CAR MODEL

So let us go forward so let us look at the equation of motion of the sprung mass first. So if we just apply newton second law what are we going to get we are going to get M_s times Z_s double dot "t" which is nothing but the acceleration of the sprung mass that is going to be equal to minus k_s times Z_s t minus Z_u t and I am just writing from the free body diagram and then we will have minus b_s times Z_s dot t minus Z_u dot t right plus F_a t.

EOM of the Sprung Mass:

$$M_s\ddot{Z}_s(t) = -k_s\big(Z_s(t) - Z_u(t)\big) - b_s\left(\dot{Z}_s(t) - \dot{Z}_u(t)\right) + F_a(t)$$

$$\Rightarrow \boxed{M_s\ddot{Z}_s(t) + b_s\dot{Z}_s(t) + k_sZ_s(t) = b_s\dot{Z}_u(t) + k_sZ_u(t) + F_a(t)} \quad (1)$$

So rearranging this equation we are going to get m_s times Z_s double dot plus b_s times Z_s dot plus k_s times Z_s t that is going to be equal to b_s times Z_u dot t plus k_s times Z_u t plus F_a t alright so this is the equation of motion for the sprung mass. So let us call this as equation 1 so similarly we can write the equation of motion for the unsprung mass.

EOM of the Unsprung Mass:

$$\dot{M}_u \ddot{Z}_u(t) = k_s\big(Z_s(t) - Z_u(t)\big) + b_s\left(\dot{Z}_s(t) - \dot{Z}_u(t)\right) - k_t\big(Z_u(t) - Z_r(t)\big) - F_a(t)$$

$$\Rightarrow \boxed{M_u \ddot{Z}_u(t) + b_s \dot{Z}_u(t) + (k_s + k_t) Z_u(t) = b_s \dot{Z}_s(t) + k_s Z_s(t) + k_t Z_r(t) - F_a(t)} \text{ — ②}$$

Equations ① and ② can be rewritten as

$$\underbrace{\begin{bmatrix} m_s & 0 \\ 0 & m_s \end{bmatrix}}_{\text{M— Mass matrix}} \underbrace{\begin{bmatrix} \ddot{Z}_s(t) \\ \ddot{Z}_s(t) \end{bmatrix}}_{\ddot{Z}(t)} + \underbrace{\begin{bmatrix} b_s & -b_s \\ -b_s & b_s \end{bmatrix}}_{\text{C— Damping Matrix}} \underbrace{\begin{bmatrix} \dot{Z}_s(t) \\ \dot{Z}_s(t) \end{bmatrix}}_{\dot{Z}(t)} + \underbrace{\begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_t \end{bmatrix}}_{\text{K— Stiffness Matrix}} \underbrace{\begin{bmatrix} Z_s(t) \\ Z_u(t) \end{bmatrix}}_{Z(t)} = \underbrace{\begin{bmatrix} 0 \\ k_t \end{bmatrix}}_{b_1} Z_r(t) + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{b_2} F_a(t)$$

So what would we get? Once again we need to equate all the forces on the unsprung mass right so we will get mu times Z_u double dot t that is the acceleration that is going to be equal to k_s times Z_s t minus Z_u t plus b_s times Z_s dot t minus Z_u dot t right minus k_t times Z_u t minus Z_r t minus F_a t right. So that is the equation of motion of the unsprung mass. So if we rearrange what we would get mu times Z_u double dot t plus b_s times Z_u dot t plus k_s plus k_t Z_u t that is going to be equal to bs times Z_s dot t plus k_s times Z_s t plus k_t times Z_r t minus F_a t okay.

So this is the equation of motion for the unsprung mass right so now these two equations can be rewritten in this form. So equations 1 and 2 can be rewritten as see we are trying to analyze the vibrations right so to the road inputs okay. So that is what we are trying to do so in order to facilitate that what we do is that like we are group these two equations and write it in a form which will enable us to do the corresponding analysis right.

So we are going to we will fill up these elements shortly but first let me write the structure so that it will become clear as to what we are after. So let us look at these terms first okay so what are we looking at so let us say you know we call this as our vector z of t which indicates the state of the system right. So because it is a two degree of freedom model so the vector z I am denoting a vector by putting a tilde underneath it right under lower case character.

So the vector z of t is the column vector that consists of the two degrees of freedom which are essentially Z_s and Z_u that is the displacement vector. So Z_s dot, Z_u dot will be the derivative of the displacement vector let me denoted by z dot right which will correspond to the velocity and this will be the second derivative of the displacement vector which will be acceleration terms okay. Suppose if we look at one and two and fill it what will be these term we will get ms 00 mu let me fill it up and then we will have b_s minus b_s minus b_s and bs and then we will have k_s minus k_s minus k_s and k_s plus k_t right.

And this is going to be equal to something times Z_r t plus something times F_a t so these are the two inputs right which are coming in right. So Z_r essentially we will have 0 and kt right from this equation here we will have 1 and minus 1 okay. So this is the general structure of this equation so this 2 by2 matrix is what is called as a mass matrix m okay so matrix we will indicate with an uppercase character that tilde underneath it okay.

So that is our mass matrix so this let us call it as the damping matrix C and this is what is called as the stiffness matrix K and let us call these two vectors as some b_1

and b_2 which are multiplying Z_r and F_a okay. So this implies that thus the governing equations can be expressed as the governing equations of the quarter car model right can be expressed in this generic form this so this will be mass matrix multiplying the acceleration vector plus the damping matrix multiplying the velocity vector plus the stiffness matrix multiplying the displacement that is going to be equal to some vector b_1 multiplying Z_r plus the vector b_2 multiplying F_a t okay.

Thus, the governing equations can be expressed as

$$M\ddot{Z}(t) + \left(\dot{Z}(t)\right) + K\, Z(t) = \, b_1\, Z_1(t) + b_3\, F_a(t) \;\rightarrow Multi\ DOF$$

Q: How can one determine the natural frequencies of this system?

Consider a 1-DOF mass-spring-damper system whose governing equation is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

$$Its\ natural\ frequency\ \omega_n = \sqrt{\frac{k}{m}}$$

The natural frequencies of the multi DOF system can be obtained by solving

$$\alpha_{et}[-\omega^2 M + K] = 0$$

So that is the governing equation alright now we are going to ask ourselves a question right so in vibration studies you know like many times we are interested in figuring out what are the natural frequencies of the system so that we can identify the modes of vibration of the particular system. So even for the suspension system to analyze ride quality you know like we are interested in what are the natural frequencies of the sprung mass and the unsprung mass components right.

So let us ask ourselves a question how can one determine the natural frequencies of this system so if we consider this in general this is a multidegree of freedom system right in general. In this case it is a two degree of freedom system but this will be the general structure for the governing equation of a linear multidegree of freedom system model okay. So we can see that it is a linear ODE but the like it is a vector matrix ODE right so that is what we have got.

Now if we want to find the natural frequencies of this system we have to figure out a generic way to do it. If we look at a simple one degree of freedom spring mass damper system let us say consider a one degree of freedom mass spring damper system alright. So that all of us are familiar with alright whose governing equation is given by let us say mx double dot t plus c_x dot t plus k_x t equals some F of t right x being the one degree of freedom that is some displacement right.

So now what is the natural frequency of the system? It is natural frequency omega n as all of us know is going to be square root of k by m alright so that is answer for a one degree of freedom system. So now the question becomes how do, we do it for this system this system. This multidegree of freedom system. So this I am going to leave it you as an exercise the natural frequencies of the multidegree of freedom system can be obtained by solving the following equation determinant of minus omega square times the mass matrix plus this stiffness matrix equals 0 okay this I will leave it as an homework to figure out how okay.

For the Quarter Car Model, $\underset{\sim}{M} = \begin{bmatrix} m_s & 0 \\ 0 & m_u \end{bmatrix}$, $\underset{\sim}{K} = \begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_t \end{bmatrix}$

$$\alpha_{et} \begin{bmatrix} -m_s\omega^2 + k_s & -k_s \\ -k_s & -m_u\omega^2 + k_s + k_t \end{bmatrix} = 0$$

$$\Rightarrow m_s m_u \omega^4 - [m_s(k_s + k_t) + m_u k_s]\omega^2 + k_s k_t = 0$$

The solution can be obtained as

$$\omega^2 = \frac{(k_s + k_t)}{2M_u} + \frac{k_s}{2m_s} \pm \frac{\sqrt{(k_s + k_t)^2 m_s^2 + k_s^2 m_u^2 - 2(k_t - k_s) m_s m_u}}{2m_s m_u}$$

So this is standard result from vibrations alright so if this were the case for our quarter car model what will happen? By the way what is the definition of natural frequency? It is the frequency at which the undamped, unforced system will oscillate when we subjected to a perturbation that is the definition one needs to use to get this formula okay. So please use the definition apply the definition to this general governing equation of an multidegree of freedom system and you will get this result so please do it as an exercise.

So for the quarter car model we immediately have this equation to be or the element of this equation as follows the mass matrix is this the stiffness matrix is a 2 by 2 matrix whose elements are these. So we have already identified them right when we derive the equations of motion. So now if we put them together we essentially get the following so we will have determinant of minus ms omega square plus k_s then we will have minus k_s minusk_s then minus mu omega square plus k_s plus k_t this is equal to zero okay.

Then if we simplify this we will get ms mu omega power four I request you to fill up the intermediate steps okay this is simple algebra. So I am just writing down the final polynomial that we will obtain please double check this result. So minus ms times k_s plus k_t plus mu k_s omega square plus k_s k_t equals zero. So this is if you simplify this is what okay pretty straight forward okay. So now we can observe that this is a what to say a polynomial on the left hand side we have a polynomial in omega of order four.

But then we can observe that we only have the what to say even powers right so we can think of it as a quadratic polynomial in omega square. so if you substitute omega hat to be omega squared you will get a second order polynomial in omega hat. So one can solve it very easily so if you please do that okay solve the solution can be obtained as okay once again please do this exercise you know I am just writing the final result omega squared is going to be k_s plus k_t divided by two mu plus k_s by two ms plus or minus we will have k_s plus k_t whole square ms square plus k_s squared mu squared minus two times k_t minus k_s ms mu okay.

The whole thing divided by two ms okay so that is the equation so you can see that it is certainly not square root of k by m that what we expect for a that what we got for a single degree of freedom system. Please note that this is only omega squared then we will have two choices for omega squared then take the positive square root for the two choices you will get the two natural frequencies. So we can see that you know like if you want to just extrapolate from the result of the single degree of freedom we are not getting something similar right. Our first intuition would be to do that extrapolation but the result is something else however right.

When the tyre stiffness k_t please do the above exercise that is going from here to here I leave it to you as homework. Once again pretty straight forward algebra okay so when the tyre stiffness k_t is much higher than this suspension stiffness k_s okay one simplification that we can do is that like we can take k_s plus k_t as almost k_t right and we can consider k_t minus k_s as almost k_t okay using this simplification

The solution can be obtained as

$$\omega^2 = \frac{(k_s + k_t)}{2M_u} + \frac{k_s}{2m_s} \pm \frac{\sqrt{(k_s + k_t)^2 m_s^2 + k_s^2 m_u^2 - 2(k_t - k_s) m_s m_u}}{2 m_s m_u}$$

When the tyre stiffness (k_t) is much higher than the suspension stiffness (k_s),

$$(k_s + k_t \approx k_t, \qquad k_t - k_s \approx k_t,)$$

$$\omega_{n1,2} = \sqrt{\frac{k_s}{m_s}}, \qquad \sqrt{\frac{k_t}{m_u}}$$

Corresponds to the sprung mass mode

Corresponds to the unsprung mass mode

what we will get is that if you do the algebra you will observe that this omega natural frequency the first and second natural frequency will reduce to square root of k_s by ms and square root of k_t by mu okay.

But this is true only if the tyre stiffness is much higher than the suspension stiffness this also can be shown very easily that is in the above equation you just wherever you have k_s plus k_t you replace as k_t wherever you have k_t minus k_s you replace as k_t once again right. So then we get a simplified equation that can be used to get a first cut idea about the two natural frequencies okay. Now we can see that this is more or less analogous to or looks to be the same pattern as what we had for the single degree of freedom system.

So, in other words when the tyre stiffness is much more than the suspension stiffness the response of the two masses alright can be analyzed in a decoupled manner okay. So that is the implication so this is what is called as the natural frequency that corresponds to the what is called as the sprung mass mode okay. And this is the natural frequency corresponding to the unsprung mass okay.

So one can immediately observe that the value of k_t by mu is going to be much higher than value of k_s by ms the tyre stiffness is going to be more and the unsprung mass is going to be lower than ms alright by and large. So we can immediately observe that the value of this omega n unsprung mass is going to be much higher than the natural frequency corresponding to the sprung mass okay so this is the simple analysis that one can do for getting an idea about the natural frequencies of this suspension system right okay.

So the there are some standards you know like which are also used to evaluate the corresponding matrix you know associated with the vibration levels since we are discussing ride comfort right.

So let us identify these standards so there is a standard ISO 2631 okay that prescribes methods to quantify whole body vibrations whole human body vibrations okay. So essentially this is for this standard deals with you know like what is the how can we quantify the vibrations that come about when human beings are using a vehicle or machinery or tools in general right. So essentially we are going to apply it for our, what to say vehicle systems analysis right.

So this is whole body vibration with respect to human health and comfort okay. So typically it is agreed upon that when we are looking at vibration levels between 0.5 hertz to 80 hertz we are interested in this range for analyzing human health and confront okay we have to analyze in this particular frequency range to decide you know what is the impact of a vehicles operation on human comfort?

And when we are looking at a frequency range of 0.1 hertz to 0.5 hertz we are considering and we will be evaluating what will be the impact on so called motion sickness right so all of us understand what motion sickness is but that is essentially falls in a low frequency range right. So we can see that there is a huge range of frequencies over which the analysis must be done okay. And one comment here please note that in the above analysis we are dealing with the angular frequency omega.

How do we convert to what to say unit frequency in hertz we have to divide by two Pi okay so this is something which I wanted to point out because sometimes you know like we may we should not forget to do this conversion right. So because we typically talk in terms of hertz right so when we talk of this frequencies.

i) Weighted r.m.s acceleration (base parameter).

$$(a_\omega)_{rms} = \left[\frac{1}{T}\int_0^T \left(a_w(t)\right)^2 dt\right]^{1/2}$$

(a_ω) ⟶ Weighted acceleration.

T ⟶ Duration of measurement.

Crest Factor ⟶ Modulus of the ratio of the maximum instantaneous a_ω to its r.m.s value. The above base measure is normally sufficient for crest factors upto 9.

So let me quickly list three parameters you know which are used to quantify this the base parameter is the weighted root mean square acceleration so what is the acceleration level felt by the driver or occupant of a vehicle right. So this is the base parameter to evaluate comfort levels right so this weighted rms acceleration levels is given by this formula. So over a time period of observation capital T this is represented according to this form okay. So where this aw is a weighted acceleration typically what is called as frequency weighting is done to get this aw and capitalT indicates the duration of measurement okay.

And in this regard there is an important parameter which is called as a crest factor what is this crest factor? It is the modulus of the ratio of the maximum instantaneous aw to it is rms value what we have calculated above okay. So that is the definition of the crest factor and by and large this base measured you know like the above base measured by and large it is people are found out that is normally valid and sufficient till crest fact for crest factors till up to nine okay. So that is this first parameter okay.

ii) Running r.m.s method ⟶ occasional shocks and transient vibration.

$$a_{rms}(t_0) = \left\{\frac{1}{\tau}\int_{t_{0-\tau}}^{t_0}[a_w(t)]^2 dt\right\}^{1/2}$$

τ ⟶ integration time for running averaging.

Maximum Transient Vibration Value [MTW] = max [$a_{rms}(t_0)$] in the period of observation.

The second parameter is when where there are speaks spikes in the acceleration levels then this base factor has to be replaced with others. So then we have what is called as a running rms method which is used when we have occasional shocks and transient vibration. So what is this running rms method how is it evaluated quantitatively. So we have a parameter which is called a_{rms} at a time t not okay which is let me write down the definition then I will briefly explain what it is?

So we essentially evaluated using this formula aw once again the weighted acceleration right frequency weighted acceleration right but here the parameter tau is nothing but what is called as integration time for running average so what we do is that like we select our instant t not alright some instant t not and then we evaluate this measure in a small range t not minus tau 2 t not. So that is why it is called as a running average so the time window keeps on shifting.

So this is the integration time for running average and using this we define a parameter called maximum transient vibration value okay which is abbreviated as MTVV which is defined as maximum of this a_{rms} t not okay in the period of observation that is the definition of MTVV okay. I will just introduce one last parameter and then we would conclude this discussion on suspensions.

iii) Vibration Dose Value [VDV]: More sensitive to peaks.

$$VDV := \left\{ \int_0^T [a_w(t)]^4 \right\}^{1/4}$$

So the last parameter is what is called as a vibration dose value so what is this vibration dose value? This vibration dose value you know like takes into account when there are peaks in the acceleration suppose let us say you know like my car is driven and then like I go into a pothole there is going to spike in the acceleration levels right. So then these vibration dose value is going to be more sensitive it is going to react to those sudden peaks right.

And the idea is to essentially use the fourth power of the weighted acceleration levels instead of second power. So that is the idea if you compare it to the rms acceleration so we are using the fourth instead of second power. So that like the peaks will be more dominant right in deciding this parameter okay so that is the definition of the vibration dose value. So people use all these parameters to evaluate ride quality and ride comfort.

Of course we need a to measure these acceleration levels okay so I will conclude my discussion on suspensions here so we have looked at broad overview of automotive suspensions different classes components, operation and a brief analysis using this quarter car model for ride analysis okay so thanks for your attention.