Control Systems

Prof. C. S. Shankar Ram
Department of Engineering Design
Indian Institute of Technology, Madras

Lecture – 09
System Response
Part – 1

In the last lecture, we were looking at the concept of transfer function. If we have a linear ODE with constant coefficient that corresponds to a linear time invariant causal SISO system. We could apply the Laplace transform, take all initial conditions as 0 and then get the ratio of the Laplace of the output to the Laplace of the input. Once we do that the quantity that we get is called as a transfer function of the system. And we defined what are called as poles and zeros of the transfer function. We are going to discuss the solution to few exercise problems and make a few observations.

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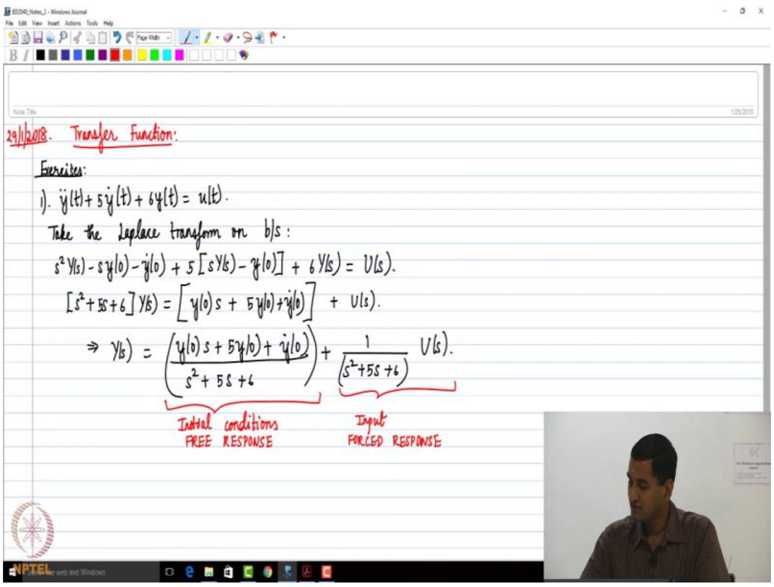

Let us take the first problem. This is the system, whose governing equation is given by

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = u(t)$$

This is a second order system. If we take the Laplace transform on both sides, we get

$$s^2Y(s) - sy(0) - \dot{y}(0) + 5[sY(s) - y(0)] + 6Y(s) = U(s)$$

We collect terms and we will get

$$[s^2 + 5s + 6]Y(s) = [sy(0) + 5y(0) + \dot{y}(0)] + U(s)$$

$$Y(s) = \frac{sy(0) + 5y(0) + \dot{y}(0)}{s^2 + 5s + 6} + \frac{U(s)}{s^2 + 5s + 6}$$

The output term has two components, the first component is due to the initial conditions, that is what we call as the free response. And the second part is due to the input that is what is called as forced response of the system. So, now when we want to get the transfer function, we take all the initial conditions as 0.

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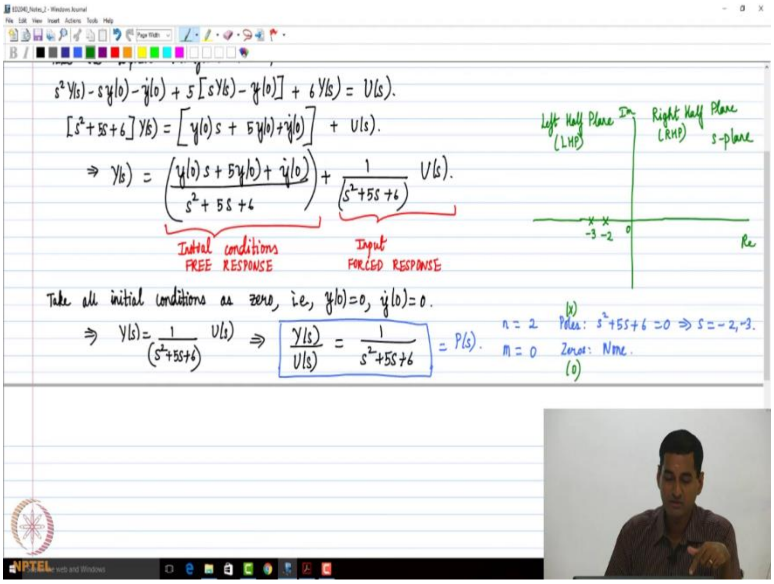

That is, in this particular problem we take $y(0) = 0$ and $\dot{y}(0) = 0$. Once we have that, the first term is going to vanish.

$$Y(s) = \frac{U(s)}{s^2 + 5s + 6}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 5s + 6} = P(s)$$

This is going to be the plant transfer function or the system transfer function $P(s)$. Here we see that the order of the denominator polynomial of the transfer function $n = 2$ and the order of the numerator polynomial of the transfer function $m = 0$. $n > m$ so we have a strictly proper transfer function. And we solve the denominator polynomial equal 0 to get the poles

$$s^2 + 5s + 6 = 0$$

The poles are -2 and -3 and there are no zeroes for this problem. And typically what we do is that we plot what is called as an s plane (the complex plane) with the real and the imaginary axis. The imaginary axis divides the complex plane into two halves called as the left half plane abbreviated as LHP and right half plane abbreviated as RHP. The poles are denoted by a cross sign (x) in the complex plane. We locate -2 and -3 on the negative real axis. The zeros are graphically denoted by a small circle (o) in the complex plane. There are no zeros in the above problem.

We know that the order of the denominator polynomial is going to be the same as the order of the system. If we look at the transfer function of the system. The order of the transfer function is 2, because the denominator polynomial is order 2.

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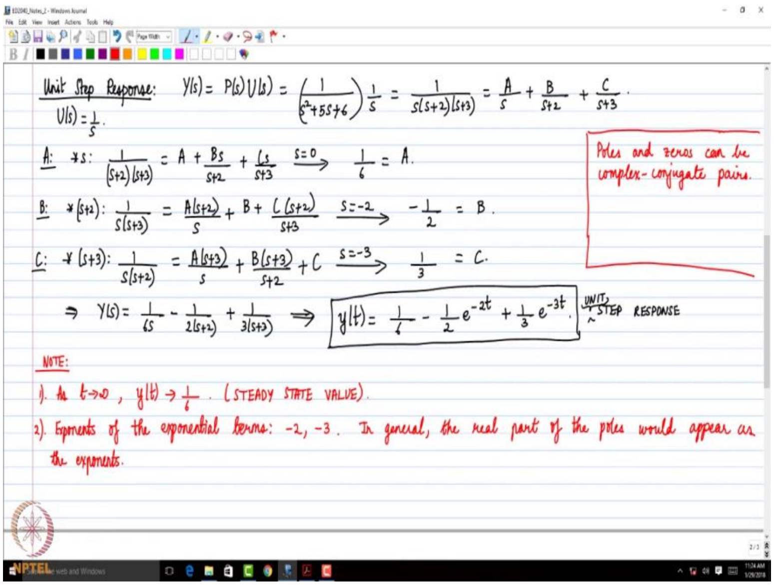

Now let us calculate the unit step response of the system.

$$Y(s) = P(s)U(s)$$

$$Y(s) = \frac{1}{s^2 + 5s + 6}U(s)$$

For a unit step response $u(t) = 1$ and $U(s) = \frac{1}{s}$.

We can write

$$Y(s) = \frac{1}{s^2 + 5s + 6}\frac{1}{s} = \frac{1}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s+3)}$$

A, *B*, *C* are called the residues. Evaluating the partial fractions, we get $A = \frac{1}{6}$, $B = -\frac{1}{2}$, $C = \frac{1}{3}$.

$$Y(s) = \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}.$$

If we take the inverse Laplace transform

$$y(t) = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}$$

This is the unit step response of the system. Suppose instead of giving $u(t) = 1$, we give $u(t) = 5$, how will this output change? We can use the property of homogeneity and say the output (unit step response) gets multiplied by 5.

Let us write down what we observe from this unit step response.

1) As, $t \to \infty, y(t) \to \frac{1}{6}$, this is called as the steady state value.
2) Initial value of $y(t)$, if we substitute to $t = 0$, we get $y(t) = 0$, because anyway we assume all the initial conditions to be 0.
3) What can we say about the exponents of the exponential solution? The exponents here are -2 and -3 which happened to be the poles of the system.

In this problem the poles happen to be real, but in general poles and zeros can be complex conjugate pairs. We can have complex numbers as poles and zeros but we need to have, the conjugate also as a corresponding pole or zero, because we are dealing with dynamic systems where the governing ordinary differential equation has real numbers as coefficients. Consequently, when we take that Laplace transform, we get a polynomial in whose coefficients are real. So, once we have a polynomial with real coefficients, we could have a complex root, but that conjugate must also be a root

The exponents of the exponential terms in general would be the real part of the poles. In this particular example -2 and -3 are real numbers, the complex part was 0. That is why we have the terms as e^{-2t} and e^{-3t}.

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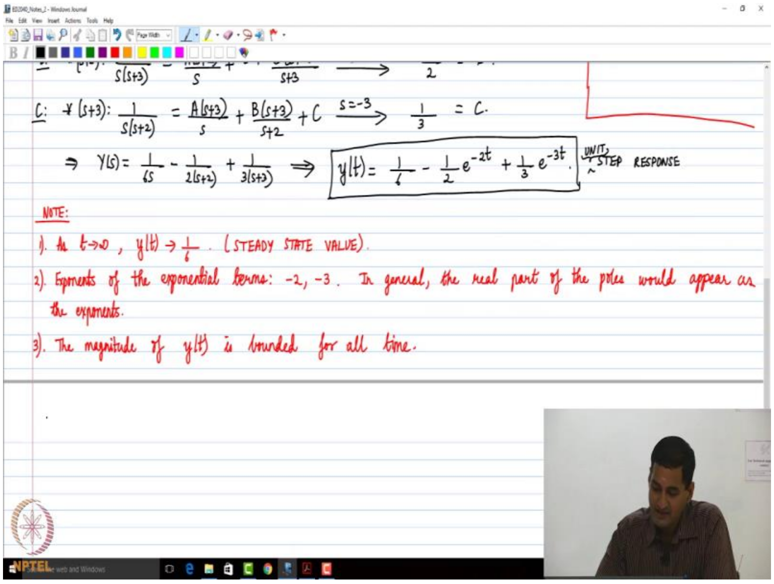

4) What about BIBO stability here? Step is a bounded input, and the corresponding $y(t)$ that we have calculated is bounded. The observation we can make here is that the system is bounded input bounded output stable.

But then it should be bounded for all possible bounded inputs. But that is a generalization which we are making here, but we will prove it more carefully later on when we look at stability.