Control Systems

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Lecture – 07
Transfer Function
Part – 1

We are dealing with single input single output systems abbreviated as SISO and linear time invariant causal dynamic systems that are characterized by linear ordinary differential equations with the constant coefficients.

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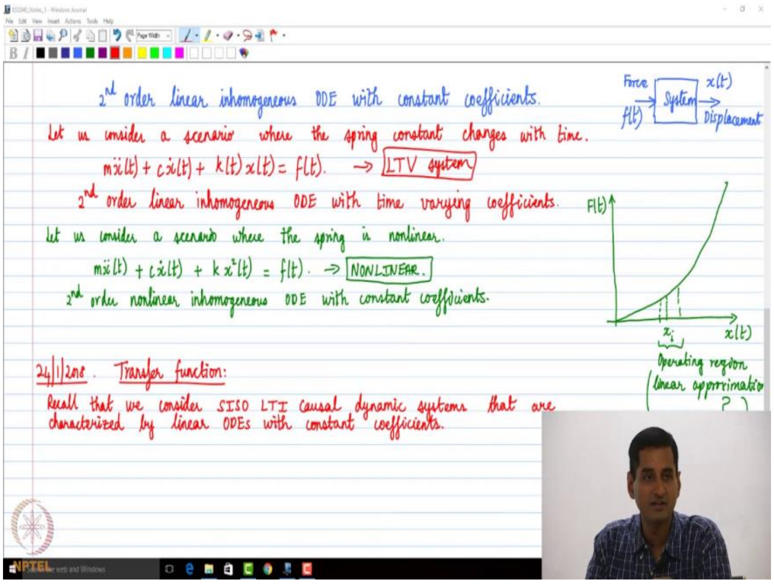

We have looked at how one could use the Laplace transform to solve such equations. What we have done till now is to have a broad overview of what we are going to study in this course and some mathematical tools that we are going to use.

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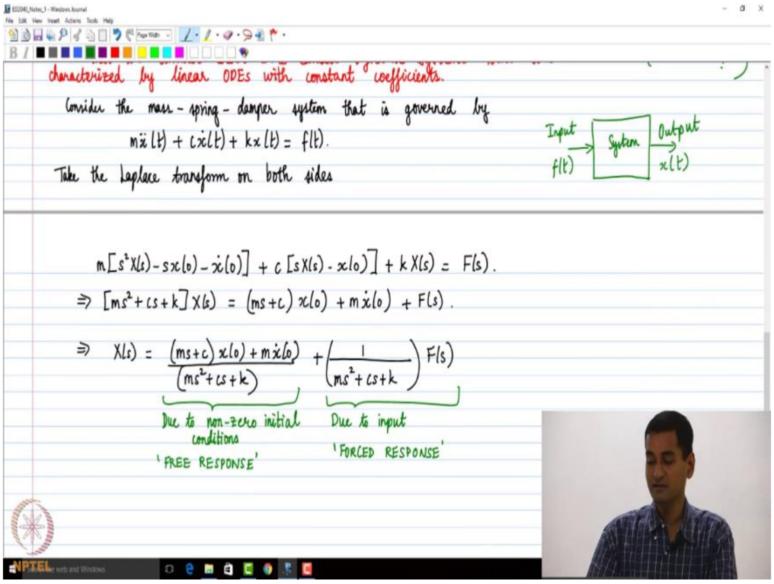

We will go into the analysis using the classical control approach. One of the main topics that we are going to learn is that of transfer functions. Let us see what a transfer function is. Let us consider the mass spring damper system that is governed by

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t)$$

This is a linear time invariant system. Let us take the Laplace transform on both sides, we are going to get

$$M[s^2X(s) - sx(0) - \dot{x}(0)] + C[sX(s) - x(0)] + KX(s) = F(s)$$

If we just rearrange the terms we are going to get

$$[Ms^2 + \mathrm{Cs} + \mathrm{K}]X(s) = [Ms + C]x(0) + M\dot{x}(0) + F(s)$$

If we now divide both sides by $Ms^2 + \mathrm{Cs} + \mathrm{K}$ we are going to get the following.

$$X(s) = \frac{[Ms + C]x(0) + M\dot{x}(0)}{[Ms^2 + \mathrm{Cs} + \mathrm{K}]} + \frac{F(s)}{[Ms^2 + \mathrm{Cs} + \mathrm{K}]}$$

The first term is the component of the output is due to the non-zero initial condition. One could immediately observe that it involves $x(0)$ and $\dot{x}(0)$. That is the displacement which is caused by any non-zero initial condition. If we have the mass initially at rest, we displace it by a small amount and leave it, it may vibrate depending on the values of

K and C and finally, it may settle down back to it is originally equilibrium state. The first part is the output due to non zero initial conditions. We call it as a free response.

The second part involves the input term $F(s)$, which is the force. Suppose we apply a time varying force on the mass, the second term provides us the displacement of the mass. That is the contribution of the input force to the output. We call it as the forced response. We are going to have these two components to the output. One due to non-zero initial conditions and other due to the input that is provided to the system.

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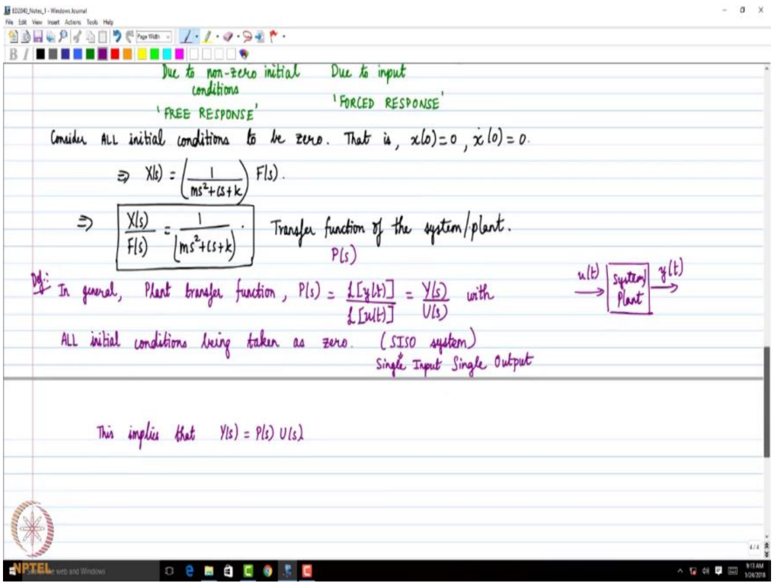

Now, let us consider all initial conditions to be 0. In this case $x(0) = 0$ and $\dot{x}(0) = 0$. Then

$$X(s) = \frac{F(s)}{[Ms^2 + \mathrm{C}s + \mathrm{K}]}$$

$$\frac{X(s)}{F(s)} = \frac{1}{[Ms^2 + \mathrm{C}s + \mathrm{K}]}$$

So, this quantity $\left(\frac{X(s)}{F(s)}\right)$ which is the ratio of the Laplace transform of the output to the Laplace transform of the input under 0 initial conditions is called as the transfer function of the system or plant. We will use $P(s)$ to denote the transfer function. To summarize, we wrote down the governing linear ordinary differential equation, then we took Laplace transform on both sides, we saw that the output term has two components one due to the non-zero initial conditions and one due to the input. We took all initial conditions to be 0. Then, we took the ratio of the Laplace of the output to the Laplace of the input, which is called as the transfer function of the system or the plant and we are going to denote by (s). In a certain sense when we use the transfer function approach we are interested in the forced response of the system because we have taken all initial conditions to be 0. Now the question becomes will initial conditions be zero all the time in practice? If not what would happen if we follow this approach? That is a question we would answer later.

Please remember that, we can take ratios of Laplace of output and input only if we have a SISO system. If we have a MIMO system, we would have a transfer function matrix. We can no longer divide a vector output by a vector input. We can still write it in another product form, where we can say the output vector is going to be equal to a transfer function matrix multiplying an input vector in the complex domain.

But we are going to deal with single input single output systems in this course. So, the notations followed holds through for a single input single output system. Consequently, $Y(s) = P(s)U(s)$.

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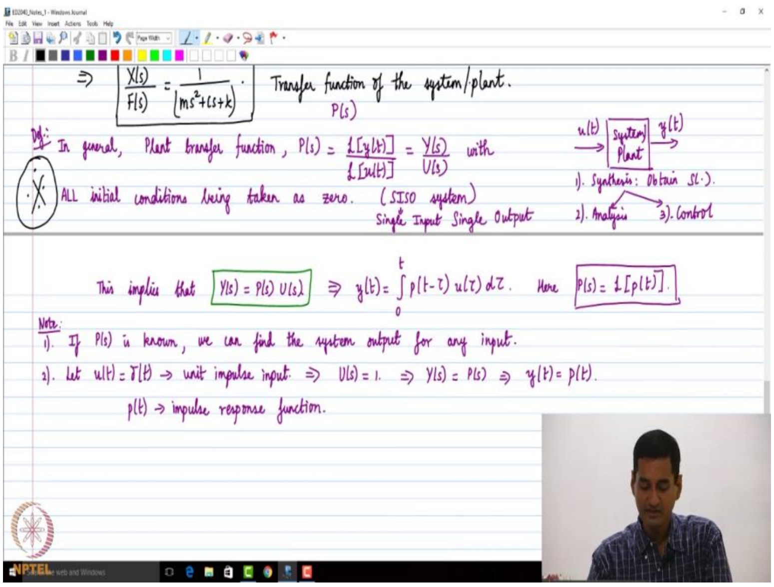

If we take the inverse Laplace transform, we have

$$y(t) = \int_0^t p(t-\tau)u(\tau)d\tau,$$

$y(t)$ is the output in the time domain if we take the inverse Laplace transform. We can note certain points. The first point is, if $P(s)$ is known, we can find the system output for any input. Another thing we can do is, if we want some output we can find out what input to give. We can do both control and analysis, what we studied as purpose of models. Prediction or analysis is where we have an input and we evaluate the output. Control is where we have a desired output and we find the input to be given. The second point is, if $u(t)$ is the unit impulse input, then $U(s) = 1$. Therefore $Y(s) = P(s)$, this implies that $y(t) = p(t)$

If you give any input and you calculate the corresponding output, that output is called as a response and the adjective can be added which corresponds to that specific input. In this particular case we are providing a unit impulse input to the system. So, consequently the output that we get is called as the impulse response and $p(t)$ is the impulse response function. And the Laplace of the impulse response function $p(t)$ is the plant transfer function $P(s)$.

This gives us a very useful idea, if we want to experimentally determine the plant transfer function, we just need to give a unit impulse input and then measure what is the output and that will be the impulse response function. From the impulse response function, we can calculate the plant transfer function. The advantage is we need to do only one experiment, but the catch here is, in practice it is very difficult to give an ideal impulse input. Then the question becomes, what approximations can we get? Is there any other method to figure out the transfer functions? We will learn as we progress. There is another method by which we can experimentally determine transfer functions by using frequency response, ok. So, we would learn that technique also. At the end of the day, if we want to figure out a plant transfer function it is going to be a mix of both approaches. For example, it is going to be a mix of deriving equations based on physics and also performing experiments and using experimental data to fit the transfer function, we need both approaches.

As far as the definition of the transfer function is concerned it is extremely important that we keep all initial conditions to be 0, then only we can talk about transfer functions and use them.