Control Systems
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Lecture – 65
Relative Stability 2
Part – 1

Let us get started with today's class; you know like we were if you recall we were discussing relative stability.

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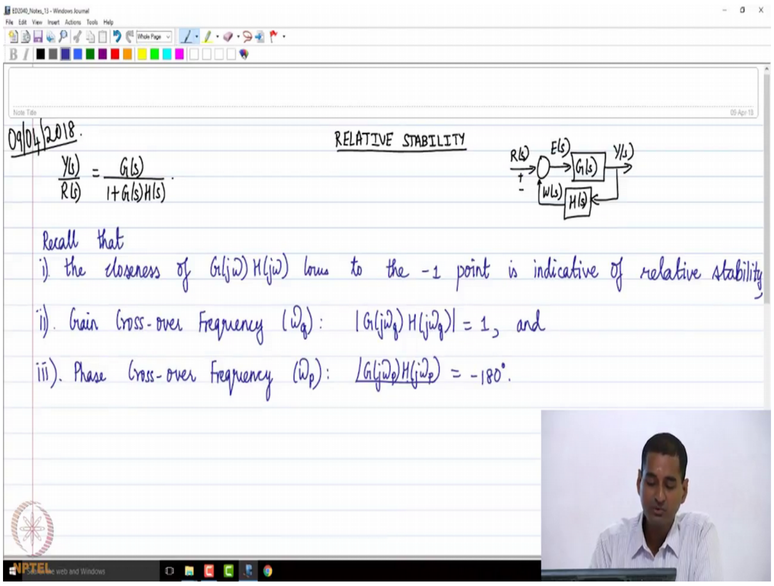

Please recall that this is the closed loop transfer function of the class of systems that we are considering right G of s divided by 1 plus G of s H of s right. And in the previous class, we discussed what was relative stability relative the notion of relative stability has to do with the fact that how robust my system is you know like as far as the location of its poles are concerned right.

So, that is if I perturb the system; if there are some un modeled dynamics or like some parametric variations, you know like how would the poles there are closest to the imaginary axis change. And is there a chance that you know the poles may reach closer and closer to the imaginary axis and perhaps crossover right at some point or other.

So, that is a notion that motivates the concept of relative stability; kindly recall that the presence of all poles in the left half complex plane implies what is called as absolute stability what we have been calling as stability is absolute stability. Now we are adding an objective before that right we are essentially learning what is called as a relative stability right. And we figured out in the last class that the closeness of the G of j omega H of j omega locus to the minus 1 point in the G of s H of s plane is indicate over relative stability right.

So, because when does a stable system become unstable when its poles crossover right; when it when they cross over S is of the form j omega right and our close to characteristic equation is 1 plus G of s H of s. So, when H S is of the form j omega; we get 1 plus G of j omega H of j omega equals 0; that means, that G of j omega, H of j omega equals minus 1 when the poles cross over.

That is why you know like how close is my plot of G of j omega H of j omega to the minus 1 point in the G of s H of s plane; that is indicate our relative stability right. So, in that sense you know like we essentially define 2 terms what was called as a gain crossover frequency and the phase crossover frequency. The gain crossover frequency is the frequency at which the open loop transfer functions; open loop sinusoidal transfer function magnitude becomes 1 or in other words the magnitude of G of j omega g, H of j omega g is equal to 1 right.

The phase crossover frequency omega p is the value of omega at which the phase of the open loop transfer sinusoidal transfer function becomes minus 180 degrees. And why are we worried about 1 and minus 180? You know knowing because that characterizes the minus 1 point right in the complex plane right. So, that is why we are focusing on that point. So, let me build on this is where we stopped in the previous class and we did a numerical example right to calculate omega g and omega p.

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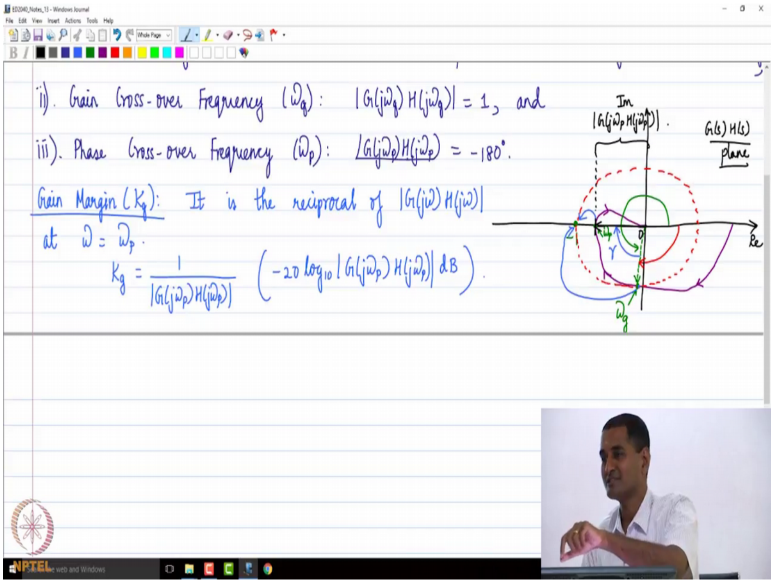

So, let us build on this. So, suppose let us say you know like I draw the G of s H of s plane ok. Now what I do is that of course, we are we are looking at the minus 1 point; let us say this is my minus 1 point right. So, and let us say this is the origin so; obviously, the minus 1 point has a magnitude of 1 and phase of minus 180 right. Suppose let us say you know like just for the sake of argument right; let us say you know like my locus of G of j omega, H of j omega is something like this ok.

So, some open loop transfer function whose locus is this way right. So, now the point is how or let me redraw it a little bit; no so sorry about that. So, let us say you know I just take some I am just taking some say a transfer function which essentially goes like this right; open loop transfer function. Now; obviously, the locus does not pass over the minus 1 point right; so, at least to begin with. Now the question is that how can I graphically locate omega g and omega p in this particular what to say a plot? This is the Nyquist plot right of this particular open loop transfer function; that is what I have drawn right.

So, I am just taking some arbitrary curve right to just convey the concept. So, now how can I locate omega g and omega p in this diagram? So, let us say you know like I draw a circle of unit magnitude right. So, sorry oops sorry I think I raise the whole thing right; so, let us just let us say you know like I have some transfer function you know like which essentially goes like this and let us say you know I am just drawing some arbitrary shape right.

So, let us say you know like I draw a circle of unit radius right and then figure out where this circle cuts the; what to say the Nyquist plot right that is my omega?

Student: (Refer Time: 05:54).

g; the gain crossover frequency ok. Now let us say at omega equals omega g; you know like I have a phase which is equal to this number or a negative phase which is equal to this number ok.

Now, let us say if I at this frequency let us say I want to make the system and tend to instability what will I do? I would just try to shift this curve such that I keep on rotating it about the origin; so that this particular point which is here should come to the minus 1 point. How can I do that? If I rotate the entire Nyquist plot; let us say by this angle gamma right. Do you agree that this point will travel on this red dotted circle of unit radius and then come and settle down at minus 1; do you agree? Does anyone agree? Ok please remember that one.

Similarly whereas omega p; where it becomes minus 180; so, in this example it is let us say this is omega p. Now let us say I want to make this point come to minus 1; what should I do? I should just scale the curve right by some number ok; in this case I am multiplying, in a general sense I should scale the curve right. So, that the phase does not get changed; I know like I am just moving the magnitudes, I am just changing the magnitude along right. So, that is what I need to do right.

So, this leads to two definitions ok. So, that is what I am going to what to say right down. So, we will refer to the this figure once again; so, the gain margin K subscript ok. So, let me write down the definition then we will figure out why it is defined the way it is ok. It is the reciprocal of the magnitude of the open loop sinusoidal transfer function G of j omega, H of j omega at omega equals omega p right. Why? Because what is this? I want to say magnitude from 0 to this point at omega p; that is a magnitude of the open loop transfer function. Now if I take the reciprocal and scale the entire open loop transfer function by the reciprocal; what will happen to this point?

Student: (Refer Time: 08:52).

It will pass through the minus 1 point right; that is the motivation behind the definition of the gain margin; is it clear why gain margin is defined like this? So, the gain margin the expression for the gain margin K subscript g is going to be 1 divided by G of j omega p, H of j omega p that is how we write ok.

So, in other words this is also indicated as minus 20 log of the magnitude of G of j omega p, H of j omega p in decibels ok; in decibel units this is the expression ok. So, you can write it in the absolute units or the decibel units is it clear; how we define gain margin? So, why am I defining gain margin like this? Because you see that if I draw a vector from the origin to the point omega p, the length of this vector ok.

So, if I take this length; so, let us say you know like I take this length that is going to be the magnitude of? G of j omega p, H of j omega p; Now let us say I i calculate a number which is the reciprocal of this number and I multiply the entire open loop transfer function with that number, what is going to happen? This entire curve is going to be scaled and this point at omega p will now go to minus 1; that means, that the system is being pushed to the verge of instability right? That is why it is called gain margin right, how much margin do I have before the system can become unstable and why is it called gain margin?

Because see sometimes the scalar parameters which come in the transfer function right can have some uncertainty right. So, then the question is that how much tolerance can I have before I push it? And even the poles and 0s also affect the gain the steady state gain of the system as we already seen right. So, then the question becomes you know like how would their locations also influence the changes in the gain of the system right. So, that is what we are essentially interested in ok. So, that is the definition of the gain margin is it clear yeah?

Student: (Refer Time: 11:32) omega equal to omega p?

p yes exactly yeah; so it is a reverse arrangement right. So, the gain crossover frequency is not used to define the gain margin, the phase crossover frequency is used to define the gain margin. We will shortly see that the gain crossover frequency is going to be used to define the phase margin ok.

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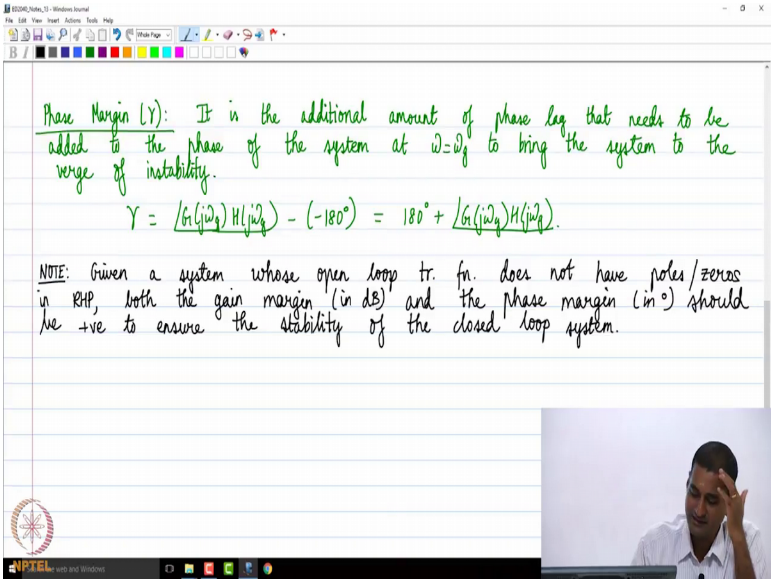

So, the next parameter which we are going to discuss is what is called as a phase margin gamma which I have just indicated that ok. The phase margin gamma; so what is the phase margin? Phase margin is the additional amount of phase lag that needs to be added to the phase of the system at omega equals omega g to bring the system to the verge of instability ok; that is the phase margin ok.

So, as we discussed this omega g right; so, the magnitude of the open loop transfer function at omega g is 1. Suppose imagine that you rotate the entire what to say Nyquist plot by this angle gamma right. So, then what is going to happen? This point is going to come to the minus 1 point; so, once again we are going to have the system being pushed to the verge of instability right; so, that is the phase margin. So, how do we calculate this? The expression for gamma is this way.

So, essentially what we do is; we calculate the phase of the open loop transfer function, add the gain crossover frequency; please remember that and subtract it from minus of 180 right because typically we deal with proper transfer functions. So, the phase of the transfer function is going to be by and large in on like. So, essentially we are even we deal with strictly open loop transfer function, the phase of the open loop transfer function by and large is going to be negative, it can be positive in some instances; we will we will see when we want to add phase to the system, but by and large I am saying ok.

So, the first number is going to be by and large negative; so, then you subtract it from minus 180 ok. So, the equation for this is going to be a 180 degrees plus the phase of the open loop transfer function, add the gain crossover frequency ok; this is the formula of a calculating the phase margin of the system is it clear ok? So, that is the expression for the phase margin yeah.

Student: (Refer Time: 14:53) gain margin.

Gain margin yeah.

Student: (Refer Time: 14:59).

Yeah, it is just that what to say that is that is essentially what you need to do is that at the phase crossover frequency, you want to push the curve so that it goes and sits at the minus 1 point right. So, the question is that how much should you scale it by? You should scale it by the reciprocal of the magnitude at that particular frequency right; the phase cause a frequency. This is what theory or rotating that is scaling you here you are essentially rotating the thing right.

So, yeah ok; so, that is what we are doing is it clear ok? Now typically now this is this is a note ok. So, for given a system whose open loop transfer function does not have poles zeros in RHP, both the gain margin in decibel units right and the phase margin in degrees should be positive to ensure the stability of the closed loop system this is result ok. So, I am just conveying this to you all right; so yeah.

So, you one may typically wonder you know like why we not want the gain margin and the phase margin to be positive this is the result right. So, that is if you want to ensure closed loop stability for a certain class of systems; you know like which do not have a poles open loop poles and open loop zeros in the RHP right, you essentially make sure that the gain margin and the phase margin gain margin decibel ok.

And phase margin in degrees need to be positive because gain margin can also be less than 1 it should be a positive scalar right. Because you can have its a reciprocal of the magnitude of the open loop the sinusoidal transfer function at omega p. Suppose let us say the magnitude of G of j omega p, H of j omega p is 10; what will happen to the reciprocal? It will become point 1 right.

So, then we need to be careful right; so, that is like we can just have what to say the take the absolute units. So, essentially that is why we take the gain margin and decibel units to be positive ok. Now, the; the another interpretation of the gain margin and the phase margin using the Bode diagram is this following ok.

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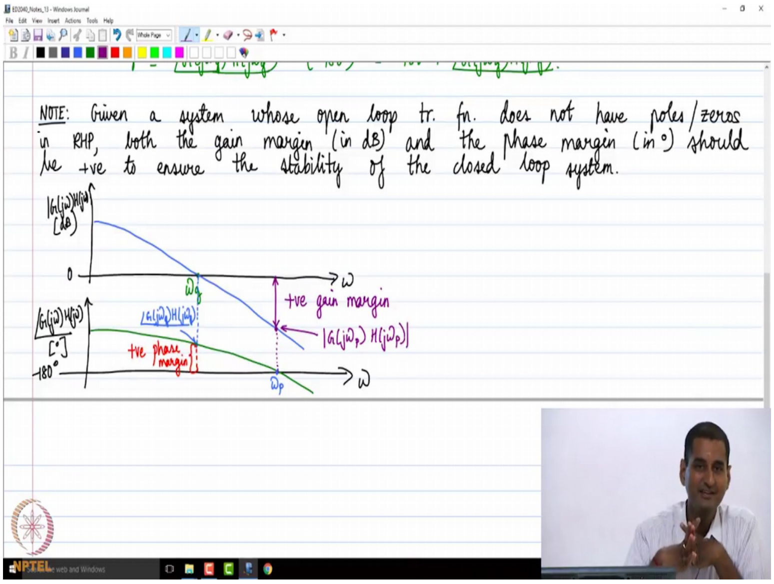

So, let us say I draw the Bode diagram of the; open loop transfer function right. So, the magnitude plot and the phase plot; so let us say I am drawing the Bode plot of the open loop transfer function ok. So, let us say this is the 0 decibel value for the Bode plot and let us say this horizontal line represents minus 180 ok.

So, now if I draw the Bode plot the magnitude plot and the what to say phase plot in this manner; where do you think is the gain crossover frequency and the phase crossover frequency? Let us say this is the Bode plot or the open loop transfer function right. So, where is the gain crossover frequency see because we interpreted gain crossover frequency phase crossover frequency, gain margin and phase margin in the Nyquist plot; let us do the same thing in the Bode plot. Suppose, let us say I give you the Bode plot you know you should be able to figure these things out right.

So, what is the definition of the gain crossover frequency? It is the frequency at which the magnitude of the open loop transfer function becomes 1 or 0 decibels. So, where do you think the gain cross or frequency will be?

Student: (Refer Time: 20:03).

Yeah. So, wherever the curve cuts the 0 decibel line; in this case since I have taken this value as the 0 decibel; so to be to essentially graphically be consistent. So, that is what that is where omega g will be. Now what about omega p? Place where the phase becomes minus 180 degrees, where will it become minus 180? Here ok.

Now, what how can we get the gain margin and the phase margin? Now if I extrapolate this right; this is going to be what the phase of the open loop transfer function and the gain margin right. So, then what will be the sorry the gain crossover frequency; now what will be the phase margin, from this angle you need to subtract minus 180.

What does it mean? That means, that this is the phase margin; so, this is a positive phase margin ok. So, if you get a Bode plot like this is the phase margin because like you are getting the difference between the phase at which are the gain crossover frequency and minus 180. Since the gain crossover frequency phase is less negative than minus 180, you are going to have a positive phase margin.

Now, how will I get the what to say gain margin? Or the phase crossover frequency I go to the?

Student: (Refer Time: 21:46).

Gain plot right; the magnitude plot, now what is this? This is the magnitude of the open loop transfer function and the phase crossover frequency. So, the gain margin is going to be the reciprocal of that or in decibel units it is going to be the?

Student: Negative (Refer Time: 22:09).

Negative of that right; so this is going to be the gain margin, is this going to be a positive gain margin or a negative gain margin?

Student: Negative (Refer Time: 22:22).

It is going to be positive right because this number is negative. So, you want to take the negative of the negative number right do you agree? Yes. So, this is going to be a positive gain margin ok; if it were the other way around right, it would be a negative gain margin right. Because of the phase crossover frequency the magnitude of the open loop sinusoidal transfer function is negative in terms of decibels.

In this example I just drew some arbitrary curve right. So, you need to take the negative of that number right; so which happens to be positive. So, you have a positive gain margin if the magnitude in the if in the magnitude plot; the magnitude of the open loop transfer function the phase crossover frequency is below the 0 decibel line; that is a positive gain margin right. You have a positive phase margin if in the phase plot; the phase of the open loop sinusoidal transfer function or the gain crossover frequency is above the minus 180 line right its negative otherwise right. So, that is how we interpret gain margin and phase margin is it clear ok; now let us go back to the example that we did right.

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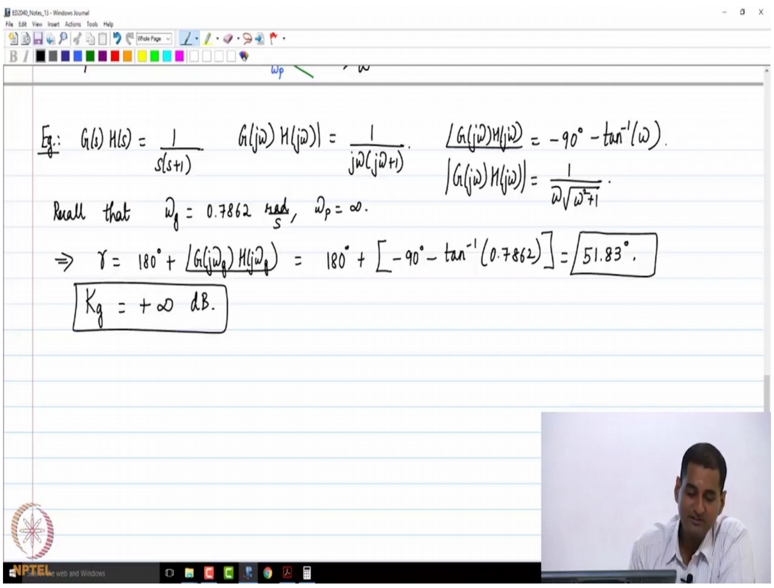

So, this is what we consider right. So, consequently the sinusoidal open loop transfer function is going to be equal to 1 divided by j omega times j omega plus 1 right; now if you recall what were omega g and omega p? We had already calculated right recall that omega g was? What was the value of omega g? Correct? Yes or no?

Student: Yes.

I think we did this example in the previous class right that is where we stopped right. So, if I recall let me go back yeah.

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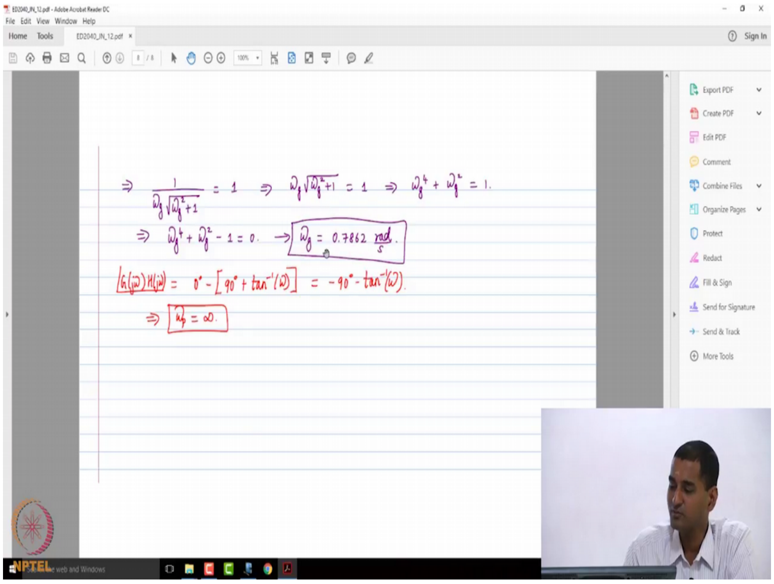

I think this is where we stopped right we did this example; we calculated omega g as 0.7862 radians per second, omega p was? Infinity right ok; Now, what does this imply gamma; the gain margin gamma is going to be equal to 180 degrees plus the phase of the open loop transfer function are the gain crossover frequency right which is this. So, what is the phase of G of j omega H of j omega; the gain crossover frequency? See in general the phase of the open loop sinusoidal transfer function is going to be from this result, it is going to be minus 90 minus tan inverse omega right by looking at this function right. So, you just plug in omega equals omega g.

So, you are going to get tan inverse 0.7862 right. So, can you calculate and tell me what you get? So, you will get around 51.8 3 degrees right; so that is the gain margin sorry phase margin for this particular transfer function. So, what about the gain margin? See gain margin is the negative of the, gain of the open loop transfer function at omega equals omega p if you go look at the measure in decibels. Now as frequency tends to infinity what do you think will happen to the magnitude plot of this particular transfer function? What are the values of n and m? n is 2, m is 0. So, what is the slope of the transfer function? Sorry magnitude plot?

Student: (Refer Time: 27:12).

Minus 20 times n minus m which is going to be minus 40 decibels per decade. So, you have a magnitude plot which is going to go like minus 40 decibels per decade at high frequency. So, at omega as omega tends to infinity; what will be the magnitude of the magnitude sorry the what is the value of the magnitude of this sinusoidal transfer function in decibels? It will tend to minus infinity right do you agree?

So, then gain margin is in decibel measure it is a negative of that right. So, then what will happen? It is going to be correct. So, you can also look at it this way right; so, the magnitude of the open loop transfer function is going to be essentially 1 divided by omega times omega squared plus 1 alright, this also give the right. So, essentially we see that data has infinite gain margin in the sense that like it is very tolerant to disturbances in the gain ok.

So, that is what happens with this particular factor right; you need to really essentially push the gain to in that is you need to scale it up by infinity to even make the system to verge of instability at the phase crossover frequency ok. So, that is going to be what to say not essentially feasible right practically. So, that is the that is the point here because the phase never reaches; reaches what to say minus 180 only asymptotically that is the that is the point here right ok. So, that is the calculation of the gain margin and the phase margin is it clear? Ok.