

Control Systems
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Lecture – 61
Nyquist Stability Criterion
Part – 1

So, let us get back to where we were previously, you know like we just started off with the Nyquist Stability Criteria. So, what is this Nyquist stability criteria? The question is you know like we are essentially trying to figure out what to say information about the stability of the closed loop system by looking at the frequency response of the open loop system. You know that that is what essentially the Nyquist stability criteria boils down too. I think in the last class, I wrote all these things around, and just restarting from the same place right, and I left you with a question right, so that is what we are trying to address today ok.

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NYQUIST STABILITY CRITERION

Recall that $\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

Closed Loop Characteristic Polynomial = $1+G(s)H(s)$

Open loop transfer function, $G(s)H(s) = \frac{n_0(s)}{d_0(s)}$

$\Rightarrow 1+G(s)H(s) = 1 + \frac{n_0(s)}{d_0(s)} = \frac{d_0(s) + n_0(s)}{d_0(s)}$

The block diagram shows a negative feedback system with input $R(s)$, error signal $E(s)$, forward path $G(s)$, output $Y(s)$, and feedback path $H(s)$.

So, let us consider the standard negative feedback system ok. G of s being the forward path transfer function, H of s being the feedback path transfer function. G of s H of s being the open loop transfer function. And the closed loop transfer function is going to be Y of s divided by the R of s , which is going to be G of s divided by 1 plus G of s H of s right. So, consequently the closed loop characteristic polynomial is going to be 1 plus

G of s H of s . And why is this important, the roots of this closed loop characteristic polynomial are the closed loop poles right. So, the open loop transfer function G of s H of s , let us consider it to be represented as n_o of s divided by d_o of s right that is essentially saying that the open loop transfer function is a ratio of two polynomials in s right.

So, consequently I can rewrite this a closed loop characteristic polynomial 1 plus G of s H of s as 1 plus n_o of s divided by d_o of s , which is d_o of s plus n_o of s divided by d_o of s ok, so that is what we have. And we are particularly interested in the numerator. Why, because the roots of the numerator polynomial are the closed loop poles right.

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Open loop transfer function, $G(s)H(s) = \frac{n_o(s)}{d_o(s)}$.

$\Rightarrow 1 + G(s)H(s) = 1 + \frac{n_o(s)}{d_o(s)} = \frac{d_o(s) + n_o(s)}{d_o(s)}$.

Zeros of $1 + G(s)H(s) \rightarrow$ Closed loop poles.

Poles of $1 + G(s)H(s) \rightarrow$ Open loop poles.

MAPPING THEOREM:

So, please recall, the zeros of 1 plus G of s H of s are the closed loop poles right, so that is that is important for us right. So, but the poles of 1 plus G of s H of s are the open loop poles right. So, why, because if you look at the denominator of the polynomial 1 plus G of s H of s is d_o of s right, d_o of s is nothing but the denominator polynomial of the open loop transfer function. So, consequently you will see that this is going at the poles of 1 plus G of s H of s are going to be the open loop poles ok, so that is what we have right, so that is something, which we already identified.

Now, the question that we are asking ourselves the place, where we stopped last class was it. Can I comment on the location of the closed loop poles, which in turn is a saying

that ok. Can I comment on the stability of the closed loop system by plotting the frequency response of the open loop system ok, or the open loop transfer function right.

So, question is can I plot $G(j\omega)H(j\omega)$ has a Nyquist plot right, the Nyquist plot of that. And then from that comment on the stability of the corresponding closed loop system ok. That is what that is the question, we have asking ourselves right. So, in order to answer that question ok, the first concept we are going to look at is what is called as a mapping theorem. So, let us look at this theorem ok, so what does this theorem tell, and how are we going to utilize this theorem for our purpose ok, so that is something, which we do.

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All the roots of the characteristic equation $1+G(s)H(s)=0$ need to have negative real parts for the closed loop system to be BIBO stable. The Nyquist stability criterion relates the behavior of the open loop frequency response transfer function $G(j\omega)H(j\omega)$, to the number of zeros and poles of $1+G(s)H(s)$ that lie in the right half s -plane. The advantage is that we can determine the location of the closed loop poles by analyzing the Nyquist plot of the open loop transfer function. We assume that the system is causal such that $\lim_{s \rightarrow \infty} G(s)H(s)$ is either zero or a non-zero constant.

In this context, we shall use the following theorem (Mapping Theorem):

Let $F(s)$ be a ratio of two polynomials in s . Let a closed contour in the s -plane contain Z zeros and P poles of $F(s)$ without passing over any of the poles and zeros of $F(s)$. Now, this closed contour in the s -plane is mapped into a closed contour in the $F(s)$ plane. Then, the total number of clockwise encirclements (N) of the origin by the contour in the $F(s)$ plane is equal to $(Z-P)$.

For example, let us consider

$$F(s) = 1 + G(s)H(s) = 1 + \frac{6}{(s+1)(s+2)} = \frac{(s+1.5+2.4j)(s+1.5-2.4j)}{(s+1)(s+2)}$$

Then, the profiles of $F(s)$ that are obtained by considering various s

So, let us go to the handout, you know like which I already shared on the course website right. So, if you look at this mapping theorem right, so what are we saying. So, let me essentially read it from here. So, we are saying let F of s be a ratio of two polynomials in s ok. So, I am discussing this mapping theorem that is what we are discussing right. So, let F of s be any polynomial a complex valued function right, which is in turn a ratio of two polynomials in s ok, it has a numerator, and a denominator ok.

Let a closed contour in the s -plane contain is a zeros, and P poles of F of s without passing over any of the poles and zeros of F of s ok, so that is what we are considering. So, what I am doing, I am considering a closed contour in a in the s -plane ok, that means

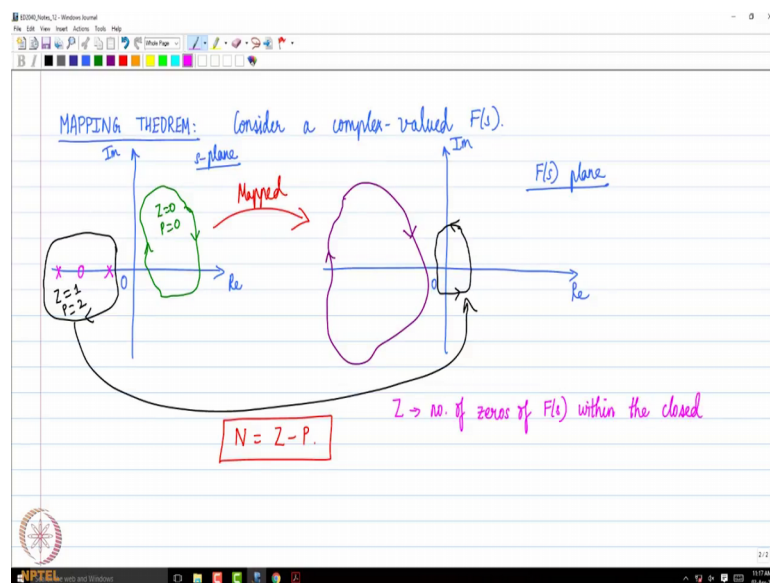
that any what to say shape any arbitrary shape that essentially is a contour in the s-plane, but does not pass over any of the poles or zeros of this function capital F of s.

Then this closed contour in the s-plane is mapped to a corresponding closed contour in the F of s plane, because see if you have a path in the s-plane of, obviously we are going to have what to say at each and every point in the contour a particular value of s that value of s you can substitute in F of s, obviously you will get a corresponding value in the F of s plane. So, you can map it to the F of s plane right.

So, once we do that what is going to happen, what is going to happen is then we are going to get a corresponding contour in the F of s plane ok. Then what we do is in the we count the number of clockwise encirclements of the origin by the contour in the F of s plane ok. We will see how to close clock on these encircles ok. What is meant by an encirclement, and how do we count them ok.

So, now, if I count the number of clockwise encirclements of the origin, by the contour in the F of s plane that is going to be equal to Z minus P ok. What pair is Z and P, Z was the number of zeros of F of s within the contour in the s plane ok, P was the number of poles of the function F of s, which was contained within the contour in the s-plane ok. So, Z minus P would be the total number of clockwise encirclements of the origin in the F of s plane of the corresponding closed contour right.

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So, let me illustrate what this is ok. So, we have read the statement, so let us say we see what it means right. So, we have a function, so let us say consider a complex valued function F of s right, so that is what we are essentially doing.

So, what we are saying in the mapping theorem is it. First you give me the s -plane I take the s -plane ok, and I take any contour let us say any arbitrary contour in the s -plane ok. This is a closed contour in the s -plane is it not right. So, obviously, the s -plane you need to mark the number of poles and zeros of F of s right that is what we have been doing till now right.

So, what this theorem says is that. So, this is going to be mapped into a corresponding contour in the F of s plane ok. So, let us say you know like some arbitrary contour you know, let us say for as the sake of argument, let us say F of s as let us say a two poles here right, and let us say a 0 here ok, so if I take this green contour you know, which arbitrary drawn right. So, what are the values of Z and P , Z should be the number of zeros of F of s in the within the contour right, P should be the number of poles of F of s within the contour. So, in this particular example Z and P are both 0 right.

So, what is going to be the number of encirclements of the origin, it is going to be 0, because Z minus P right. So, I will have some arbitrary contour in the F of s plane, which may be like this you know some something like this. I am just drawing an arbitrary contour ok, so which essentially gets mapped this way ok. This s contour gets mapped to this F of s plane, and it will be this way ok.

Now, on the other hand, let us say I take a contour you know like which essentially is like this ok. So, what happens here? So, in this case, Z was 0 and P was 0 rights. In this case, what happens, what is what are Z and P , Z is 1, P is 2 right within the contour. So, what is Z minus P minus 1?

So, what does a negative value mean that means that see what was N , N was a number of clockwise encirclements of the origin, and negative value means you know it is anti-clockwise or counterclockwise encirclement that is what it is right. So, this black contour may be mapped to something like this. I am just drawing an arbitrary shape without. So, you see that if you stand at the origin, it is encircling the origin, ones in the counterclockwise direction that may be the mapping here ok.

So, what this mapping theorem essentially says if you write the mathematical statement, it essentially says N is equal to Z minus P ok, so that is the equation ok, the mathematical statement of this mapping theorem ok. It essentially says it look you know like if I have a closed contour in the s -plane, which does not pass over any of the poles or zeros of F of s an arbitrary function F of s .

Then that contour in the s -plane gets mapped to a corresponding closed contour in the F of s plane, and the number of encirclements of that contour in the of s plane in the clockwise direction N is going to be Z minus P . And what is Z ? Z is the number of zeros of F of s within the closed contour in the s -plane, P is the number of poles within the closed contour and the s -plane ok, so that is what this mapping theorem is ok. N is equal to Z minus P ok. So, please remember that yes.

Student: (Refer Time: 11:15) initial contour that you had correction during (Refer Time: 11:19).

Yeah. So, typically you would take a clockwise contour ok. So, yeah it can get mapped to a clockwise or counterclockwise control like a depending on the function right.

Student: Even that Z (Refer Time: 11:30) that is means that (Refer Time: 11:34).

Z minus P is 0, that means that number of clockwise encirclements or the origin is F of F of s plane is 0 that is what it is right, so that is what happens right.

Student: (Refer Time: 11:48) circle.

Yeah you can have.

Student: How you (Refer Time: 11:50).

Yeah yeah we we will we are going to look at examples exactly ok. Yes.

Student: What is capital (Refer Time: 11:56)?

(Refer Slide Time: 12:30)

$N_c = Z - P$

$Z \rightarrow$ no. of zeros of $F(s)$ within the closed contour in the s -plane.
 $P \rightarrow$ no. of poles of $F(s)$ within the closed contour in the s -plane.
 $N_c \rightarrow$ no. of clockwise encirclements of the origin in the $F(s)$ plane.

Capital N capital N is the let me write down the meanings of all these terms ok. So, capital S is essentially the number of zeros of the function F of s within the closed contour in the s-plane that is Z ok. What is P? P is the number of poles of F of s that lie within the closed contour in the s-plane that is P ok. So, N is the number of clockwise encirclements of the origin in the F of s plane of the origin by the corresponding contour in the F of s plane right ok.

So, you map a close contour in the s-plane to a corresponding contour in the F of s plane, and then you get this result ok, so that is a mapping theorem. If the value of N is negative that means, it is counterclockwise, N is positive means, it is clockwise right. If N is 0, there is no encirclement ok, so that is what it means ok.

Student: (Refer Time: 13:57) related to the there also Capital N (refer Time: 14:00).

No no no that is the different ok. Good, good point Good point. You know like, so that is a good observation. So, let me just maybe used use a different N here ok, let me put N c some N contour right. So, did we used Z and P for something else right. I do not recall right ok, I hope everyone understood this question you know like, this is not the same Capital N as we wrote for s power n in type 0, type 1, type 2 system ok, this is different ok, so that is why I am just putting N c that is why good observation right. Yeah ok.

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In this context, we shall use the following theorem (Mapping Theorem):

Let $F(s)$ be a ratio of two polynomials in s . Let a closed contour in the s -plane contain Z zeros and P poles of $F(s)$ without passing over any of the poles and zeros of $F(s)$. Now, this closed contour in the s -plane is mapped into a closed contour in the $F(s)$ plane. Then, the total number of clockwise encirclements (N) of the origin by the contour in the $F(s)$ plane is equal to $(Z-P)$.


For example, let us consider

$$F(s) = 1 + G(s)H(s) = 1 + \frac{6}{(s+1)(s+2)} = \frac{(s+1.5+2.4j)(s+1.5-2.4j)}{(s+1)(s+2)}$$

Then, the profiles of $F(s)$ that are obtained by considering various contours in the s -plane are shown in Figure 1.

Application to stability analysis:


Let us consider the characteristic polynomial $1 + G(s)H(s)$. We know a closed loop system is BIBO stable if all the roots of this polynomial (i.e., its left half s -plane. Let us consider a contour that encloses the entire right



So, let us look at some examples ok, so that is what I have presented in the handout ok. In the handout, you know like, essentially what I have done is a following I have just taken this particular function for F of s oops right. So, essentially one may wonder why this example has been taken you know, I will explain to you once we discuss ok. So, let us say you know we take F of s to be this function ok, which has two zeros at minus 1.5 plus or minus 2.4 j right, and two poles at minus 1, and minus 2 right.

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The slide displays four diagrams illustrating the mapping of contours from the s -plane to the $F(s)$ plane. The top row shows contour (a) in the s -plane, which is a rectangle with vertices at -4 , -2 , 0 , and 2 on the real axis, and $j\omega$ and $-j\omega$ on the imaginary axis. The corresponding $F(s)$ plane shows the image of this contour, with points labeled A, B, C, D and A', B', C', D' . The bottom row shows contour (b) in the s -plane, which is a rectangle with vertices at -4 , -2 , 0 , and 2 on the real axis, and $j\omega$ and $-j\omega$ on the imaginary axis. The corresponding $F(s)$ plane shows the image of this contour, with points labeled A, B, C, D and A', B', C', D' .

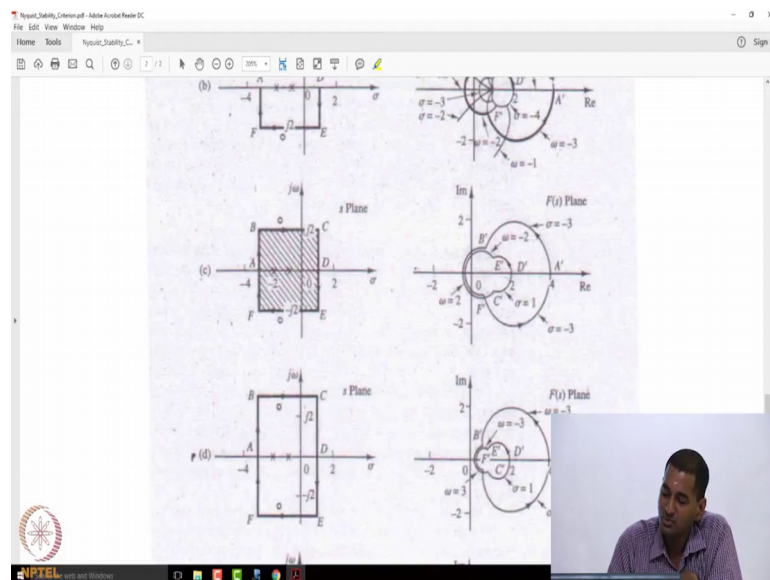


Let us look at the function F of s , in that form current t right. Now anyway of course, I had given the references as Ogata ok. So, with this example has been taken from that book right. So, if you look at the first figure right, so where can you see the two poles, you know like the two poles all right minus 2 and minus 1 right. So, we already know that that right.

And the two zeros are at are complex conjugate at these points ok. So, to begin with if you take a contour A, B, C, D , in the s -plane so happens that if you plot it, I leave it to you as an exercise ok, to do it in matlab. You take an corresponding contour, this is not a closed contour by the way ok, we are not analyzing mapping theorem yet right. So, this is an open contour, if you look at A, B, C, D it is not a closed contour.

You can see that that gets mapped to a contour A' , B' , C' , D' in the F of s plane right. So, the corresponding character of the prime is the corresponding value in the F of s plane right. And we get a different shape altogether right that depends on the function right. So, similarly if you have the contour D, E, F, A that gets mapped to a contour D' , E' , F' , A' ok, so that is how it just turns out right.

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So, now if you take a closed contour A, B, C, D, E, F, A , I hope you do not agree it that is a closed contour in the s -plane. Now, you can see that that gets mapped to the contour A' , B' , C' , D' , E' , F' back to A' ok, so that is what happens with this particular contour right. How do you get it, once again you need

to plug in and generate the curve ok, and is there a gap between these two curves right should there be a gap, do you think you know like, when you plot C for example, when you go from D prime, E prime will it overlap with this part of this B prime. I want you to figure it as homework ok, so please do it. Then you come back and tell me what your observations are I hope my question is clear right.

So, you see two for example, arcs, you know like, which seem to have a constant gap in between right would you have a gap there or would it not be there, if it would not it if it is not there, what is the reason right. So, I think, you need to figure it ok. Please take this take an arbitrary contour, and map it right using matlab yes.

Student: How many (Refer Time: 18:12) matlab.

How do we.

Student: Representing something with rectangular (Refer Time: 18:17).

We are just taking a some arbitrary contour ok.

Student: What is contour (Refer Time: 18:21).

Contour, contour is a just a curve right, it is just a domain in the s-plane.

Student: (Refer Time: 18:28) input.

It is not an input per say, it is input to this mapping theorem right, so that is it. So, it will give you see a what is a set of values of s right put it naught.

Student: What is a (Refer Time: 18:40) a in this case.

A is a point. See A is a point. Let us say you start from $3 s c$ equals minus 3 is a right that is it ok. You just draw an arbitrary contour and give some values of.

Student: (Refer Time: 18:53) values.

Yes and then plug into F of s, you know of s already, and then generate that contour in the F of s plane yes.

Student: How is this (Refer Time: 19:02).

We are coming good point. We are going to answer that right. How is this useful, you know like that is something, which we are going to learn. How are we going to apply to our subject with this control systems that is a good question, we will come there ok. So, please bear with me for another five minutes ok, I will explain this.

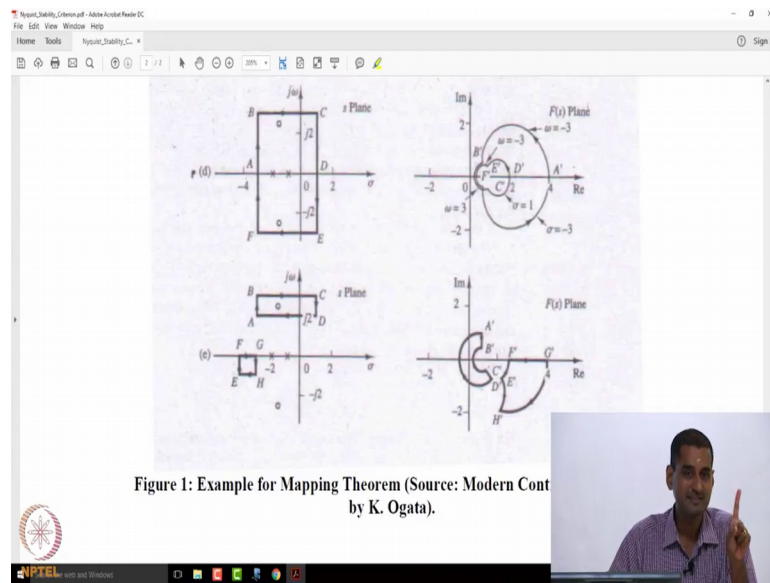
So, now within this closed contour in the s-plane how many zeros and how many poles of F of s are present. What are what are the values of Z and P , Z is 0 right, because the two zeros lie outside right, what is the value of P , two, so what is Z minus P , minus 2.

So, now, you stand at the origin in the F of s plane right, and then start counting the number of encirclements of the origin as you stand at is the origin in the F of s plane, and sweep through the entire mapped contour in the F of s plane. So, you look at it you draw a vector from the origin to A prime, which is your starting point, and then start sweeping. So, if you come to A prime, B prime, C prime, D prime, you have one encirclement right, you would have rotated three 360 degrees around the origin in the counterclockwise direction. Let us not worry about the magnitude, the magnitude changes. But, then the angle will come to 360 degrees, do you agree?

Now, you go D prime, E prime, F prime, back to A prime, you sweep another 360 degrees right in the counterclockwise directions. So, how many counterclockwise encirclements do we get? 2 in the F of s plane right. So, what should be the value of N_c minus 2, and that is what you get from the mapping theorem right.

So, the mapping theorem says that $N_{\text{subscript } c}$ should be Z minus P . So, Z here in this example was 0, P is 2. So, Z minus P becomes minus 2, and that is what you can observed from the diagram in the f of s plane ok. In figure D, if you look at it, the entire contour has been stretched a little bit, so that you enclose both the zeros within the contour right. So, now what are the values of Z and P , 2 and 2? So, number of encirclements should be Z minus P should be 0. So, you look at it, you look at the map contour in the F of s plane does it encircle the origin, no right. The map contour does not encircle direction ok, so that is what it is right.

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So, there are a few more examples in this case A, B, C, D has only one 0, you can have contours like this also right, and no poles. So, $Z - P$ will be 1 here. So, you look at A prime, B prime, C prime, D prime, it encircles origin once in the clockwise direction, because N_c will be 1 here plus 1 right. So, you can see that A prime, B prime, C prime, D prime will encircle the origin once in the clockwise direction. If you take E, F, G, H does not encircle any, it does not contain any of the zeros or poles, it just gets mapped to E prime, F prime, G prime, H prime ok, so that is what happens right yeah.

Student: (Refer Time: 22:28) if N_c equal to $Z - 3$, then instead of just one (Refer Time: 22:36) taking every every s points here in just one.

But, it would have basically encircle the origin three times that is why I am saying please do it, and convince yourself yeah. So, because we cannot do it by hand right. So, we need a this particular example you need to use matlab, or any other numerical tool to generate the contour. So, good point that is why I am saying kindly a program you know, I am sure you can you guys can do that right. So, just give a sequence of values of s right, and then take this F of s , calculate F of s plot it in matlab, then you will see how the contour comes up right.

So, of course, the finer the would say the closer the values of s , the most smoother will be the profile in the F of s plane right, which you can do numerically, and then like

convince yourself right, please do that, you experiment with different contour ok, so that is a mapping theorem.