Control Systems
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Lecture – 06
Mathematics Preliminaries
Part - 2

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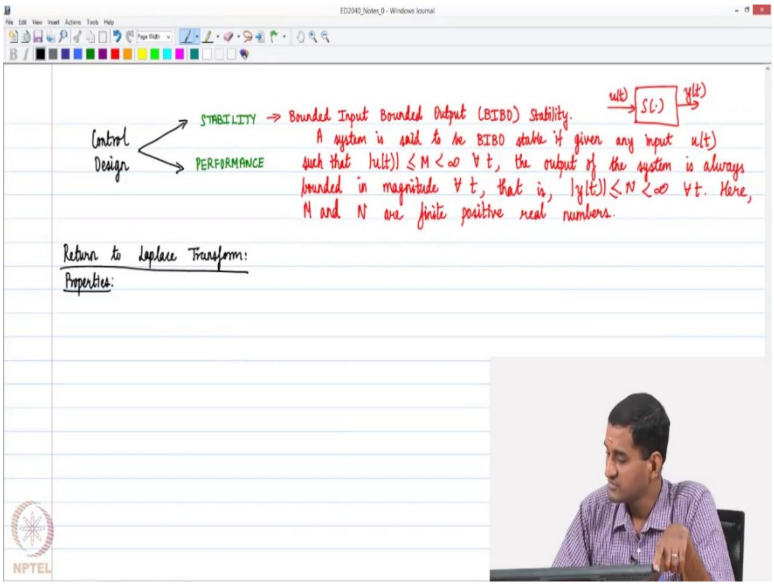

So, now let us come back to Laplace transform. So, let us return to Laplace transform. So, we learned what are the lap, what to say, the, what is the definition of the unilateral Laplace transform. So now, let us look at some properties, like which are which we are going to use. The first one is transform of derivatives.

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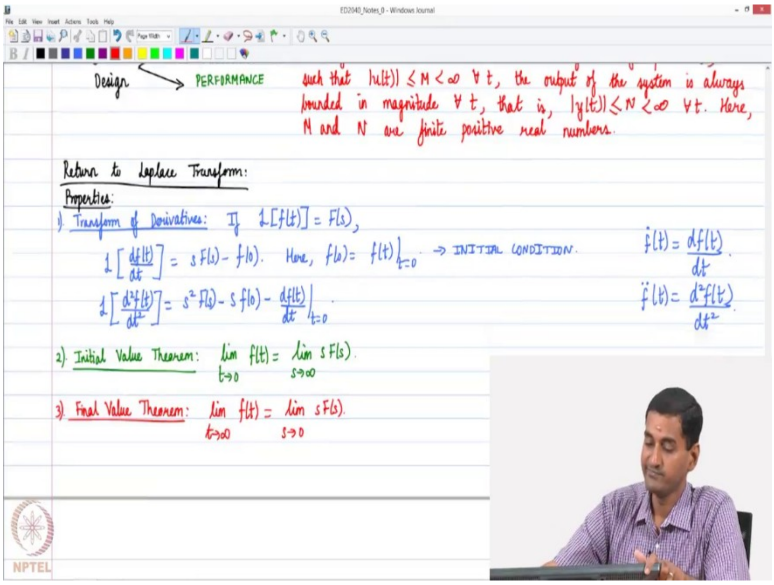

So, first let us see how derivatives of functions transform when we apply the Laplace transform. So, for example, if $F(s)$ is equal to the Laplace transform of or let me write the other way, let us say if I take a function $f(t)$, if the Laplace transform of $f(t)$ in the time domain is indicated by capital $F(s)$ in the complex domain, then the Laplace of $\frac{df}{dt}$ is going to be equal to $sF(s)-f(0)$.

So, here what is $f(0)$? $f(0)$ is nothing but $f(t)$ evaluated at t=0, that is the initial condition. Similarly the Laplace transform of the second derivative of f which is $\frac{d^2f}{dt^2}$ is going to be $s^2F(s)-sf(0)-\frac{df}{dt}(0)$. So, that is the Laplace transform of the second derivative, ok

Now, we have 2 initial conditions right f and $\acute{f}$, ok. So, as far as this particular course is concerned if we use dot over a variable, that means it is one derivative with respect to time, ok. So, that is the notation we are going to use. Similarly, if we use, if we use 2 dots it is a second derivative and so on.

So, the number of dots over a variable indicates the order of the derivative, right, with respect to time. So, you see that, as we keep on going to derivatives of higher order we are going to get as many initial condition terms, right, in the time domain. So, that is the first property, we, which we would be using. The second property is what is called as the initial value theorem.

The initial value theorem essentially says that $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$. That is what was called as an initial value theorem. And there's another theorem which is called as the final value theorem. So, what is this final value theorem? $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$. So, that is the final value theorem.

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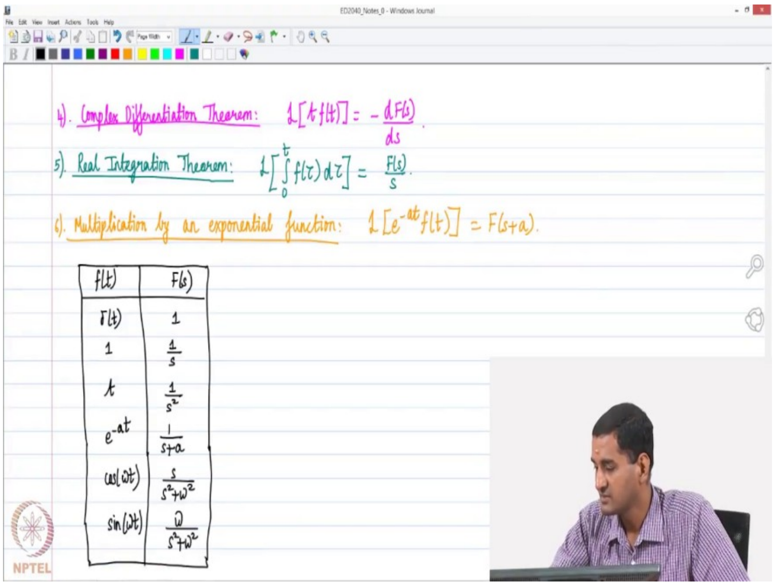

Then the 4th property that we would be using is what is called as a complex differentiation theorem. It just states that $L(tf(t)) = \frac{-dF(s)}{ds}$. that is like we take the derivative in the complex domain, right. That is why it is called as the complex differentiation theorem.

So, then there is something called as the real integration theorem. So, what is this real integration theorem? $\int_0^t f(t) = \frac{F(s)}{s}$. So in fact, like if you use tools like MATLAB and Simulink, this there is a reason why we see the block s for derivative and 1/s for integral, integrals, right. So, we if you use Simulink, this is the graphical user interfaces that we would have for a differentiator and integral block, right. So, that is because of this Laplace transform because when you take derivative we introduce s in the complex domain and we multiply by s and then when we integrate we divide by s, right. So, that is why we have these blocks.

Another theorem which we are going to use is multiplication by an exponential function. And this is, I am just pointing out what are the properties which are useful, this just says that Laplace of e power minus a t $f(t)$ it is going to be equal to $F(s+a)$. So, let us say if I multiply the function $f(t)$ by e^{-at} and take the Laplace transform of the product, the complex domain wherever I have s, I just substitute s+a, ok.

So, typically we have a table of Laplace transform, ok, like I am just going to write a table of Laplace transform. I strongly suggest that you derive this table, based on the definition of the Laplace transform that we have written down.

So, if you have the unit impulse, the Laplace of the unit impulse is one, and if you have the unit step that applies transform the unit step is $\frac{1}{s}$. If you have the unit ramp that Laplace transform the unit ramp is $\frac{1}{s^2}$. If you have an exponential function e^{-at} the Laplace transform of the exponential function is $\frac{1}{s+a}$. So, if you have $\cos(\omega t)$,

(ωt)
cos¿
¿
L¿ .

If you have $\sin(\omega t)$, Laplace transform is going to be $\frac{\omega}{s^2+\omega^2}$, all right. So, that is what we are going to have. So, this is just a preliminary table of Laplace transform. Of course, we can add more functions as we go along, but I am just listing whatever we are going to use very frequently.

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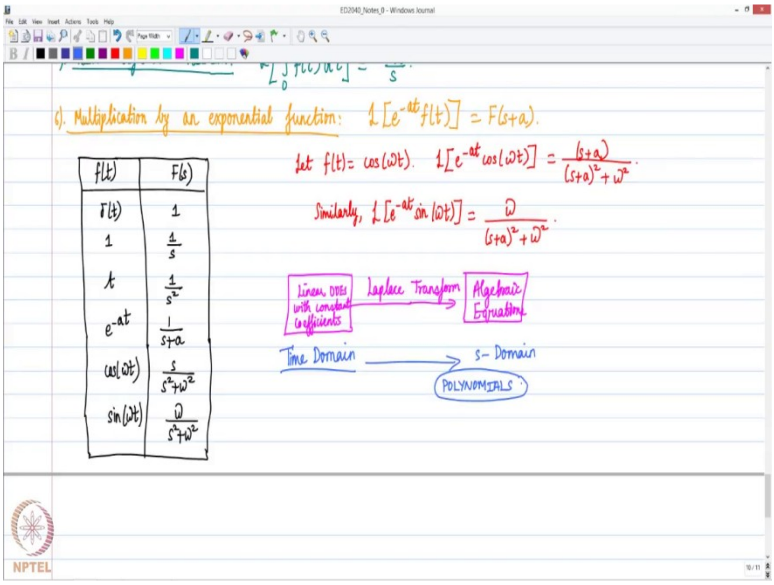

So, just to demonstrate a few theorems, right; So, even if I want to demonstrate this, suppose let us say let $f(t)=\cos(\omega t)$, then what will happen is that

$L\left(e^{-at}\cos(\omega t)\right)=F(s+a)$, right. That is, (ωt) $\cos$ ¿ ¿ L ¿. So, what this theorem tells me is that wherever you have s, substitute s+a. So, I am going to substitute s+a, wherever we have s, so the Laplace transform is going to be $\frac{s+a}{(s+a)^2+\omega^2}$, right.

Similarly, we have $L\left(e^{-at}\sin(\omega t)\right)=\frac{\omega^2}{(s+a)^2+\omega^2}$.

We will keep on adding to these properties, and let us say some other Laplace transform as we go further in this course, but from a big picture level what we are going to do is that we are just going to apply this Laplace transform operator on linear odes with constant coefficients, ok. So and how do we get these linear odes with constant coefficients? As we discussed, the typical mathematical models using the modeling approach that we use for the class of systems that we are considering in this particular course take the form of linear odes with constant coefficients, right.

So, that is our rationale. So, we use a Laplace transform, and then convert them into algebraic equations, which we can then, analyze, ok, but there is a catch here, right what is the catch? When we are dealing with linear odes with constant coefficients we are in the time domain, but that goes to the s domain, right. So, when we apply the Laplace transform we convert the problem into a problem in the complex domain, right.

And so, here we will shortly see that we need to have a basic understanding of polynomials to essentially work with the algebraic equations that we get in the s domain, right. So, that is also something which we will look at as we go along. So, a couple of more concepts in Laplace transform that we wish, we would recap is that how to take the inverse Laplace transform, alright.

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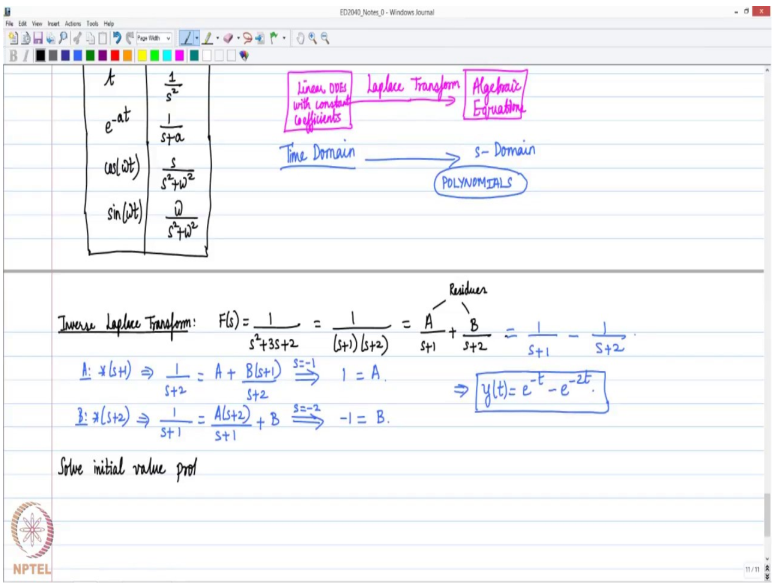

So, how do we take the inverse Laplace, right. So, suppose let us say I am given a Laplace transform function which is $F(s)=\frac{1}{s^2+3s+2}$. so, then what happens? What we need to do is that like we need to what is called as partial fraction expansion and then like take the inverse Laplace transform, right. So, we see whether we can factorize the denominator polynomial. In this case, I can factorize s^2+3s+2 as $(s+1)(s+2)$, and this can be factorized as $\frac{A}{s+1}+\frac{B}{s+2}$, ok. So, that is what we have.

So, then A and B are what are called the residues of the 2 factors, ok. So, how do we find A and B? Of course, we can use LCM and then like we can essentially equate the coefficients of the numerator polynomial on both sides and then we can solve. There is also another method to solve for A and B which I will just teach you.

So, if I want to find A, what I do is that I multiply both sides by the factor s+1 which is in the denominator. So, this implies that the left hand side will become one divided by s+2 the right hand side will just become A, and then I will have $B\left(\frac{s+1}{s+2}\right)$.

So, what will happen to this? Now, this equation is valid for any value of s right. So, what I do that is, I substitute the root of the factor that I have multiplied with, ok. So, what is the root of s+1? A root of a polynomial is, indicates those values of s at which the polynomial is 0, right. So, that is, that is essentially the meaning of the word root, right. So, the root of s+1 is going to be $s=-1$, right.

So, I substitute $s=-1$, and the left hand side becomes 1, the right hand side I have A and B(s+1) that term vanishes, right? Because s+1 at s=1 is 0. So, that B term vanishes so, we see that we very easily get the value of A. Similarly, I hope now you get the point, right so, if I want to find B I multiply both sides by s+2, we are left with $\frac{1}{s+1}$ on the left hand side.

Then I will have $A\frac{s+2}{s+1}+B$ and of course, now I need to substitute the root of s+2 that is going to be a s = -2. So, what I am going to get is minus 1 on the left hand side. A(s+2) vanishes at s=-2. So, I am just going to be left at B, ok. So, as a result what is going to happen is that this partial fraction expansion is going to be $\frac{1}{s+1}-\frac{1}{s+2}$.

Now, once we have broken down into blocks that we can easily process, we look at the table of Laplace transform, right. Immediately we can see that if I look at the table of Laplace transforms, the Laplace inverse of $\frac{1}{s+1}$ I can write as e^{-t} and $\frac{1}{s+2}$ I can re-write as e^{-2t} , ok. So, that is the way we will do the inverse Laplace transform, ok.

So, this is what is called as partial fraction expansion. So, these quantities A and B are what are called the residues of the partial fraction expansion process. And one important application of Laplace transform for us is to solve initial value problems, ok. So, that is essentially one important application from our course perspective, because we are going to deal with models that take the form of linear ordinary differential equations with constant coefficients.

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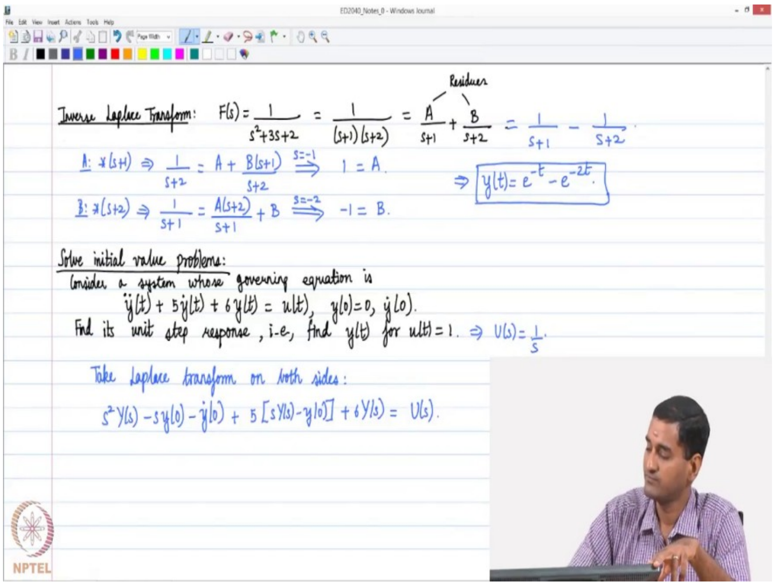

So, let me consider an example. Let us say I have a system whose governing equation is governed by $\dot{y}(t)+5\dot{y}(t)+6y(t)=u(t)$. Let us say $(0)=0$, $\dot{y}(0)=0$, right. So, let us consider a system whose governing equation is, ok, is the following. So, let us say, that is, find y(t) for u(t) =1, right. So, that is what we want. So, the solution what we do is that like we take Laplace transform on both sides of this particular equation.

So, what happens? So, for $\dot{y}(t)$ you will get $s^2Y(s)-sy(0)-\dot{y}(0)$. Then we will get $5\dot{y}(t)$ we will get $5(sY(s)-y(0))$ plus for $6y(t)$ we will get $6Y(s)$ and that is going to be equal to U(s), all right. So, if u(t) is 1 this implies that $U(s)=\frac{1}{s}$, right. So, we know that the Laplace transform is of one is $\frac{1}{s}$, right. Given this, let us process this. So, what we are going to get is the following.

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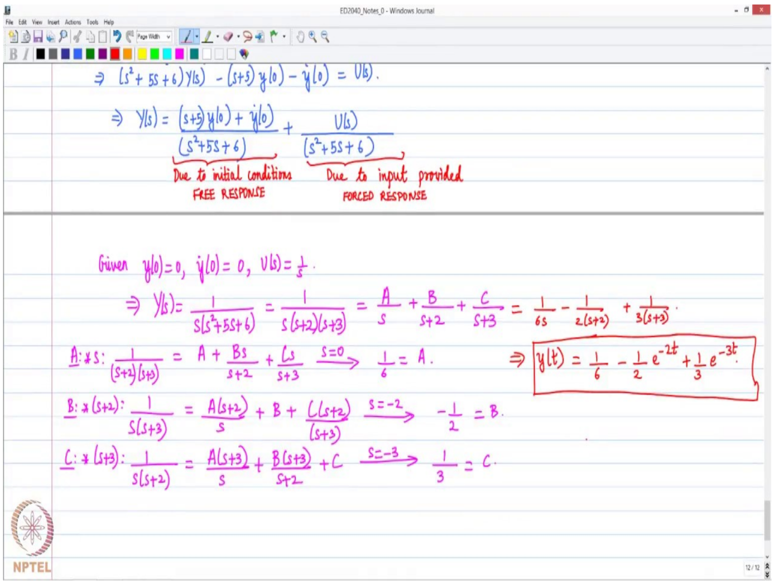

So, if I group terms involving $Y(s)$, let us say, I will have $(s^2+5s+6)Y(s)-(s+5)y(0)-\dot{y}(0)=\frac{1}{s}$. So, this will imply that $Y(s)$ = $\frac{s+5y(0)+\dot{y}(0)}{s^2+5s+6}+\frac{U(s)}{s^2+5s+6}$. That is what we are going to have. So, immediately we see that the first term is the response due to initial conditions, right, I am repeating it again. So, this is what is called as a free response right.

The second term is the output due to the input that we provide, right, due to input provided. So, that is called as the forced response.

So now we substitute $(0)=0$, $\dot{y}(0)=0$, the free response term vanishes so, the force response term is one what we are left with.

So, we will just get $\frac{s}{s(¿¿2+5s+6)}$ $\frac{1}{¿}$. If I factorize for s^2+5s+6 can rewrite this as (s+2)(s+3). So, $\frac{s}{s(¿¿2+5s+6)}=\frac{1}{s(s+2)(s+3)}$. $\frac{1}{¿}$ So, this can be rewritten as

$$\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+3}$$.

So, if I want to find A, I multiply by the factor involved in the denominator of A, which is s. So, $\frac{1}{(s+2)(s+3)}=A+\frac{Bs}{s+2}+\frac{Cs}{s+3}$. And here I substitute the root of s which is s = 0 right. So, consequently I will get 1/6 on the left hand side and just A, right. Similarly, if I want to find B I multiply it by the factor the denominator of B which is s + 2. So, I will get $\frac{1}{(s)(s+3)}=\frac{A(s+2)}{s}+B+\frac{C(s+2)}{s+3}$.

And here I substitute the root of s + 2 which is s = -2. So, I am going to get $\frac{-1}{2}=B$

So, similarly if I want to find C what I will do is that I will essentially multiply both sides by s + 3, the left hand side becomes $\frac{1}{(s)(s+2)}=\frac{A(s+3)}{s}+\frac{B(s+3)}{s+2}+C$. Now, we will substitute the root of s + 3 and that is - 3. So, we are going to get $C=\frac{1}{3}$ So, that is what we will have.. So, if I take the inverse Laplace transform just by looking at the table of Laplace transform, $\frac{1}{6s}$ will give me $\frac{1}{6}$, right and $\frac{-1}{2(s+2)}$ will give me $\frac{-1}{2}e^{-2t}$. And $\frac{1}{3(s+3)}$ will give me $\frac{1}{3}e^{-3t}$. So, that is what I will have, ok.

So, we see that the process of calculating the solution to the ordinary differential equation becomes pretty simple because we are converting the ode into an algebraic equation and then we process that. So, that is how we are going to use the Laplace transform for our advantage, right. So, to once again give a big picture perspective we are dealing with, SISO LTI causal systems which are going to be modeled using spatially homogeneous dynamic continuous time deterministic models that will typically result in linear odes with constant coefficients.

And once we get linear odes with constant coefficients we are going to use Laplace transform and convert them into algebraic equations. And then we calculate the output or response of the system to various standard inputs and we analyze them, ok. So, that is the big picture viewpoint. That is why we have studied all the concepts still now.

So, following this we will go further, we will see how we represent the system, and then we see whatever I call as transfer functions and then how do we use transfer functions in our analysis, ok. So, we are going to look at the big picture viewpoint following this. But I am just going to leave you with a few problems that you can do as exercises, ok.

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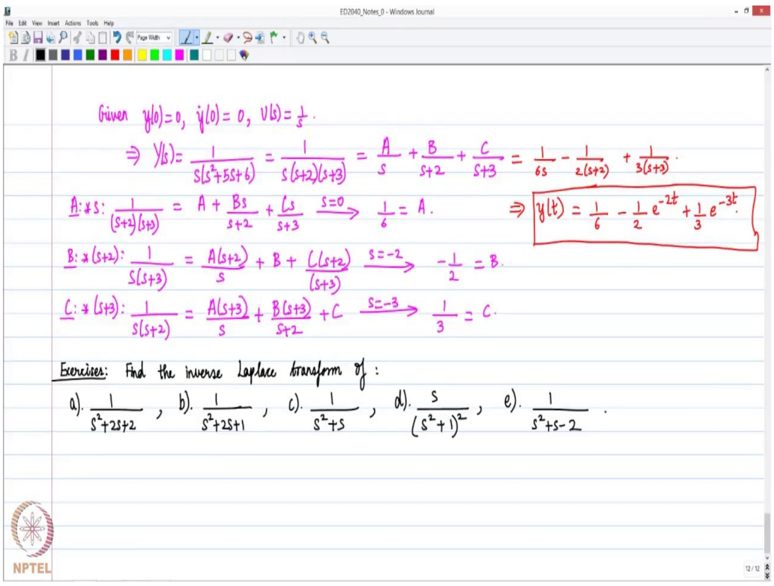

So, that like we become familiar with the inverse Laplace transform. So, once again I am sure all of us would have learnt it in engineering mathematics course, this is just to help you with the recapping all those concepts, right. So, please go back and let us say find the inverse Laplace transform of the following functions:

$a¿\frac{1}{s^2+2s+2}$ $b¿\frac{1}{s^2+2s+1}$ $c¿\frac{1}{s^2+s}$ $d¿\frac{s}{¿}$ $\frac{s}{(¿¿2+1)^2}$ $e¿\frac{1}{s^2+2s-2}$

Please go back and work out the inverse Laplace transform of these functions, ok. And then like we would continue from here.

Thank you.