

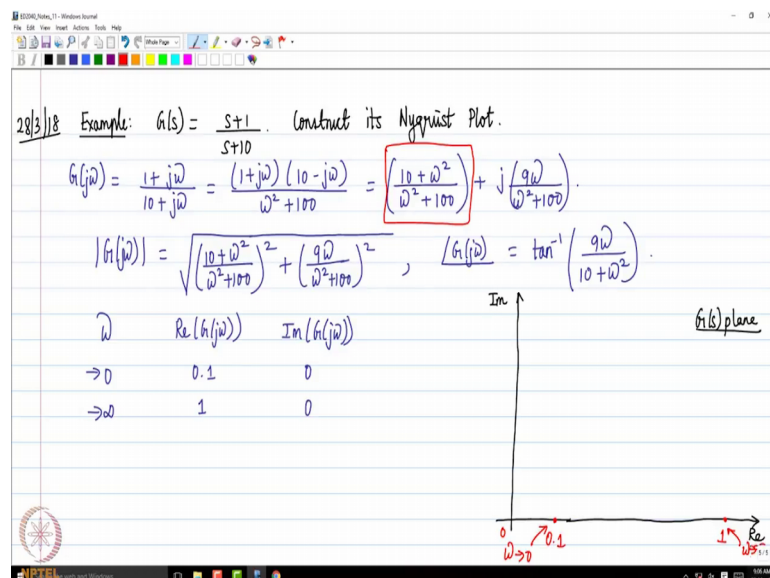
**Control Systems**  
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**Lecture – 59**  
**Nyquist Plot 2**  
**Part -1**

Ok, let us get started today's class. So, we were looking at the Nyquist plot right. So, the Nyquist plot is essentially the plot of the sinusoidal transfer function in the complex plane that is it is a plot of the real part of the sinusoidal transfer function, which is the imaginary part as omega is varied from 0 to infinity ok, so that is the Nyquist plot.

And we looked at the Nyquist plot of some basic factors yesterday. So, we are essentially going to do one example today ok, so that is what I am going to do. Just to show you, you know like what is the broad procedure in constructing a Nyquist plot ok, some observations that we can make. And then like we will start discussing what is called as a Nyquist stability criteria ok. So, how we are going to use Nyquist plot for control systems right, so that that is something, which we are going to study today.

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So, let me what say do an example. We are essentially going to start off with an example in constructing the Nyquist plot. Let us say that we consider a transfer function, which is

of the form  $s + 1$  divided by  $s + 10$ . So, the essentially the question is construct its Nyquist plot that is the question that we are asking ourselves.

So, now how do we start doing it. You know like, so essentially first we figure out the sinusoidal transfer function  $G$  of  $j\omega$  is going to be  $1 + j\omega$  divided by  $10 + j\omega$  right, so that is what I am going to get. So, of course, now I what do I do, I just multiply and divide by the conjugate of the denominator factor. So, I will get  $1 + j\omega$  multiplying  $10 - j\omega$ , and this denominator will become  $\omega^2 + 100$  right, so that is what I will get ok.

So, if I now multiply, and then like simplify this, what am I going to get in the numerator, I will get  $10 + \omega^2$  divided by  $\omega^2 + 100$ . And what will be the imaginary part, it is  $j$  times  $9\omega$  divided by  $\omega^2 + 100$  right, so that is what we will have, correct.

So, in terms of magnitude and phase, what are we going to have? So, this same thing, if I write the magnitude and phase of the sinusoidal transfer function, the magnitude is going to be anyway the square root of the real part square, and some plus imaginary part square right. So, we are going to get  $10 + \omega^2$  divided by  $\omega^2 + 100$  whole square plus  $9\omega$  divided by  $\omega^2 + 100$  whole square ok, so that is the magnitude.

And what about the phase, the phase is going to be  $\tan^{-1}$  of the imaginary component divided by the real component. So, we will get it as  $9\omega \tan^{-1}$  of  $9\omega$  divided by  $10 + \omega^2$ , am I correct right. So, these are the various things that we will get right. So, once I once we process the sinusoidal transfer function right.

And as we are discussing, you know like when you essentially want to plot the Nyquist plot, what you do is that like you also create a table, and then like start plugging or substituting different values of  $\omega$  that you think are critical. And start calculating the real part imaginary part and so on right, so that you get an idea as to what would happen ok. So, as these what to say, these functions tend to those frequencies right.

So, let us look at  $\omega$  the real part of  $G$  of  $j\omega$ , and the imaginary part of  $G$  of  $j\omega$  ok. So, let us see what happens at various frequencies right. So, now if I have

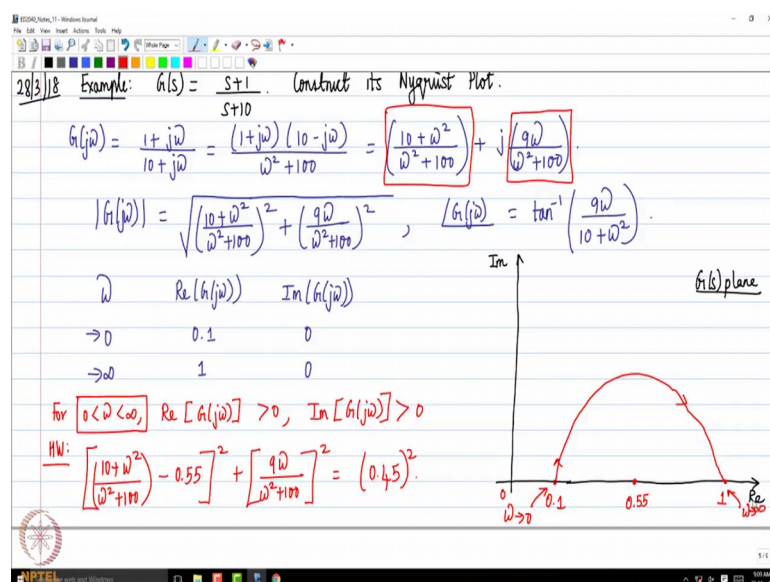
omega to be tending to 0, what will happen, what do you think will be the real part and the imaginary part, please note, the real part is this. What will happen to the real part as omega tends to 0?

Student: (Refer Time: 05:02).

0.1 right, so that is what is going to happen right. So, as omega tends to 0, the real part is going to be 0.1. What about the imaginary part, it is going to be 0. Now, as omega tends to infinity, what do you think will happen to the real part, it will tend to 1 right. And what will happen to the imaginary part, 0 once again right. So, if you plot the G of s plane here, for this particular transfer function. So, this is the real axis ok, so this is the imaginary axis, and this is the G of s plane ok.

So, let us say, we plot this sinusoidal transfer function right. So, what we are going to get is the following right. So, let us say this is 0, so let us say this is 0.1, and let us say this is 1 all right. So, at as omega tends to 0, the transfer function values tend to I am sorry, I think let me scroll down a little bit ok, the transfer function values, you know like value or the transfer function tends to 0.1. And the as omega tends to infinity, the transfer function tends to 1 ok. The real part tends to 1, imaginary part is 0.

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So, now question is that like what happens to the locus of the what to say, this Nyquist plot for this particular transfer function, as omega goes from 0 to infinity ok. So, please note that, it is going to be the plot of the imaginary part, and the real part right.

So, you can see that the real part is always going to be positive, for all omega between 0 to infinity right. So, one can always see that for omega between 0 and infinity, you can immediately see that the real part of  $G$  of  $j\omega$  is always greater than 0 that is something, which is pretty obvious right. And the imaginary part is always going to be for omega greater than 0, it is also going to be greater than 0 right, and less than infinity.

Of course, if I put omega greater than or equal to 0, it is I have to put it as imaginary part greater than or equal to 0 all right. So, please note what am I doing, I am taking omega greater than 0, and less than infinity. So, so essentially the imaginary part will be infinitesimally positive ok. So, you see that immediately the Nyquist plot is going to lie in the?

Student: (Refer time: 07:47).

1st quadrant ok, so that is something, we can immediately see. And as I told you, you know like by enlarge, if we go, and look at it right. So, sometimes you know like what you can do that you can use intuition right. So, and by enlarge, when you have this first order factors, and when you have such transfer functions, you could try out an equation of a circle, you know like for seeing whether that will be the Nyquist plot right. So, if I want to have a semi-circle, which whose endpoints are 0.1 and 1, what should be its center?

Student: 0.5 (Refer Time: 08:22).

Point?

Student: (Refer Time: 08:23).

0.55 right. And what should be its radius?

Student: 0.45.

0.45, right. And it, so happens that this I leave it to you as homework ok. So, if you check this expression,  $10 + \omega^2$  divided by  $\omega^2 + 100$ , which is

a real part, minus 0.55 whole square plus the imaginary part square, which is 9 omega divided by omega square plus 100 the whole square that is going to be equal to 0.45 square ok. You will see that this equation is going to be satisfied right.

So, you will see that the center is somewhere here at 0.55. And we are going to have a semi-circle ok. So, is this ok, yeah it looks like circle. Yes, any questions, ok. So, it has the maximum value at 0.55 ok, so that is what we have right. So, this radius is going to be 0.45 ok, so that is the what to say Nyquist plot ok. So, please do this as homework, you know like you can double check that this is the case, this is the equation in this scenario.

Now, if you want to find out at what omega, the phase angle of this particular sinusoidal transfer function is maximum, how do you do that, how would you calculate the phase of this G of j omega at any omega?

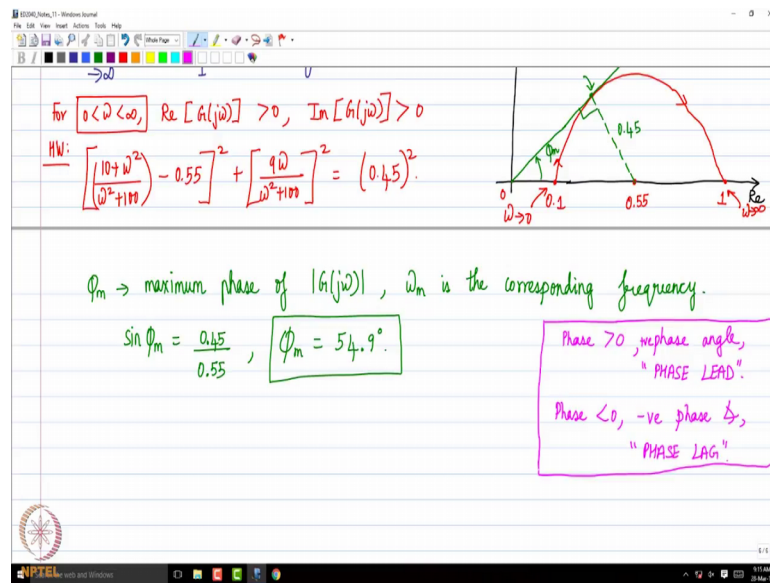
Student: (Refer time: 10:12).

Graphically let us say, you have the Nyquist plot right, what do you do?

Student: (Refer Time: 10:18).

Yeah. So, at any omega, what you do is that you draw a vector from the origin to the to that point, and then calculate what is the angle made by that vector from the positive real axis, in the counter clockwise direction that is taken to be positive convention right. So, as you said if I want to figure out the maximum angle, I need I need to draw the tangent all right.

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So, from the origin I see, which point at which point essentially I can get a tangent to the semi-circle right. So, essentially let us say, this is my tangent right. So, this is the omega,. What I call as omega m is the frequency at which I get the maximum phase phi m ok. So, phi m is the maximum phase of G of j omega right.

So, the Nyquist plot helps us with all these things. And we will see why this is important ok. I have chosen this example with a with a motivation right. All the examples I am choosing finally, lead to some consequence ok, we will see ok. In maybe in the next few classes, we will see, why this is going to be important. Why did I choose this particular example, and why am I talking about the maximum phase contribution right, or phase angle of this particular transfer function.

So, you see that phi m is the maximum phase of G of j omega ok, omega m is a corresponding frequency ok. Now, how can I calculate phi m, is there a way to calculate phi m. Even using geometry, phi m is that angle right, I am sure all of us agree right. You just using geometry, can you calculate phi m right.

Student: (Refer Time: 12:38).

Phi m is this angle right. So, see let us say of course, if I draw this, you will get the answer right. Let us say I draw a line segment from this point to the center of the circle semicircle right, what is the length of this line segment?

Student: (Refer Time: 13:00).

The radius. What can you say about the angle made by the tangent and the radius, 90. Now you do you have the answer. What is  $\sin \phi_m$ , this is 0.45, then so we notice that  $\sin \phi_m$  is going to be 0.45 divided by 0.55. Can you calculate  $\phi_m$ , and tell me. So, this is 54.9.

So, essentially you can see that this is the maximum phase that is contributed by this particular transfer function ok. And by enlarge, if this phase angle is positive, that means, that there is a phase lead. And if the phase angle is negative, that means, there is a phase lag ok.

So, in general so this is a just an aside ok. So, if phase is greater than 0 ok, we call it as a positive phase right, so positive phase angle. So, it is also called as phase lead so essentially or lead factor. So,, but if you have a negative phase angle ok, phase is greater than less than 0 that essentially do not serve negative phase angle that is what is called as a phase lag.

So, you see that this particular factor adds positive phase right. So, essentially it provides a phase lead and the maximum phase lead that it can provide is around 55 degrees. We will see why this becomes important as we go later. Please remember this ok. So, we will we will use it later on ok.

Now, how do you find  $\omega_m$ ?

Student: (Refer Time: 15:38).

Of course, there are you know the equation for the phase angle. And then like you plug it in and then calculate  $\omega_m$  all right. You know  $\tan \phi_m$  that is going to be  $9 \omega_m$  divided by  $10 + \omega_m^2$  right is it not right.

(Refer Slide Time: 16:06)

The image shows a digital whiteboard with handwritten mathematical work. At the top, there are calculations for phase margin:  $\sin \phi_m = \frac{0.42}{0.55}$  and  $\phi_m = 54.7^\circ$ . Below this, the phase margin is calculated as  $\phi_m = 90^\circ - 54.7^\circ = 35.3^\circ$ . The phase margin is then related to the gain margin  $G_m$  via  $\tan \phi_m = \frac{9 \omega_m}{10 + \omega_m^2}$ , which is solved to find  $\omega_m = 3.162 \text{ rad/s}$ . A note indicates that a phase margin of  $0^\circ$  corresponds to a phase lead, while a negative phase margin corresponds to a phase lag. A homework problem is assigned: "Plot the Bode diagram of  $G(s) = \frac{s+1}{s+10}$ ". The transfer function is then factored into  $G(s) = \frac{s+1}{s+10} = \frac{s+1}{10(\frac{s}{10}+1)} = 0.1 \frac{(s+1)}{(\frac{s}{10}+1)}$ . The corner frequencies are identified as  $1 \text{ rad/s}$  and  $10 \text{ rad/s}$ . The magnitude plot is shown as  $\log_{10} |G(j\omega)| = \frac{1}{2} [\log_{10}(1+\omega^2) - \log_{10}(1+(\frac{\omega}{10})^2)] = \frac{1}{2} [\log_{10}(1+\omega^2) - \log_{10}(1+\frac{\omega^2}{100})]$ . A Bode plot is partially shown with corner frequencies at 1, 3.162, and 10 rad/s.

So, for finding omega m you know that tan phi m is going to be equal to 9 omega m divided by 10 plus omega m square. So, if you solve this, you will get omega m to be equal to 3.162 ok, of course, radians per second. So, you please do this as homework once again ok. So, this is also a homework problem so that something which I wanted right. Please do this.

And one more thing which we are going to do is an I am going to leave you with another homework problem plot the bode diagram of this G of s which is essentially you took it as s plus 1 divided by s plus 10. So, the same transfer function plot its bode diagram. Please do this as homework ok, please do this and compare right. I think we have already discussed bode diagrams in detail we have done an example in class also [, but let me get you started. So, how do you plot the bode diagram?

Student: (Refer Time: 17:34).

Of course you need to divide into particular factors right. So, what do we do? If G of s is going to be this, I can rewrite this as I need to rewrite in the standard form right. I can pull 10 out from the denominator, so I am just going to get 0.1 s plus 1 divided by s by 10 plus 1 correct. Do you agree? Ok, so it has three factors.

So, this means that it has a 0.1 scalar, and then you have a s plus 1 and there is an s by 10 plus 1 ok, three factors are there. So, you plot the bode diagram for all the three



individual factors and then write the magnitude plot and the phase plot. So, what are the corner frequencies of these factors? If you consider  $s + 1$  what is the corner frequency, corner frequency?

Student: 1 (Refer Time: 18:43).

1? Right, and what about the corner frequency of  $s + 10$  plus 1?

Student: (Refer Time: 18:53).

0.1 or 10?

Student: 10.

10 right, because its  $T s + 1$  right. So, and corner frequency is 1 by  $T$  capital  $T$  is a coefficient of  $s$ . So, here is 1 by 10 then you take the reciprocal. So, the corner frequency is going to be 10.

So, now can you relate 3.162 to 10 and 1 no of course, I am just playing with numbers, but then you will see that it translates into generality for such factors right. So, so I chosen these numbers for due to a particular motivation, which will become clear as I told you. So, how can you relate 1 and 10 to 3.162?

Student: (Refer Time: 19:36).

Ah square root of essentially 10 and 1 right. So, when you use a logarithmic scale, whereas 3.162 going to be? Say let us say you use a logarithmic scale right for omega. So, you have 10 and 1 where is it going to be?

Student: (Refer Time: 20:05).

Midpoint right so exactly. Why, because if you if you want to essentially do 1 times 10 right. So, you take the log to the base 10. So, first of all you have log of square root is 1 by 2 of log correct. Do you agree? And then log of a times b is going to be log of a times sorry log of a plus log of b. So, this is going to be of course we are doing log to the base 10.

So, what are we going to get, we are going to get the midpoint right. So, when we do this 1 in the logarithmic scale. So, you see that for this particular factor, the maximum phase

is going to be at the of course, what do you call as is let us say I give you two numbers, you take the product and you take the square root of that what do you call that as?

Student: Arithmetic (Refer Time: 21:06).

Arithmetic mean?

Student: Geometric mean.

Geometric mean? Is it, or arithmetic mean or something else?

Student: Geometric mean.

Geometric mean right. So, you see that the phase that is frequency at which you get the maximum what to say phase contribution from this is going to be at the geometric mean of the two corner frequencies right in this particular case right, so that is what we are getting. And this particular factor you know particular transfer function you know we will generalize and we will use it in control design ok, so that is why I am I am doing this in this example in so much of detail.

So, please go back you know like I cannot be what to say essentially more I cannot emphasize this more. Please go back and draw the bode plot, because you will see that you will need it in control design using frequency response. Kindly do that. Yeah, yes any questions ok. So, please go back and plot the bode diagram and then compare it with the Nyquist plot so that is what I want you to do as homework right.

So, so we see that you know like what we have done is we have looked at the frequency response, we have looked at what to say the bode diagram or the Nyquist plot which are two essentially methods to visualize the frequency response of a transfer functions ok. There is a third plot which is called as a Nikolas plot which I leave it to you as homework. So, we are not going to use that in this course, so but if you are interested you can go back and look at that. So, once again its just a visual representation of the sinusoidal transfer function that is about it.

Student: So, it is a geometric mean result is good for all  $s$  plus  $a$  by  $s$  plus (Refer Time: 23:00)

Ah all  $s$  plus  $a$  divided by.

Student: s plus b.

Ok, so a factors of this form will give you a that is what I am saying you know like I am not going to generalize here, it depends on the value of a and b ok, we will come there. So, please wait for a few more classes ok. So, that is why I am being careful to not to extrapolate right now ok. Please hold on that is a good question ok. He is asking if you take a transfer function of the form s plus a divided by s plus b right, is this result going to hold in general, partly yes, partly no, depends on the value of a and b ok. We will see as we go along, fine. Yeah.

Student: s plus 10 (Refer Time: 23:43).

Oh, yes, yes thank you yeah sorry about that.

(Refer Slide Time: 23:54)

HW: Plot the Bode diagram of  $G(s) = \frac{s+1}{s+10}$ .  $n=1$ ,  $m=1$

$$G(s) = \frac{s+1}{s+10} = \frac{s+1}{10\left(\frac{s}{10}+1\right)} = \frac{0.1(s+1)}{\left(\frac{s}{10}+1\right)} = (0.1) (s+1) \left(\frac{1}{\frac{s}{10}+1}\right)$$

Corner Freq. = 1 rad/s      Corner Freq. = 10 rad/s

$$\log_{10} |G(j\omega)| = \frac{1}{2} [\log_{10}(1+\omega^2)] = \frac{1}{2} [\log_{10}(\omega^2) + \log_{10}(1+\frac{1}{\omega^2})]$$

$G(s) = \frac{n(s)}{d(s)}$  → order  $n$  As  $\omega \rightarrow \infty$ ,  $\angle \rightarrow -90^\circ (n-m)$   
decade      min. phase system.

Now, let me erase this yeah this should be 1 divided by s by 10 plus 1. You are right, thank you. Yeah, sure ok, so that is what you have But then like you can also make some general observations right. What are the values of n and m? It is essentially 1 and 1. So, is this a minimum phase transfer function, it is right.

See, because typically see by the way I think I should highlight this point there are two groups or two what to say ways in which people look at a minimum phase transfer function or a minimum phase systems right. Some people associated only with the zeros ok, because they implicitly they assume a stable system to begin with.

See, if you take any transfer function which is of the form  $n$  of  $s$  by  $d$  of  $s$  right. How did we define a minimum phase transfer function, 1 whose zeros are in the left of complex plane you know it was implicitly assumed that the poles are in the left of complex plane. Because when we talk about frequency response, you recall that we are talking about stable systems then only frequency response make sense right. So, implicitly we are assuming all the poles to be in the left of complex plane. Another school of thought essentially tells that you know minimum transfer function is 1 where all its zeros and poles are in the left of complex plane ok.

So, I just wanted to highlight that difference to you these this alternative notion our definition to you. The reason why I wrote the definition as what to say minimum transfer function being 1, where all zeros are in the left of complex plane was that because I implicitly assumed that all the poles are already in the left of complex plane. Is it clear ok? So, there is a subtle difference I want wanted to a highlight.

So, this is a minimum transfer minimum phase transfer function, if you recall a minimum phase transfer function. What does the what to say in general the slope of the high frequency asymptote? Did you recall our do you recall our discussion? Slope of their magnitude curve see if this is of order  $m$ , and this is of order  $n$ , what is the slope of the high frequency asymptote?

Minus  $20$  times  $n$  minus  $m$  decibels per decade right. Is that satisfied here? Yes, right, why because  $s$  plus  $1$  will contribute plus  $20$  decibels per decade;  $1$  divided by  $s$  by  $10$  plus  $1$  will contribute minus  $20$  decibels per decade. What is going to happen both are just going to cancel out each other right that is what is going to happen that is point number  $1$ . And what can you say about its phase as  $\omega$  tends to infinity? What is going to happen to for a minimum phase transfer function you remember? The phase tends to minus  $90$  times  $n$  minus  $m$  right.

So, this is these are the results as  $\omega$  tends to infinity the magnitude tends to this  $1$ , the phase tends to this for a minimum phase system right. So, what is going to be the value here  $0$ . And Does it tend to  $0$ ? It does right because  $s$  plus  $1$  will give you a phase contribution of plus  $90$   $1$  by  $s$  by  $10$  plus  $1$  will give you a phase contribution as minus  $90$  as  $\omega$  tends to infinity. So, the sum is going to be  $0$ . And can you observe it from

the Nyquist plot? You can right. Because what is the what to say transfer function, where is the transfer function as omega tends to 0?

Student: (Refer Time: 28:09).

1

Student: (Refer Time: 28:13).

1?

Student: (Refer Time: 28:14).

Not 0.1, 1; see as omega tends to what to say infinity is this where is it at omega tending to infinity it is at 1 right. If the transfer function is at 1, what is the angle made by that? See how do you get the angle you draw the vector from the origin to the point right. So, you are going to have a vector which is aligned along the positive real axis. So, what is the angle, 0 degrees all right. So, I am just what to say we are just double checking the results. Is it clear? Ok, so you can see that the results that we discussed hold true. Yeah.