Control Systems
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Lecture – 05
Mathematics Preliminaries
Part – 1

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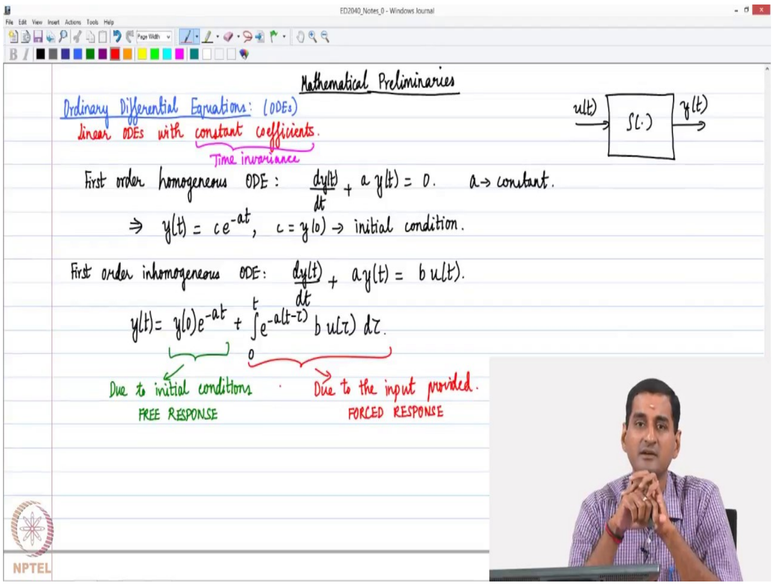

Having obtained a big picture view point of what we are going to study in this course; let us have a brief recap of what mathematical tools that we would be requiring. As we discussed, we are going to deal single input single output linear time invariant and causal dynamic systems that we are going to characterize using spatially homogeneous dynamic continuous time deterministic models.

Typically, these models take the form of an ordinary differential equation. So, the, first set of a mathematical tools that we would require for this course is the basic working knowledge of ODEs. Specifically we are going to consider linear ODEs with constant coefficients. Why linear ODEs with constant coefficients? Typically linearity comes into play because we are dealing with linear systems and because of time invariance, we are dealing with constant coefficients. That is why we are considering linear ODEs with constant coefficients.

Let us consider some specific ordinary differential equations. But I suggest that a recap of engineering mathematics course on ordinary differential equations would really be helpful. I am just going to recap a few important concepts. If we consider a first order homogeneous ODE of the form, of the form

$$\frac{dy(t)}{dt}+ay(t)=0,$$

where a is a constant parameter (real number). The solution to this ODE is of the form

$$y(t)=ce^{-at}.$$

We can write $c=y(0)$, where $y(0)$ is the initial condition.

This is an initial value problem; given that the independent variable is time. I can write the same solution as $y(t)=y(0)e^{-at}$. Why do we have an exponential solution? How do we get this? We just substitute $y(t)$ of the form e^{mt} and then we get $m=-a$. why should we have exponential solutions? Why should we substitute an exponential function to begin with? Because it has been shown that, for linear ODEs with constant coefficient, the solution exists and the unique solution takes the form of exponential functions. In other words the exponential function is the only solution for linear ODEs with constant coefficients, That is why we are substituting a solution form that corresponds to an exponential function to solve this equation.

If we now consider, a first order inhomogeneous ODE of the form

$$\frac{dy(t)}{dt}+ay(t)=bu(t),$$

What is the solution to this equation?

$$y(t)=y(0)e^{-at}+\int_0^t e^{-a(t-\tau)}bu(\tau)\,d\tau$$

We know that it is going to comprise of two parts. The first one is due to the initial condition. This part is typically referred to as free response. The second part is the solution, due to the input provided. This response is what is called as forced response. Please remember we are looking at a system to which we provide an input $u(t)$ and an output $y(t)$. So equations are taken in terms of the input on the output variables.

So, we see that in general, in, you know like, we have two components to the output function, ok, like, when we model systems using this class of equations, like, the first part is, what is called as a free response, which comes in, you know like, due to the nonzero initial conditions the second part is what is called as the forced response, which comes due to the input that is provided to the system, ok.

Given these responses, the question is how we use this to analyze the class of systems under study. Once we model the systems, we would get a linear ODEs with constant coefficients. And then we see that given any input, we can use solutions to the ODE to calculate what would be the corresponding output from the system.

In this process, we are going to use Laplace transform that helps us to go from the time domain to the complex domain and convert problems involving ordinary differential equations into problems involving algebraic equations. Then we take the inverse Laplace transform to come to the time domain.

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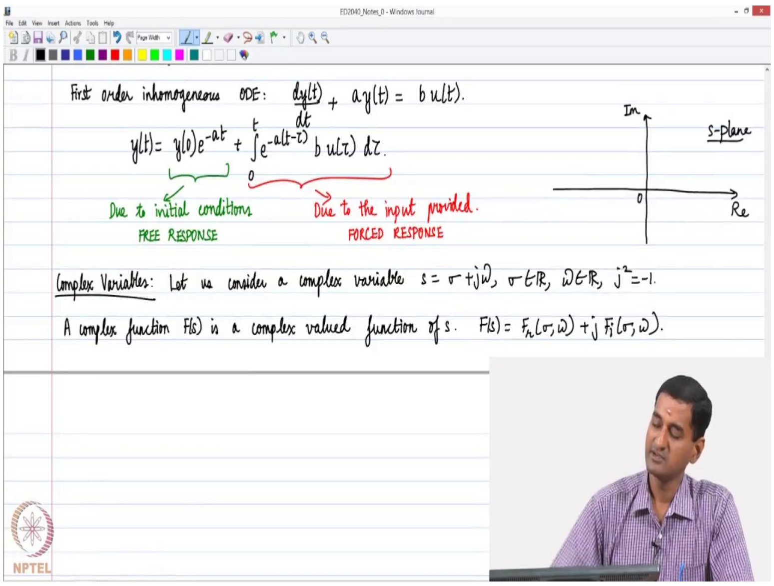

Let us look at complex variables, then we will come back to ODEs and we will see how we use Laplace transform on ODEs and process them. We consider complex variable of the form $s=\sigma+j\omega$, where σ is some real number ω is a real number and $j^2=-1$.

Typically, we will draw what is called as an S plane, which can be used to graphically, represent a complex number. We will have two axis called as the real axis and the imaginary axis and we use this to graphically represent any complex number.

And a complex function $F(s)$ is a complex valued function of s. We write

$$F(s)=F_r(\sigma,\omega)+jF_i(\sigma,\omega)$$

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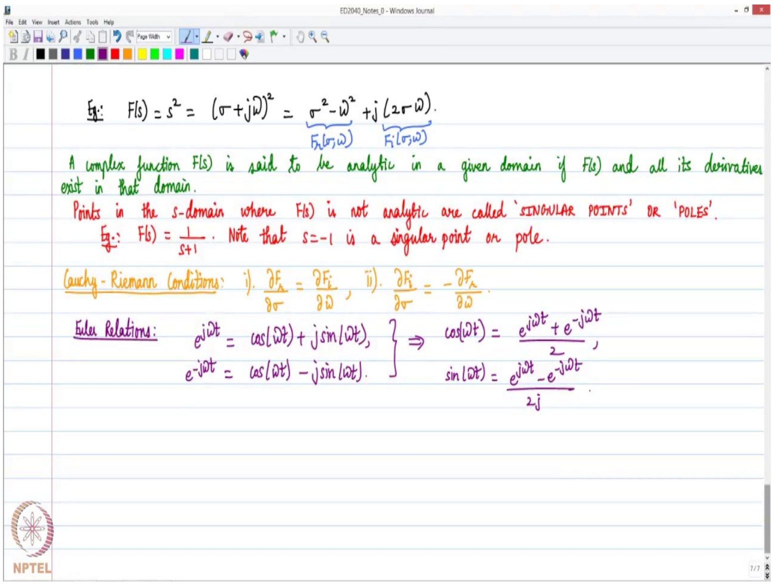

For example; we consider $F(s)=s^2$. We can rewrite this as

$$F(s)=(\sigma+j\omega)^2 \quad = \quad \sigma^2-\omega^2+j(2\sigma\omega)$$

Here $F_r(\sigma,\omega)=\sigma^2-\omega^2$ and $F_i(\sigma,\omega)=(2\sigma\omega)$

A few definitions:

A complex valued function $F(s)$ is said to be analytic in a given domain if $F(s)$ and all its derivatives exist in the domain. The points in the s domain where $F(s)$ is not analytic are called singular points or poles. Consider, $F(s)=\frac{1}{s+1}$, we can immediately note that, $s=-1$ is a singular point or pole.

There are what are called as Cauchy Riemann conditions, which are used to evaluate whether a given function $F(s)$ is analytic. The first condition is

$$\frac{\partial F_r}{\partial \sigma} = \frac{\partial F_i}{\partial \omega}$$

The second condition is

$$\frac{\partial F_i}{\partial \sigma} = \frac{-\partial F_r}{\partial \omega}$$

Another set of equations are called Euler's relationships. They are

$$e^{j\omega t} = \cos\omega t + j\sin\omega t \quad \text{and} \quad e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

We can immediately see that

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \quad \text{and} \quad \sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \quad .$$

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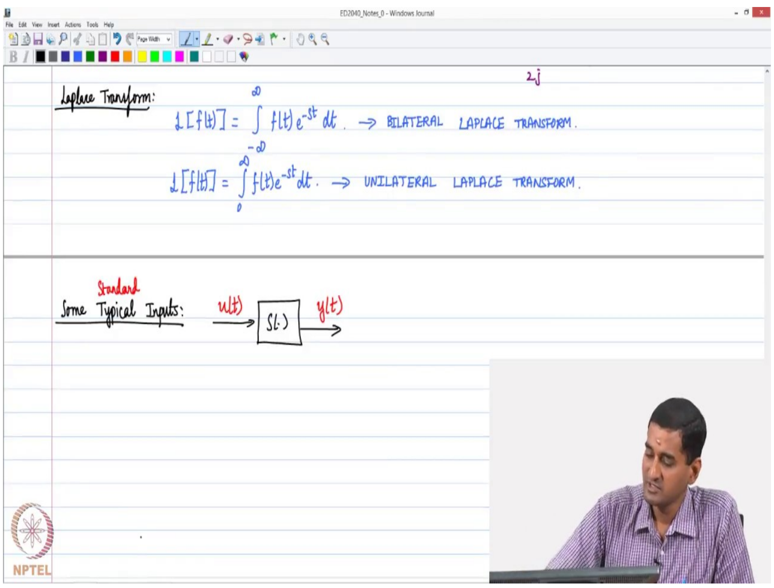

Now we look at Laplace transform. Let us write down the definition of a Laplace transform of a real valued function $f(t)$.

$$L[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-st}\, dt \quad \text{– Bilateral Laplace Transform}$$

$$L[f(t)] = \int_0^{\infty} f(t) e^{-st} dt \quad \text{– Unilateral Laplace Transform}$$

In this course, we would use unilateral Laplace transform, because we are going to start analysis of a system from time $t=0$. So, we make the lower limit as 0 and consequently we will go from 0 to ∞ .

There is something called as the inverse Laplace transform, which maps any given function in the complex domain back to its function in the time domain. But we calculate the inverse Laplace transform using what is called as a partial fraction expansion rather than using the definition per say.

Before we move on to learn how we are going to apply Laplace transform in our course, let us point out a few important functions that are going to be useful to us. Let us look at some standard inputs. If you want to have a proper analysis done, we need to give some standard inputs, so that we evaluate the system response. If I design, n systems for our application, I can evaluate them provided, I give the same input to all the designs. For that purpose we use some standard inputs. Let us look at each one of them.

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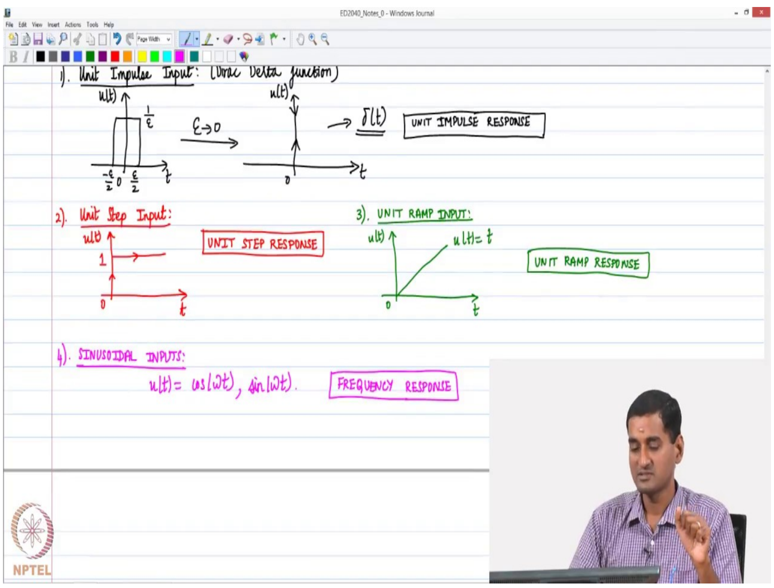

First we look at unit impulse input. This is also called as a Dirac delta function denoted by $\delta(t)$. We consider, $u(t)$ to be a signal of the form shown in figure, around time

$t=0$ for a very-very small time interval $\frac{-\epsilon}{2}$ to $\frac{\epsilon}{2}$, the magnitude of the signal is $\frac{1}{\epsilon}$. We see that the area of this rectangular signal is 1. That is how the adjective unit comes in to be.

Then, we shrink, this epsilon and make it tend to 0 $\epsilon \to 0$. Now the input is going to just jump to a very high magnitude instantaneously and come back to 0 instantaneously. If I provide the unit impulse input as an input to the system. The corresponding output is what is called as the unit impulse response.

Another input which is typically used in systems analysis is unit step input. As the name suggests, what we are going to do is that at time t equals 0, we give a step input is of magnitude 1, to the system. That is what is called as a unit step input. The output that we get from the system is what is called as a unit step response. We will see that the impulse response and the step response are going to be very valuable to us.

The third standard input that we would consider is a unit ramp input. Unit ramp input is $u(t)=t$. The input the scales like *t*. The slope is 1 and that is why we call it as a unit ramp input. When we provide the unit ramp as the input, the output that we get is what is called as a unit ramp response.

The fourth standard input that we provide are sinusoidal inputs. Sinusoidal inputs as the name suggests, we provide sinusoidal functions as inputs of varying frequencies $\cos\omega t$ and $\sin\omega t$. If we have a stable LTI system, and we provide a sinusoidal input, the steady state output is also going to be sinusoidal of the same frequency. This information is used in analysis called as frequency response analysis.

These are four common inputs that are typically provided impulse, step, ramp, and sinusoids. In general, an input can be in any arbitrary combination of these inputs.

There are two reasons why we study the response of systems to standard inputs. We have a uniform baseline to study systems and also we extract and define parameters for system performance, which are based upon one type of input. And we can quantify system performance. We are going to use step response and frequency response to define parameters that quantify system performance, which is one reason.

And second, in general, any arbitrary input, can be usually written as a linear combination of standard inputs. So, since we are dealing with linear systems once we know the output to these standard inputs, in theory at least, we can calculate the output to any arbitrary input.

There are two important aspects that are critical in control design.

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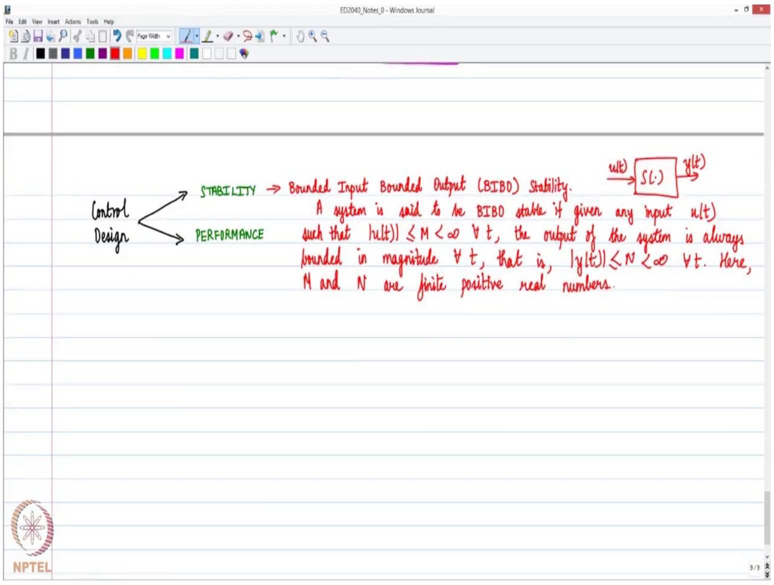

The first one is what is called as stability. We want to design stable systems and if a system is unstable to begin with, we want to stabilize systems using feedback. That is one reason why we design control systems. Whenever we design dynamic systems we want those systems to be stable. We will shortly define, what notion of stability we are going to use in this course.

The second thing is, once we design stable systems, we want to evaluate how those systems perform. We are going to be interested in performance. For us stability is paramount; once we have stable systems, we worry about performance.

The notion of stability that we are going to use in this course is called as bounded input, bounded output stability. It is abbreviated as BIBO. What has meant by bounded input, bounded output stability? It means that if I provide any bounded input to the system, the corresponding output should be bounded in magnitude for all time.

The system is said to be a BIBO stable, if given any input $u(t)$ such that $|u(t)| \leq M < \infty$ $\forall t$, output of the system is always bounded in magnitude. That is $|y(t)| \leq N < \infty$ $\forall t$

, where M and N are finite positive real numbers. This is the notion of bounded input bounded output stability that we will consider in this course.

These ideas are extremely important. When we design controllers, we want to ensure that the closed loop system that we design is first of all stable and then it performs as desired. We will learn about them as we go along.