Control Systems
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Lecture – 47
Frequency Response
Part-1

Ok. So good morning, let us get started right ok. So, yesterday if you recall, we were looking at the state space representation right. So, we looked at how to rewrite a nth order ODE as a set of n first order ODEs, and the state space representation had two equations in general right what was called as a state equation, which is a first order vector ODE that provided the time evolution of the state variables. And then we had what is called as an output equation right, so that related the system output to the state vector and the system input ok. So, those were the two equations in the state space representation, and we essentially did an example, where we rewrote the mass spring damper system governing equation in the state space form right.

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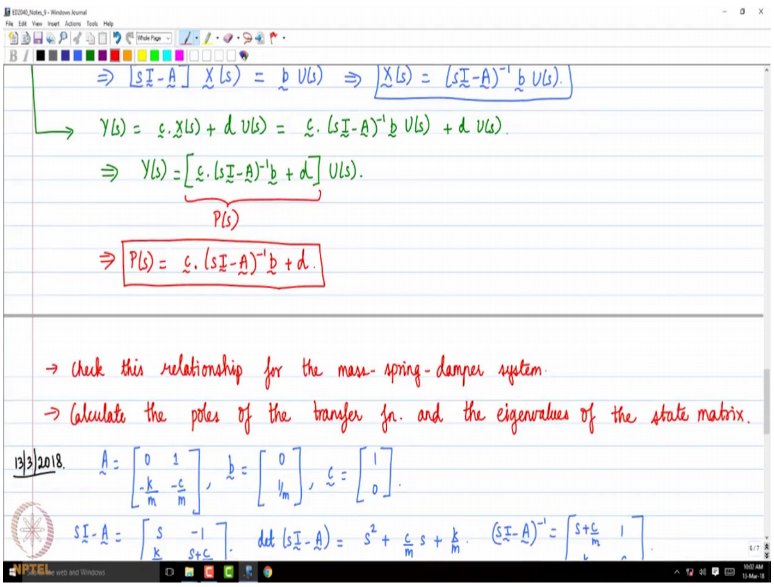

And, then we got an expression between the transfer function and the state space representation, which was this equation P of S was C dot s I minus A inverse b plus d. And I left you with two homework's, homework exercise problems right.

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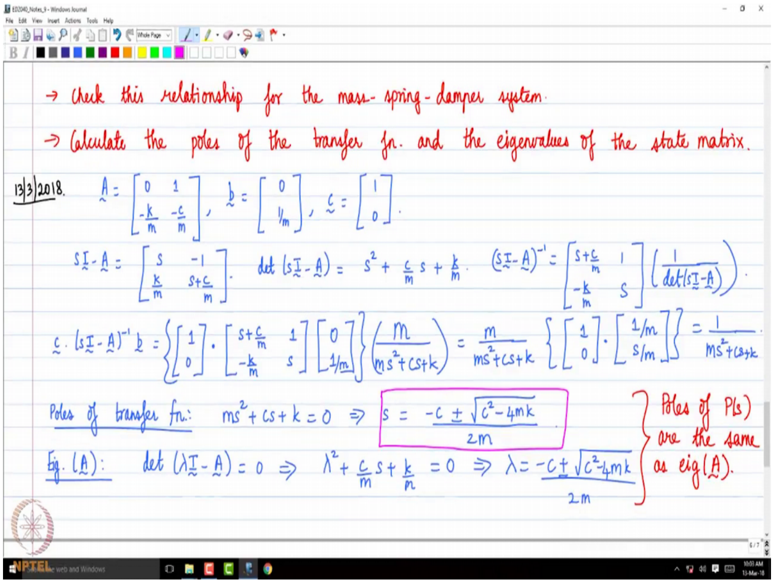

So, I asked you to first check the relationship for the mass spring damper system and that is what I just worked out here. So, this is the realization. So, the matrix A, the vector b and vector c are these for the mass spring damper system. So, these are quantities which you already derived yesterday. So, if you just work it out, calculate SI minus A and then like SI minus A inverse, of course SI minus A inverse is going to be of course this multiplied by 1 by determinant of SI minus A right, so that is the inverse ok, so that is going to be SI minus A inverse and c dot SI minus A inverse b is going to be this. And if you do the algebra, you will see that you will get the transfer function itself. So, just that it is just an exercise to verify you know like our check the formula that we derived for this particular example obviously, it must be true right.

And then like I asked you to calculate the poles of the transfer function and the eigen-values of the state matrix. So, the poles of the transfer function are these ok. So, these are the poles of the transfer function minus c plus or minus square root of c squared minus 4 mk divided by 2m. And we can see that the eigen-values of the state matrix also turn out to be the same ok. So, in fact so to generalize of course, I am not telling the complete picture, but then like by and large you know like for what are called as minimal realizations ok. The poles of the transfer function and the eigenvalues of the state matrix are the same ok.

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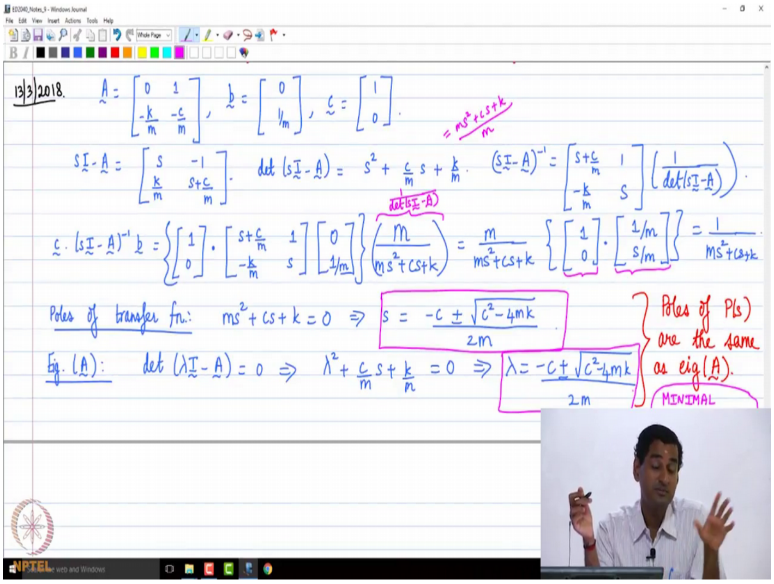

Of course, this is true for what are called as minimal realization ok. So, we will see what this is when we go to advanced courses on controls you know particularly on what is called as modern control theory or state space theory ok. Yes, please.

Student: Sir, how was (Refer Time: 03:33).

How did you do that?

Student: (Refer Time: 03:36).

How did I do? What do you want, right.

Student: (Refer Time: 03:45).

Ok.

Student: (Refer Time: 03:51).

Ok. So first thing is that like you have SI minus A inverse to be a 2 by 2 matrix, first multiply it by the vector b ok, which is a column vector 0 and 1 by m. So, if you multiply that, you get this 1 by m and S by m right. And, then you have c, which is 1 0 you take the dot product with that you will get 1 by m. See what is the dot product of 1 by 1 0 and 1 by m s by m? You will get 1 by m; 1 by m there is a m in the numerator you m and m gets cancelled right. Look at 1 by m s squared plus c s plus k ok. So, this is the determinant right.

See, what I have written here is the determinant right. So, the of course, this is the reciprocal of the determinant this is 1 by determinant of SI minus A right. So, so what I have done is that like determinant of SI minus A is s squared plus c by m s plus k m. This can be rewritten as just s square plus sorry, m s square plus c s plus k divided by m all right. So, I am just taking m as LCM. So, when you flip it, you will get m divided by m s squared plus c s plus k and the m and 1 by m gets cancelled, so that is why you get 1 by m s square plus c s plus k simple algebra ok.

Student: Sir.

Yeah.

Student: (Refer Time: 05:18).

So, that I am not touching upon here ok. So, essentially as I mentioned you know like this concept of the set of poles or the plant transfer function being the same as eigen-values of state matrix is true if the realization is minimal. So, what is a minimal realization, I preemptively give you the answer, but then like you will understand it better when you go to let us say modern control theory or an advanced course of control that deals with state space. A minimal realization is one which is both completely controllable and completely observable ok. So, then the question becomes you know what is completely controllable and completely observable. So, you will see that that is going to have a cascade effect, so we have to go deeper and deeper right.

So, essentially if you have a non-minimal realization by and large, you know like a the one would be a subset of the other ok. This two sets will be exactly the same when you have what is called as a minimal realization. So, here you see that there are two poles of the transfer function and two eigen-values of the state matrix both are the same, so the set is the same all right, so that is that is something which we can observe ok, yeah, fine.

So, but why is this important because you will immediately see that the what to say the concept of stability right you in fact like we will learn what is called as stability about an equilibrium state, when you go to state space based control. But, then you can immediately feel that for a BIBO stability, we wanted all the poles to be the left of complex plane right, if we use the transfer function approach.

So, similarly if I use the state space approach right for asymptotic stability what should I have I should have all the eigen-values of the state matrix to be in the left half complex plane, because the set is the same right. So, the task of ensuring asymptotic stability a false boils down to the location of the poles of the transfer function or the eigen-values of the state matrix ok, so that is that is that is the equivalence which I wanted to show you here. Is it clear, what was the purpose behind this particular derivation right.

So, of course, we can go deeper and deeper, but I am not going to do that here because our course is mainly focused on transfer function based analysis, but I just wanted to give you an introduction to state space and we would learn more about this in advance course of controls right ok.

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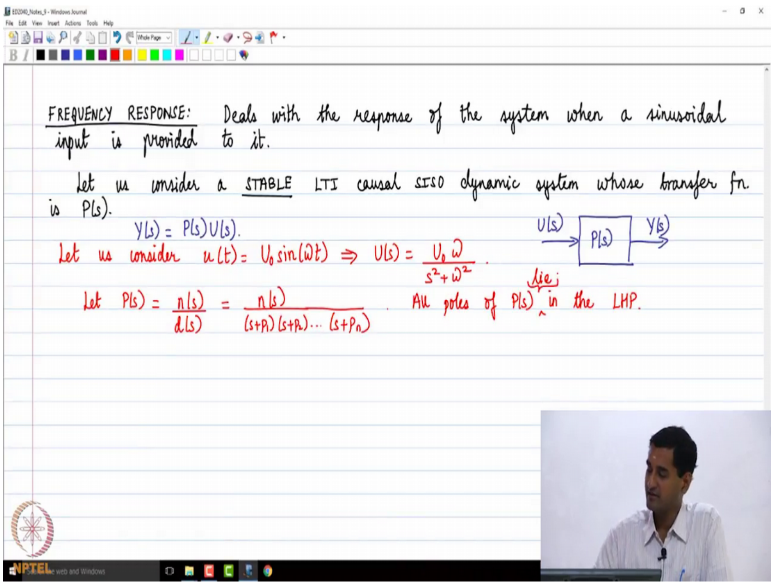

So, let me now come back to our discussion using the transfer function ok. So, we are going to discuss what is called as frequency response that is going to be something which we are going to do for the next few classes, right. We are going to discuss what is called as frequency response and how that is that can be used for control design ok, so that is going to be the objective for this particular discussion right.

So, till now you know like if you recall what we have done right, predominantly we have looked at the step response right. So, we have looked at the step response of dynamic systems to essentially come up with performance parameters like definitions like time constant, you know like rise time, settling time, peak overshoot and so on correct. So, now we are going to analyze what happens when we provide a sinusoidal input to the system and that is what frequency response deals with ok.

So, the term frequency response deals with the response of the system when a sinusoidal I am sorry, input is provided to it ok, so that is what we are going to look at. By and large we are going to look at what to say the steady state response ok, we will say why ok. So, what we will first consider is that like let us consider a stable this is very important of course, we can only talk about performance only if we have a stable system to begin with right there is something which we already discussed right.

So, let us consider a stable LTI causal SISO dynamic system whose transfer function is P of S ok. So, let us say we consider a plant or a system with a transfer function P of S. So, what do we have, so we have the system transfer function to be P of S, so we provide an input U whose Laplace transform is U of s and we get an output Y, so whose Laplace transform is Y of s right. So, immediately we know that Y of s is going to be equal to P of s times U of s ok.

So, let us consider u of t to be sum U naught sin omega t ok, so that is like we are considering a sinusoidal input, so where u of t is sum U naught sin omega t ok. So, the input has a frequency omega and a magnitude of U naught right. And consequently what is going to happen to the Laplace transform so we are going to get U of s to be equal to U naught omega divided by s squared plus omega squared ok. So, that is what we are going to have right, if I take the Laplace transform of this particular signal U naught is the magnitude all right, so that is a constant positive real number right. So, on the frequency of the input is omega so that is what we are going to have for U of s right.

And, so let us say P of s be sum n of s divided by d of s so this I can rewrite as n of s divided by let us say s plus p 1, s plus p 2, s plus p n ok. But, and what do we know about all the poles of P of s?

Student: (Refer Time: 12:34)

All poles of a P of s lie in the left of complex plane right. Why, because we are assuming a stable system to begin with right. So, all poles of P of s lie in the LHP right, so that is something which we already have presumed right in this particular discussion ok.

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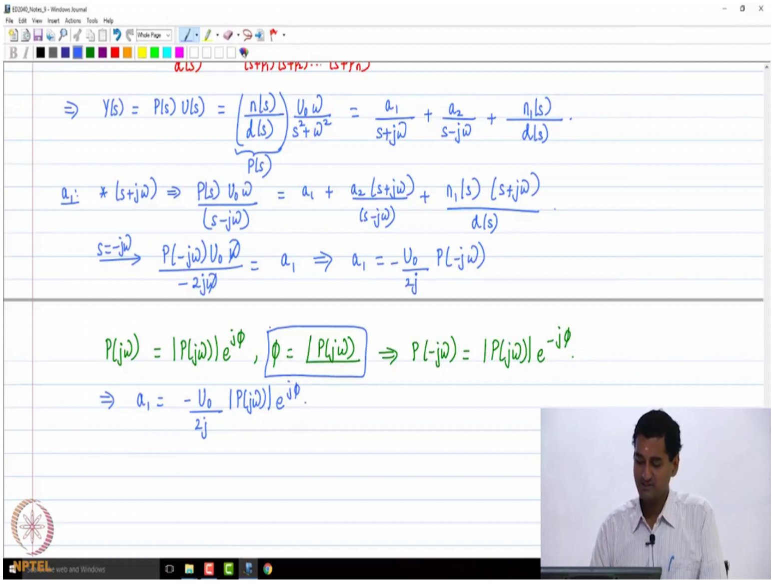

So, let us now you know like process this particular equation. So, this implies a Y of s is equal to P of s times U of s that is going to be equal to sum n of s divide d by d of s times U naught omega divided by s square plus omega square correct, so that is something which we. So, this I can rewrite the as sum a 1 divided by s plus j omega plus a 2 divided by s minus j omega plus sum n 1 of s divided by d of s ok. So, I am just start doing partial fraction expansion right.

See, because what has happened like so the plant transfer function is n of s divided by d of s then I multiply it with a second order term denominator is s squared plus omega square. So, what I am doing is that I am writing s squared plus omega squared as s plus j omega times s minus j omega I can do that right.

So, essentially then we are doing partial fraction expansion. So, I will write for s plus j omega a 1 b y s plus j omega and then s minus j omega the residue I take it as a 2, so I get a 2 by s minus j omega plus sum n 1 of s. You know whatever is left behind divided by d of s d of s is the original denominator polynomial s plus p 1 all the way till s plus p n ok, so that is my output right, so and that is essentially this ok. So, this is nothing but please note then this is P of s ok, please remember that ok.

So, if I want to find a 1 and a 2, what should I do now? Let us say I want to find a 1, what should we do, we multiply both sides by s plus j omega right. So, if I do this, what will I get, I will get P of s U naught omega divided by s minus j omega that is what will happen to the left hand side. On the right hand side, I will get a 1 a 2 times s plus j omega divided by s minus j omega plus sum n 1 of s which is left behind right times s plus j omega divided by d of s right. So, this is what I will get right, if I multiply both sides by this one.

So, now what should I do, I should substitute the root right of that factor which is s equals minus j omega, I can do this. So, then what will I get, I will get P of minus j omega U naught omega divided by what will happen to the denominator it will be minus j omega minus j omega I will get minus 2 j omega and that is going to be equal to a 1 right ok. So, this implies that the residue a 1 is going to be equal to U naught divided by 2 j right, omega and omega cancelled now times P of minus j omega ok.

So, what is P of minus j omega P of minus j omega is the plant transfer function evaluated at s equals minus j omega, obviously that is going to be a complex quantity right. So, and we know that any complex variable has a magnitude and a phase right. So, then there are various representations of a complex variable right. I can have a representation in terms of real and complex part, I can have a magnitude and a phase representation also for the same complex variable right.

So, now what we do then, if I have a complex variable P of j omega, I can rewrite this as the magnitude of P of j omega times e power j phi, where phi is the phase of P of j omega right. This is one representation of a complex variable. Do you agree? Yeah.

Student: (Refer Time: 17:38) a 1 minus (Refer Time: 17:39).

Yes, I missed it minus thank you, you are right, thank you ok, so yeah, so that is minus u naught by 2 j p of minus j omega right. So, I get this right. So, this immediately implies that if I have P of minus j omega what am I going to have that is going to be P of j omega e power minus j phi ok, so the phase just becomes negative of P of j omega all right, so that is the representation which we are going to follow all right for the complex valued function p of j omega ok.

So, once I have this what is going to happen to a 1 this implies in a 1 is going to be minus U naught divided by 2 j the magnitude of P of j omega e power j phi ok. Please remember what is phi, phi is the phase of P of j omega, obviously phi depends on omega right you vary the frequency obviously the phase also varies right so that is what we have for a 1 ok.

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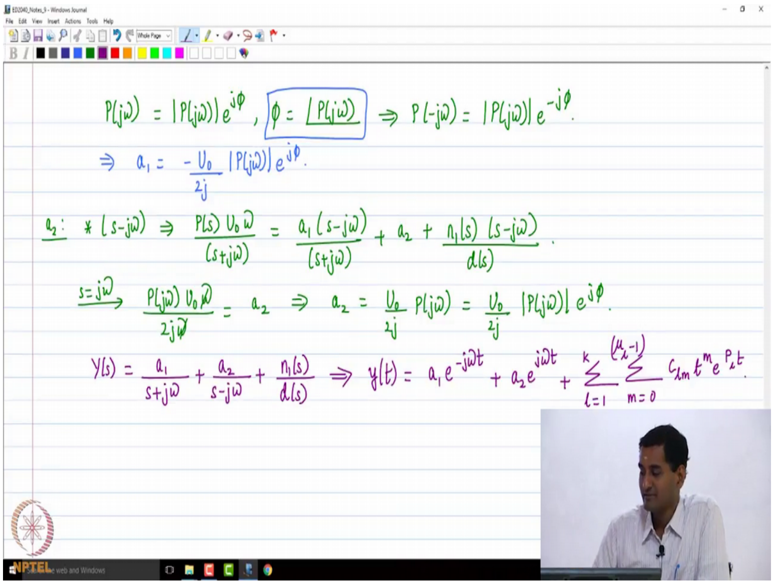

Now, similarly let us figure out a 2 right. So, let us get the expression for a 2. So, if I want the expression for a 2, what do I do, I multiply both sides by s minus j omega all right. So, this implies that I will get P of s U naught omega divided by s plus j omega that is going to be equal to a 1 plus a 2 sorry a 1 s plus sorry what is going to happen I multiplied by s minus j omega right. So, I will have a 1 s minus j omega divided by s plus j omega plus a 2 plus n 1 of s minus j omega divided by d of s all right so that is what I will have correct. I hope everyone agrees.

Now, what should I do, here I substitute s equals j omega right that is the root of this particular factor. So, what will I immediately get, I will get P of j omega U naught omega divided by 2 j omega that is going to be equal to a 2 all right. So, omega and omega will immediately cancels. So, a 2 is going to be u naught by 2 j P of j omega then I just represent P of j omega as the magnitude of P of j omega e power j phi right. So, we are almost done ok. So, we are almost there just a few more steps ok.

Now, let us go back here right. So, our Y of s is going to be a 1 divided by s plus j omega plus a 2 divided by s minus j omega plus n 1 by d of s right. So, let me write it down. So, our Y of s is going to be a 1 divided by s plus j omega plus a 2 divided by s minus j omega plus n 1 of s divided by sum d of s this immediately implies there. If I take the inverse Laplace transform, what will I get, what will be the inverse Laplace transform of the first term I will get e a 1 e power minus j omega t right correct. And then I will get for the second term a 2 e power j omega t right.

And what is going to happen to the next term you know like so essentially let there be k distinct poles, this is something which we already done right for a nth order system. Let us say let there be k distinct poles. The inverse Laplace transform of this term is going to be the following right, so this is something which we have already done, so please go back and look at that. So, let us say let there be k distinct poles with mu I being the multiplicity of each pole we have already figure out what is the inverse Laplace transform right so this is what we will have all right.

So, the main thing is that like we are going to have the poles as the exponents right. So, this is something if you recall which we did when we analyze stability, if you recall. And this is the result where we use to figure out that if my poles lie in the left of complex plane, the system is going to be BIBO stable right. So, if you recall that is what we did sometimes some classes back. So, essentially I am just going to get the same structure right, so that is very important as far as this is concerned.