Control Systems
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Lecture – 45
State Space Representation
Part – I

Let us get started with today's class right, so what we are going to do today is to learn about what is called as the State Space Representation ok. So, this is like an alternative representation of the class of dynamic systems that we are studying. So, this cause is more towards analyzing systems, when we represent them using what are called as transfer functions right.

So, the state space representation is another commonly used method to essentially do the same thing right, so as what we are doing with transfer function. So, but then the main difference is that like we stay in the time domain right rather than go going to the complex domain, as we do when we talk about the transfer function of the system ok. So, let us learn what this is and then like how one can get the state space representation and in a certain sense of ha how it is it equivalent to the transfer function that we have been looking at.

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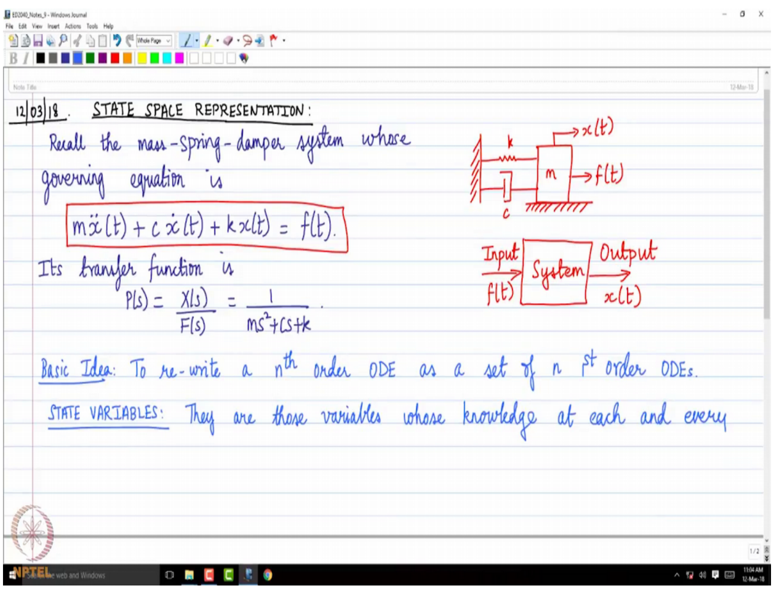

So, let us start with an example so that like just we continue from where we left off right. So, let us say I have a mass spring damper system you know like which we already are familiar with and we already know that the governing equation of the mass spring damper system is going to be mx double dot plus cx dot plus kx equals f of t ok. So, for this system or plant the input is a force f of t which we give to the mass, let us consider that the output of the system is the displacement x of t.

So, that is my input output as far as this particular system is concerned and we have already derived it is transfer function, the plant transfer function was a Laplace of the output by Laplace of the input and all the initial conditions were 0 and we already know that for this particular mass spring damper system the transfer function is 1 divided by m s square plus cs plus k ok, so all this is something we already are aware of ok. So, now what is this state space representation? So the basic idea behind the state space representation is to rewrite a n th order ODE, ODE meaning Ordinary Differential Equation right as a set of n first order ODE s ok, so that is the basic idea right behind these state space representation.

So, you give me a n th order ODE can I rewrite it as a set of n first order ODE s ok. So, that is the question we are going to ask ourselves. So, the answer is it is possible we are going to see how so to essentially do that we define we need to first define what are called state variables. So, what are state variables so for example, let me give an example let us say in this mass spring damper system you know I can consider the displacement of the mass and the speed of the mass as state variables right.

Because let us say suppose someone gives me information about the displacement of the mass and the speed of the mass at each and every instant of time, I would be able to completely characterize, what is happening to that mass right. So, in a certain sense state variables are those variables, whose knowledge at each and every instant of time can completely characterize that system ok, so that is the what is a notion of state variables ok.

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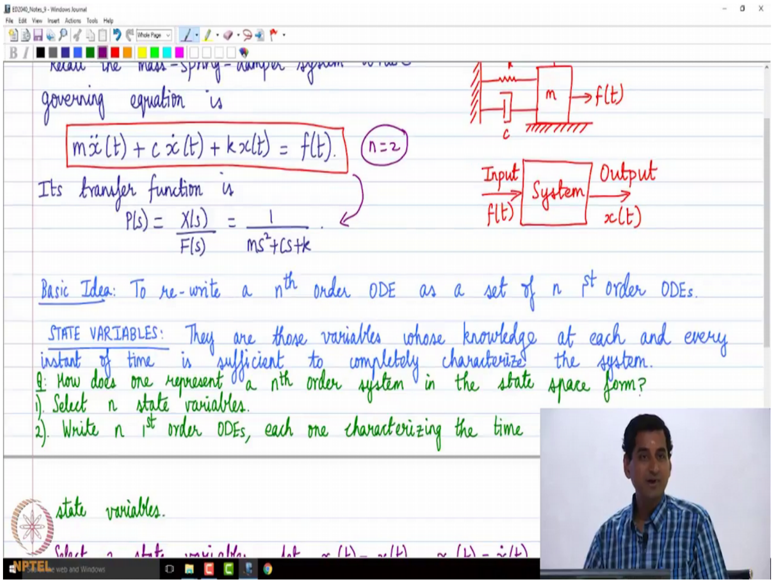

So, let me write that definition down, so state variables are those variables whose knowledge at each and every instant of time is sufficient to completely characterize the system understood, sufficient to completely characterize the system or plant. So, that is the definition of state variables ok. So, as we discussed in this particular mass spring damper system I could take the displacement of the mass and the speed of the mass right, as the 2 state variables 2 characterize a system we are going to use that to our advantage.

As we as you go deeper you know like I am only going to give you a broad introduction to state space representation, in advanced courses and controls you know like you would go deeper into state space representation and where we would realize that you know the state space representation for the same system can be non unique and by and large you know like that is used to our advantage when we do analysis ok. So, all these factors you know like would be further studied when you go to advanced courses ok, but I will briefly touch upon those once we complete our analysis today.

So, how do I get the state space representation of the system right. So, the main step is select ok, so if I want to essentially write the governing equation of any system in the state space representation what do I do right. So, first thing is that first thing is a select n state variables so because, in general we are dealing with a n th order system right. So, the question is what am I answering as far as this step is these steps are concerned, the question that we are asking ourselves is that how does one represent a n th order system in the state space form, so that is the question that we are asking ourselves right.

So, let us say we already know how to represent in the transfer function form right. So, we are giving an intersystem for the class understudy you know like we can we know that we can characterize it as using an n th order linear ODE right with constant coefficients, then we take Laplace transform then substitute all initial conditions to be 0 then we get the transfer function of the system.

So, now the question that we are asking ourselves is that like how can I do the same thing as far as the state space representation is concerned. So, you are given a n th order system the first step is to choose n state variables right. So, second step is a so write n first order ODEs each one characterizing the time evolution of the n state variables ok so, that is the second step.

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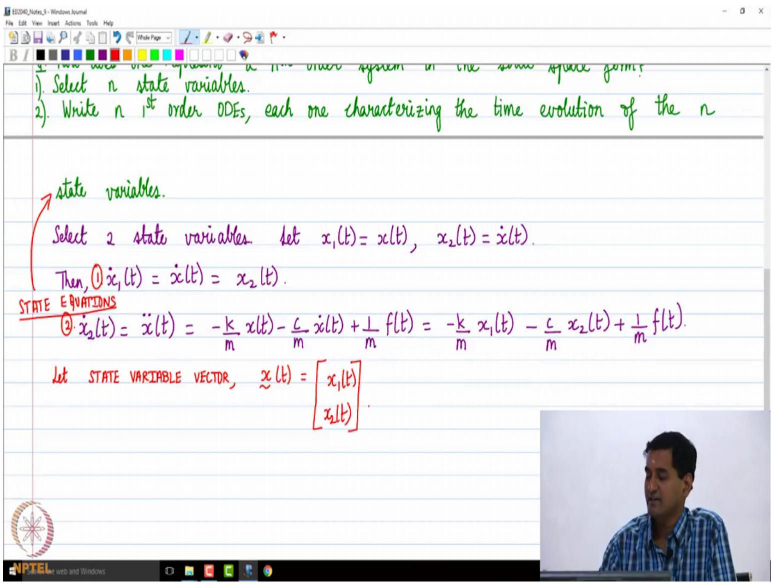

So, we have n state variables what do you is it like essentially you essentially write down n first order ODEs each one characterizing the time evolution of each state variable ok, we will understand what it is shortly so and then like we collect them. And in fact, the main mathematical tool which will be useful in the state space representation or analysis using state space representation is a linear algebra in general right. So, that is why you know like a good working knowledge of linear algebra is important ok.

So, if you want to work with trans state space representation, for transfer function we needed a good working knowledge of Laplace transform on complex variables right, for the state space we need a good understanding of linear algebra ok. So, to work see we will see why right, so let us go back to for a to an example what we do is that like we essentially select 2 state variables, why am I selecting 2 state variables because, the value of n is 2 here right, so for our mass spring damper system the value of n is 2 right.

So, that is something which we already know correct yeah, so I have to select 2 state variables, so by and large state variables are denoted by the character x ok. So, let the first state variables first state variable x 1 please do not confuse this x with the displacement x ok. So, in by enlarge is a convention to writes the state variable using the character x ok. So, the first state variable x 1 let us take it as the displacement x of t right and let us say the second state variable x 2 let us say we take it as speed x dot of t, so let us say we choose these 2 as the state variables right. So, then we can immediately observe that x 1 dot t which is the first derivative of the first state variable x 1 right, that is going to be equal to x dot of t right. If I differentiate the first state variables I want to get x dot, but what is x dot x dot is going to be equal to x 2 t right.

So, the idea is essentially to write these time evolution equations in terms of the state variables themselves and the inputs that is what that is why coming to right. So, that is what we get from the first equation, I hope it is clear what I did right. So, I defined 2 state variables as the displacement x and the speed x dot and then like what we are doing now is that like I am taking the first derivative of each state variable ok. So first state via first derivative of x 1 is x 1 dot that is x dot by our definition and x dot is nothing but x 2 once again by the choice of our state variables right.

Then we have x 2 dot, what is x 2 dot it is going to become x double dot ok. So, now I need to rewrite x double dot which is the acceleration right in terms of the state variables themselves right. How can I do that? So, of course I cannot write it as x 1 double dot right because what will happen if I do that then the order of this equation will become 2 right. So, the idea is to write what to say equations you know like which characterize a time evolution of the state variables in terms of first order ODEs right. So, how can I write for x double dot I can use the governing equation right, so yeah at some point we need to use a governing equation of motion right.

So, see how did we get the transfer function? The transfer function came from the governing equation of motion; we use the governing equation of motion for the system to derive the transfer function did we not, similarly we have to use it for the state space representation also. So, from the governing equation of motion I can write x double dot as the following, I can write x double dot as minus k divided by m x of t minus c divided by m x dot of t plus 1 divided by m f of t, am I correct I just rearranged jobs.

So, let me tell you how I got it, I use the governing equation of motion, so if I take cx dot and kx to the other side you get a negative sign and then you divide throughout by m is it alright so very simple algebra. So, if I do that I get x double dot as minus k by m x of t minus c by m x dot of t plus 1 by m f of t. Now what is x of t? It is a first state variable right, so I can re replace x of t as x 1 t and x dot of t is going to be x 2 t anyway f of t is the input I retain it as it is fine. So, these 2 equations equation 1 and equation 2 are the; what we call as the 2 state equations.

So, what are the state equations? State equations are these first order ODEs that talk about the or characterize the time evolution of each state variable ok, so that is those are what are called as state equations. So, that is what we talked about in the second step right. So, I told that we need to write the end first order ODE each one characterizing the time evolution of the n state variables, what we have done is that we have written 2 first order ODE s right for this particular example, so those are what are called as a state equation. So, state equations are essentially first order differential equations which characterize the time evolution of the state variables that is the state equation fine ok.

So, these two are the 2 state equations scalar state equations here right, now what we do is that we define let the state vector ok. So, some people call it as a state variables vector or the state vector or the vector of state variables ok. So, depending on which reference you have different terms, but essentially what we have is that we have a state vector x of t, of course my notation is it like typically vectors are denoted by a lowercase characters in bold font right. So, because since I cannot write bold font here you know what I am putting a tilde underneath just to in indicate that that quantity is a vector right, so that is the notation I am following here.

So, let us say I choose the state vector as a column vector where I arrange the 2 state variables x 1 and x 2 ok, so that is what I do ok. So, I take the state variable vector as fine.

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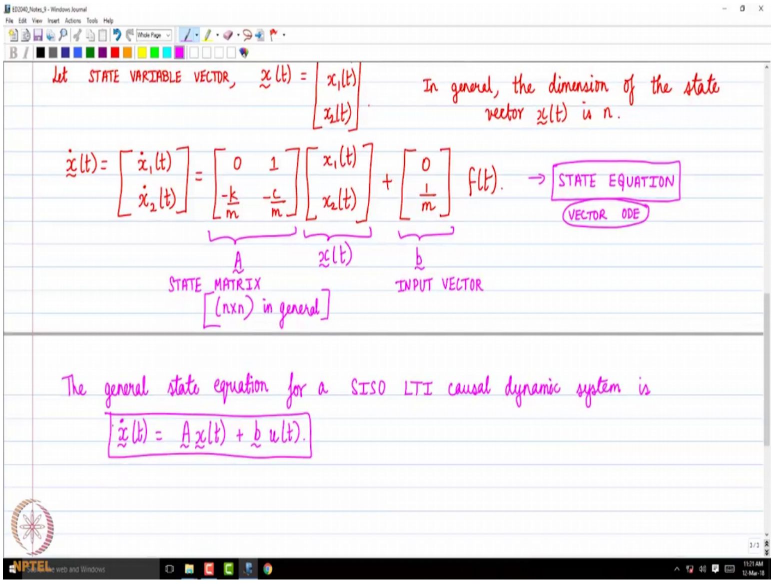

Now, this essentially tells me that x dot which is a first derivative of the state vector right, it is essentially x 1 dot x 2 dot that is nothing but something multiplying x 1 t x 2 t plus something multiplying the input f of t, we will see that typically we can write in this form.

Let me show you how to write in this form ok, so what I have done is that I have corrected the n state variables as a n th order state vector ok. So, in general the dimension of x of t is n right, the dimension of the state vector x of t is n ok. So, that that is easy to see right because you just arrange the n state variables in this case, it is 2 we have 2 stable variables.

So, now what is x 1 dot, x 1 dot is x 2, so I can immediately say that I can write the first equation in this way right if I say you know like in the first term which multiplies x 1 and x 2. Let us say it right the first row is 0 1 and in the term multiplying the input f of t I write 0 as a first entry, if you do the algebra do you get x 1 dot to be equal to x 2 we do right. So, the row 0 1 multiplies the column x 1 x 2 you get x 2 right and 0 multiplying f anyway will not contribute anything additional right, so we are getting the first state equation.

So, then how will I get x 2 dot I can rewrite this as minus k by m minus c by m 1 by m right, if I do this can I get the second equation I do right; you multiply the second row with the column state vector and then 1 by m multiplies f of t you win the second equation ok. So, this is what is called as a state equation of course, we already define what are called state equation right. So, what is called as a state equation? It is essentially an equation that gives me the time evolution it is a first order ODE which gives me a time evolution of the state variables right. But this state equation is a vector state equation right because, my x is a vector, the state vector is a vector so this is an vector ODE alright. So, this is a vector ODE right so that is what we have got as a state equation.

So, in general what happens is that this column vector is my state vector x of t and that is typically multiplied by a matrix A which is called as the state matrix. So, in general the dimension of the state matrix will be n by n in general the size of the state matrix would be n by n in general ok. So, for SISO systems this vector is denoted by a vector b which is called as a input vector ok. So, that is the input vector which is denoted by lowercase b.

So, the general state equation for a SISO LTI causal dynamic system is you get x dot which is the state vectors first derivative that is going to be written as A x t plus b times u t ok, that is the state equation in general ok. So, this is the state equation, so what does the state equation you know like some points to not state equation please know that it first of all it is a vector ODE. But a first order vector ODE right and what does it tell me, you know it tells me how the state vector would evolve with respect to time ok. In relation to the input that we are providing to the system because, extra t is a time a time derivative of x that is related to the state vector itself and the input provided to the system.

So, you have some initial condition for the state and you give me an input to the state system sorry, then the question becomes you know like what happens to the state variables right with time. So, that is what the state equation will tell right. So but we are not done yet right because ultimately you know like far us our visualization of a system or a plant is an entity to which we give an input and we get an output right, we are not even looked at the output yet right. So, this is the state equation, this is only the first equation in the state space representation.

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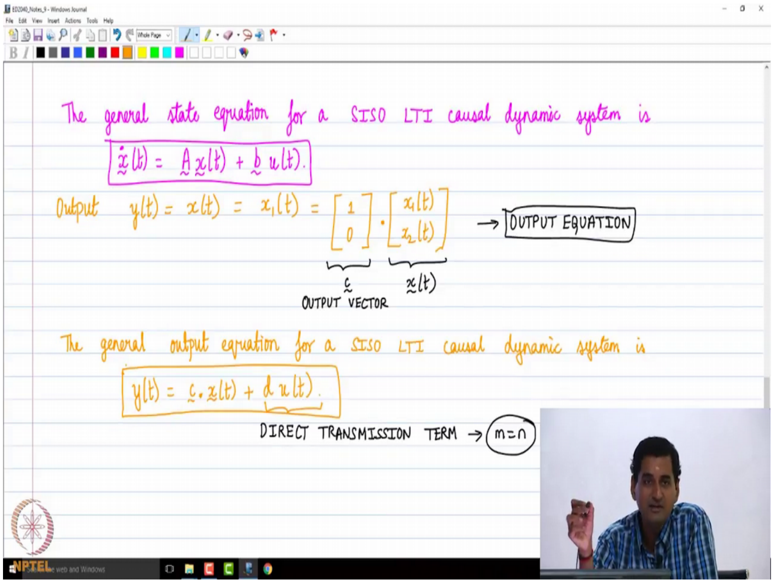

Now, what is our output in this case let us say we call the output as value y of t, for our example the output is a displacement x right. So, what is x in terms of the state variable it is x 1 correct, so how can I get x 1 from the state vector ok. So, in this example my output is a displacement x which is the first state variable, suppose I want to generalize right and rewrite it as what to say some function of the state vector itself right. How can I extract x 1 from the state vector x 1 x 2, if I take the dot product of 1 0 with x 1 and x 2 do I get x 1 right correct do you agree.

So, we immediately see that my output is equal to the first state variable in this example, which can be obtained as a dot product of a vector and the state vector ok. I choose the first vector 1 0 appropriately depending on what my output is right. So, since we are dealing with a linear system the output is going to be a linear combination of the state variables ok. So, the out the vector which with which I take the dot product will be appropriately varying in this example it is 1 0 ok. So, this is what is called as the output equation, this vector is denoted by a vector c which is called as a output vector and this is my anyway my vector x of t alright.

So, this equation is what is called as a output equation ok, this is what is called as a output equation. So, similarly the general output equation for a SISO LTI causal dynamic system is going to be y of t is equal to c dot x of t plus d u t. What is this d u t, c dot x of t we are already seen right c is output vector, x is the state vector, what is this d u t term?

d u t is what is called as a direct transmission term which comes in the output equation then m is equal to n.

We get this term when m is equal to n do you know what are m and n, if you recall the general governing equation of the class of system under study the ODE, n was the highest deri n is the highest derivative the output on the left hand side right, m has the highest derivative of the input term right. So, in this example m is 0 n is 2, so we did not get a direct transmission term in the output equation for this example.

But in general this is going to be the output equation right y the y of t is equal to c dot x of t plus d times u of t ok, that is going to be the equation in general.

Student: (Refer Time: 27:40).

Of which one?

Student: Direct (Refer Time: 27:45).

Direct transmission term because, typically what will happen is that when you have m is equal to n the state vector will be affected by the input that you are getting. But the output term b in addition b to being influenced by the state vector there will be one component which will be directly influenced by the output input.

So, we will do an example I will maybe give you an example, so that you can understand it better right; see what is going to happen is that if you are having m equals n think about the transfer function approach right. So, we did not do any case where m is equal to n yet alright. But let us say if m where equal to n for any transfer function what do you think will happen you know when you take the let us say partial fraction expansion and inverse Laplace transform, you will have 1 constant right which will come in ok.

Because, why you are dividing let us say in the numerator you are a second order polynomial denominator also you will have a second order polynomial, if you recall de division of polynomials you can have a remainder right a quotient and remainder right that quotient will be a constant right and then you will have a remainder. So, that quotient will automatically multiply the input, so you will have one term which will directly come from the input, other term will be from the dynamics of the system so that is a physics right.

Similarly, here the direct transmission term will essentially imply the term that is directly affecting the output through the input right and c dot x comes from the dynamics of the system ok, so that is the interpretation that we have ok. So, this is the output equation in general, of course at this point one aside which I should say is that like almost all textbooks references in controls would write this output equation as y of t is equal to c transpose x of t plus a d u of t.

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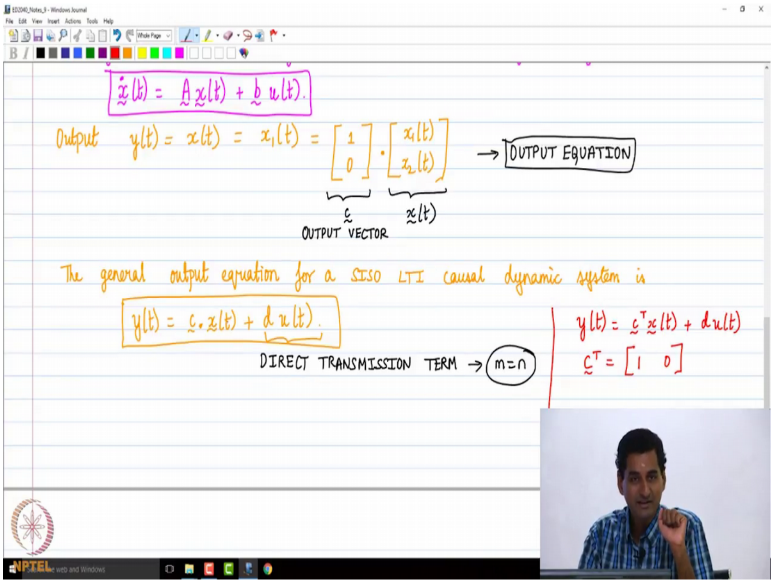

Because what they will do is that like they will take the output vector to be a row vector. So, that like it is like a matrix operation right you multiply a what is a row with a column and then you get it, you are free to choose any notation that you want to follow. So, if you chose that notation you know like you will have c transpose to be a 1 0 ok, rather than I wrote it as a column vector c as a column vector 1 0 and put a dot product you know like. If you want to follow this notation which is there and almost all textbooks for, so state space representation yeah you can also follow that absolutely.

Student: Sir d is a vector?

d is a scalar for SISO system right because ultimately we are dealing with a SISO system right. So, essentially y is a scalar so; obviously, on the right hand side also you should have a scalar right, so u is a scalar so d is a scalar and c is a vector x is a vector you take dot product you will get a scalar right scalar value function. So, you can see that it is the terms agree dimensional right, so that is something which is very important.