Control Systems
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Lecture - 38
Root Locus 3
Part – 2

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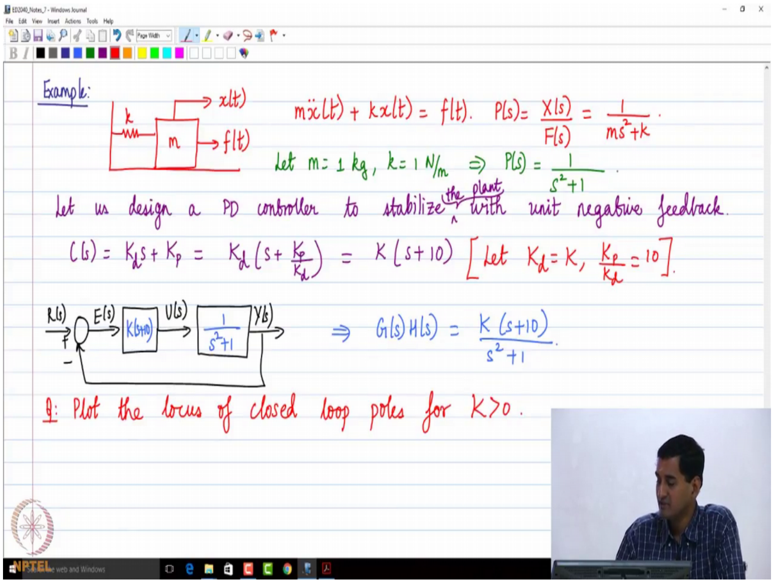

So, now I am going to do an example where we just to begin with we consider a mass spring system ok. Let us say you know I apply a force f of t and let us say we get displacement x of t the spring being some k ok. Let us say we have a frictionless surface right. So, we already know that the governing equation is going to be m x double dot plus k x t is equal to f of t ok. The plant transfer function is going to be X of s divided by F of s is going to be 1 divided by m s square plus k right; this is these are things which we already know right.

So, just for simplicity let m be 1 kilogram and k be 1 Newton per meter just for simplicity right. So, this implies that the plant transfer function is going to be equal to 1 divided by s square plus 1 right. So, the poles of the plant transfer function are going to be a plus and minus j. So, is this system BIBO stable? Yes, no. So, what did we discuss you know like if you have non-repeating poles on the imaginary axis right, for one class of inputs the system output is going to blow off to infinity right. So, in this example for what input your system output will go to infinity in magnitude cos t and sin t right.

So, for imaginary roots if you have a sinusoidal input whose angular frequency is going to be the magnitude of the pole, however, output magnitude goes to infinity right as t tends to infinity; that is what we have right. So, so, we want to stabilize the system right. So, let us design a PD controller ok, like if you recall we did one problem like this a proportional controller does not work. So, let us we saw that PD controller works for such cases, if you recall from what we did previously to stabilize the closed loop system right stabilize with negative feedback ok; let us say with the unit negative feedback ok, stabilize the plant for unit negative feedback ok.

So that means, that C of s is going to be equal to K d s plus K p right. So, let us say you know I pull K d common out of the term so, I get s plus K p by K d alright ok. So, let us consider the ratio of K p by K d as 10 just for an example and let us say I call K d as K. You know so, that I do not need to write the subscript d all the time right. So, what we have done? So, let K d be denoted by K and K p by K d be equal to 10.

So, if we were to draw the block diagram, what will what would that be. So, I have a plant ok, we have an input and an output, we have a controller right, you have a summing junction. We have a reference input R of s, this is E of s, this is U of s, this is Y of s and then we have unit negative feedback right. So, what are the what is the plant transfer function? 1 divided by s square plus 1. What is the controller transfer function? K times s plus 10. So, this implies that what is the open loop transfer function? K times s plus 10 divided by divided by s square plus 1 ok.

Now, the question that we are asking ourselves is a plot the locus of closed loop poles ok, for K greater than 0 that is the question we need to answer ok. I hope it is clear how we are formulating it has a root locus problem right. So, this will be pretty useful for us right because, I want to see how the root locus varies right. The closed loop poles vary as a change a controller parameter. See previously using Routh's criteria what we did, we you only got a ranges of the controller parameters that stabilized the closed loop system. But now, I want to look at more things right. I want to see how the branches troubles in the complex plane has vary a parameter so, that I can design my controller in a better manner right. So, that is the advantage of a root locus ok that is how we are going to use it for control design ok.

So, I am taking very simple examples so, that we can work the mode by hand right. So, see if I take a higher order system you know hand calculations are going to become more challenging ok; that is why I am taking second order systems to a illustrate the point. So, let us run through the steps ok, like you tell me what would be step 1.

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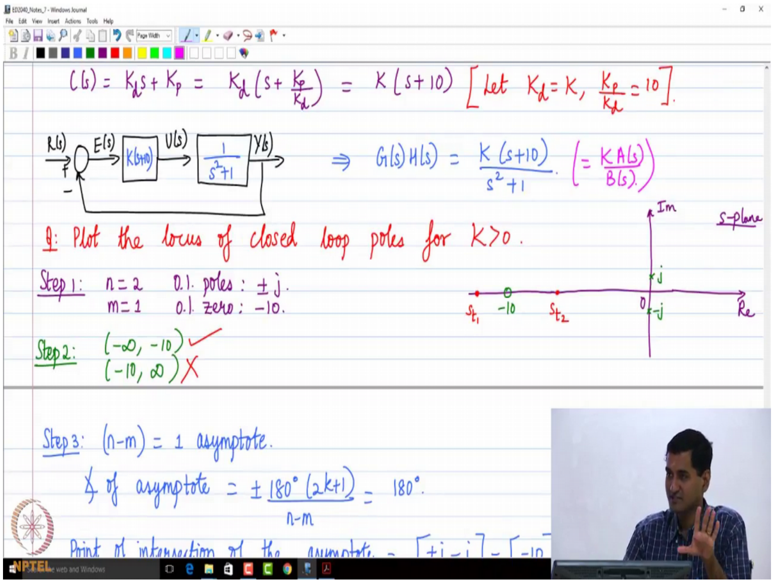

What is step 1? So, step 1 is to first determine the or locate the open loop poles and open loop zeros. So, in this example the value of n is 2. What is the value of m? 1 right. And what are the open loop poles? Plus or minus j and we have an open loop zero at minus.

Student: 10.

10. So, first let us let us mark those ok. So, let me first extend this 2 minus 10. So, let us say you know like I have a 0 at minus 10 and then I have an open loop pole at plus j and minus j ok. Those are my open loop poles and open loop zeros fine.

Now, what is step 2? What is step 2? Step 2 is locate the part of the real axis that lying on the root locus. So, if you look at the real axis are there any open loop poles or open loop zeros on it, there is one right. There are no open loop poles on the real axis, but there is an open loop zero on the real axis. And that cuts the real axis into two sub-regions right. What are they? Minus infinity to minus 10 and minus 10 to plus infinity, right. Now, the question is that like which which part lies on the root locus stood to figure it out what will I do, I choose a test point s t 1 here and I look to the right.

How many open loop poles and open loop zeros do I have? 1, which is an odd number. So, minus infinity to minus 10 lies on the root locus. What about minus 10 to plus infinity? I look to the right, let us say I take a test point s t 2 in the region from minus 10 to infinity I look to the right and there are 0 open loop poles and open loop zeros right to the right of it. So, this region does not lie on the root locus.

So, the only region that lies on the root locus is the region between minus 10 to sorry minus infinity to minus 10, right so, there is a region. What is step 3? Asymptotes right so, there will be n minus m asymptotes in this case n minus m is 1. So, there is going to be 1 asymptote ok. And so, what is the angle of the asymptote? That is going to be equal to plus or minus 180 degree times 2 k plus 1 divided by n minus m ok.

So, that essentially becomes 180 degrees right because I do not have any other, it is right n minus m is only 1. So, I need only 1 angle which is 180 degrees right. You can call it plus 180 or minus 180, anyway the angle will be the same. Now, the question is that like where is asymptotes. See minus 180 degree means, I can have lines parallel to the negative real axis. But where in the s plane, that will be answered by the point of intersection although, it is trivial you know like let us let us do this right.

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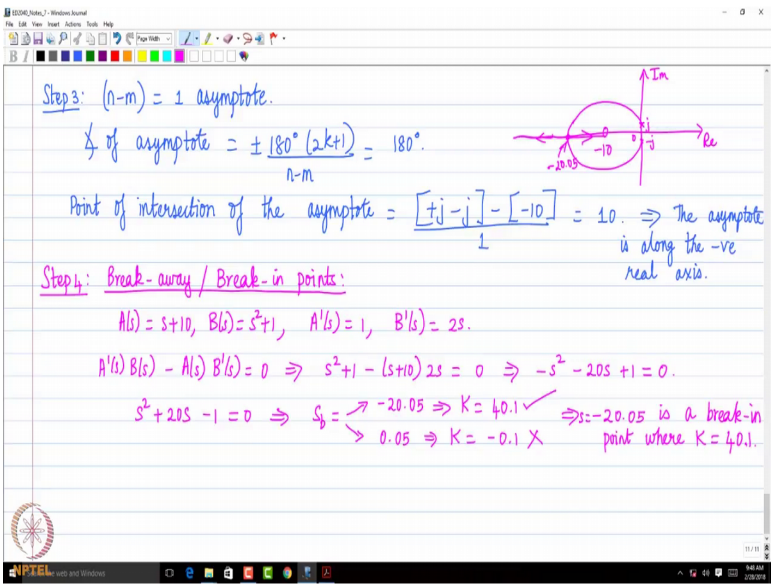

So, point of intersection of the asymptote with the negative real axis is going to be, what was that? Sum of open loop poles minus sum of open loop zeros divided by n minus m right. So, open loop poles are plus n minus j minus sum of open loop zeros that is going to be at minus 10 divided by n minus m. So, what are we going to get? Plus 10.

So, what does this really mean? Does it really means that the negative real axis is the asymptotes right? See the asymptote goes along the negative real axis instead of any line parallel to the negative real axis, the negative real axis is a asymptote right. So, the asymptotes is along the negative real axis that is what oops this analysis can control ok. So, that is step 3.

Now, what is step 4? What is step 4? Break-away or break-in points so, let us let us look at them. So, how do we calculate break-away, break-in points? We look at the open loop transfer function, right, the open loop transfer function is of the form K times A of s divided by B of s right. So, what is A of s here? s plus 10, B of s is s square plus 1 right. See as I told you just need to follow the steps ok; if you follow the steps methodically you know like we will be able to get the root locus right. So, that is it ok. So, A of s is going to be s plus 10, B of s is going to be s square plus 1.

Student: How do we (Refer Time: 12:45) negative (Refer Time: 12:45)?.

Negative real axis because the point of intersection of the asymptotes with the real axis is plus 10. So, what is the only line which will intersect the real axis at plus 10 and still have an angle of 180 degrees with the positive real axis? So, the asymptote has to necessarily go along the negative real axis right.

Student: Because the other locus (Refer Time: 13:07). So, it is (Refer Time: 13:11).

See, what is an asymptote? The asymptote is indicates the line along which the root locus will tend to as it goes to infinity right, that is the meaning of asymptotes. Now, when we say the asymptote makes an angle of 180 degrees with the positive real axis and intersects the real axis at plus 10; the only choice is the real axis itself right. So, only the real axis only if the asymptote is along the negative real axis you know like you can satisfy these two conditions right. So, that is why the asymptote goes along the negative real axis right.

Student: Plus minus tend to be angle.

Yeah, plus minus both are the same right, I just took one that is it right. You can take plus minus 180 degrees plus minus 540 degrees does not matter ok, you will get the same direction right. So, here so on ok. So, A of s is s plus 10, B of s is s squared plus 1. So, consequently A prime s is going to be 1, B prime s is going to be 2 s. And what is the equation that we need to solve for break-away points? It is going to be A prime s B of s minus A of s B prime s equal to 0. So, this would give me s squared plus 1 minus A of s is going to be s plus 10 times 2 s; this is going to be 0.

So, this will give us what I will have minus s square then minus 20 s plus 1 equals 0, right. Am I correct? So, can you calculate the roots of this equation and let me know. So, I can rewrite this as s squared plus 20 s minus 1 equals 0. So, this implies that the potential break-away points are going to be what; there are going to be 2 roots right. What are the 2 roots? You will get the 2 roots as minus 20.05 and 0.05 ok, please check it independently, ok.

So, s b equals minus 20.05 would imply a K of 40.1 and s b equals 0.05 implies a K of minus 0.1, right. How do we get this? K of s is equal to minus B of s divided by A of s right, if you solve that you will get this. So, by looking at this which is a break-away point, this one right this is not. So, minus 20.05 s equals minus 20.05 is a, is it a break-away point or a break-in point? In this case it is a break-in point, I will explain why, where K is going to be equal to 40.1; that is why that is what we will get from this analysis ok.

Now, the question becomes why is this a break-in point because, you can see that see I am going have two branches starting from plus j and minus j ok. They are going to be complex conjugates right and ultimately one branch is going to come and terminate at minus 10 right, because that is one open loop zero. Another branch is going to go to infinity along the negative real axis that is to minus infinity along the negative real axis. So obviously, both points should intersect somewhere on the real axis right that is going to be at minus 20.05. So, two branches will come from the complex conjugate plane, hit at minus 20.05 and then go to one to the right one, one to the left ok. So, that is what is going to happen.

So, in this case what is going to happen is the following. So, you have an open loop pole, open loop zero at minus 10 ok; let us say I am not drawing to scale, but you get the idea right. So, and what is going to happen is it your you are going to have two branches which are going to start from the two open loop poles and come and hit the negative real axis at minus 20.05. And then one branch will go like this and one branch will go like this ok, that is why it is called as a break-in point right because it breaks into the real axis ok. So, that is why it is a break-in point ok; is it clear why we that becomes a break-in point.

Now, what is the next step? 5th step. What is step 5? It is to calculate the angle of departure right. So, in this case we need to calculate the angle of departure at because, we have a bunch pair of complex open loop poles right.

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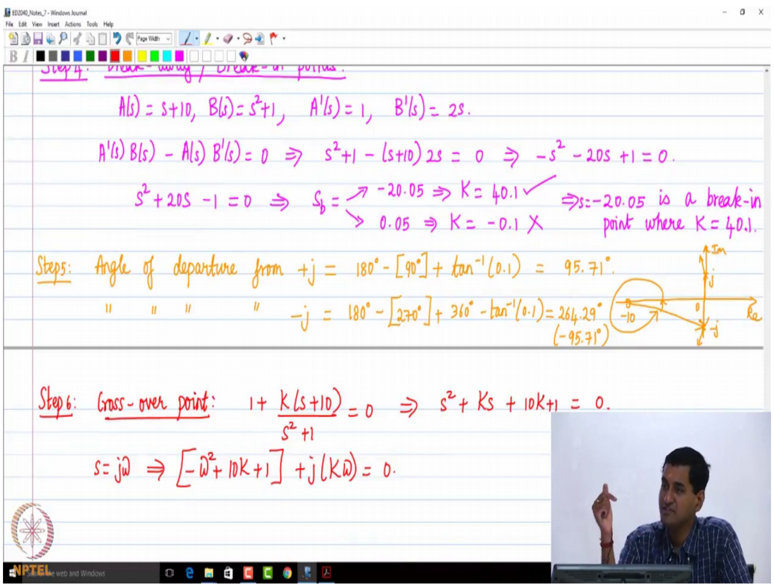

So, let us calculate the angle of departure from plus j. What is the formula for angle of departure from plus from an open loop pole? It is 180 degrees minus the sum of the angles made by the vectors from other open loop poles to this open loop pole right. So, if you look at what is the only other open loop pole, see we are considering plus j right. What is the only other open loop pole?

Student: Minus.

Minus j so, if I draw a vector from minus j to plus j; what is the angle made by that vector 90 degrees. So, I just write 90 there are no other open loop poles plus some of the vectors angles made by the vectors drawn from open loop zeros to this open loop pole. What is the only open loop zero here? Minus 10, you draw a vector from minus 10 to plus j. And what is the angle made by this vector?

Student: Tan inverse of 1 by 10.

Tan inverse of.

Student: 1 by 10.

1 by 10 and if you calculate tan inverse of 1 by 10, so, this let me write it as tan inverse of 0.1 ok, tan inverse of 0.1 and if you evaluate this this is going to come as, the answer is going to come as 95.71 degrees that is what you will get. So, this is the angle of departure from the plus j port. So, what it really means is that so, if I consider the s plane once again and if I consider the pole at plus j; the root locus is going to depart at an angle which is 95.71, right. So, then what can you say about minus j, it is going to be symmetric right, it is a mirror image. Why? Because you can only have complex conjugate poles right. So, you can immediately figure it out that the angle of departure from minus j is going to be either 360 minus this or minus 95.71. But anyway I will calculate so, that like we will convince ourselves right. So, you have a 0 at minus 10 now, we are looking at minus j right.

So, what is the angle made by the vector drawn from the other open loop poles to this pole? It is going to be minus 90 or 270 you can write it either way, let us write it as 270, right. Then what is the angle made by the vector drawn from the open loop zero to this pole? It is going to be this angle right, which I can write as 360 minus tan inverse 0.1, right. Because, I want to look at this angle right I take angles to be positive counter clockwise from the positive real axis. You can you can write it as either positive thing or negative minus tan inverse of 0.1.

So, if you calculate this you will get the answer as 264.29 which is the same as calling it as minus 95.71. So, I am just doing this to convince you that when you have a complex conjugate open loop pair the angles of the departure are just going to be negatives of each other or this they will sum to and even multiple of pi alright. So, that is that is important ok. So, they are just symmetrical about the mirror images about the real axis ok. So, that is something which we need to remember ok.

So, what I want you to do is do the final step and then like construct the root locus right final step 6 is the cross-over point right. So, how do we get the cross-over point? Consider the characteristic equation right, close loop characteristic equation is going to be 1 plus K times s plus 10 divided by s squared plus 1 equals 0. So, this is going to be s square plus K s plus 10 plus K. Am I correct? So.

Student: 10 K.

ah 10 sorry yeah sorry 10 K plus 1 right yeah. So, if I substitute s equals j omega in this equation. What am I going to get? I am going to get minus omega squared plus 10 K plus 1 plus j times K omega is equal to sorry j times; did I do something wrong here. Is this correct? Right j times K omega equals 0. So now, if you equate both real parts and imaginary parts to 0, what do you get? See K omega equals 0 means K can be 0, omega can be 0 or both can be 0. K equals 0 means you know like see this will give me two cases; if K is 0 from the real part I will get omega to be what plus or minus 1. So, plus or minus j is anyway the open loop poles that is where I start when K is 0 right, I get the same answer.

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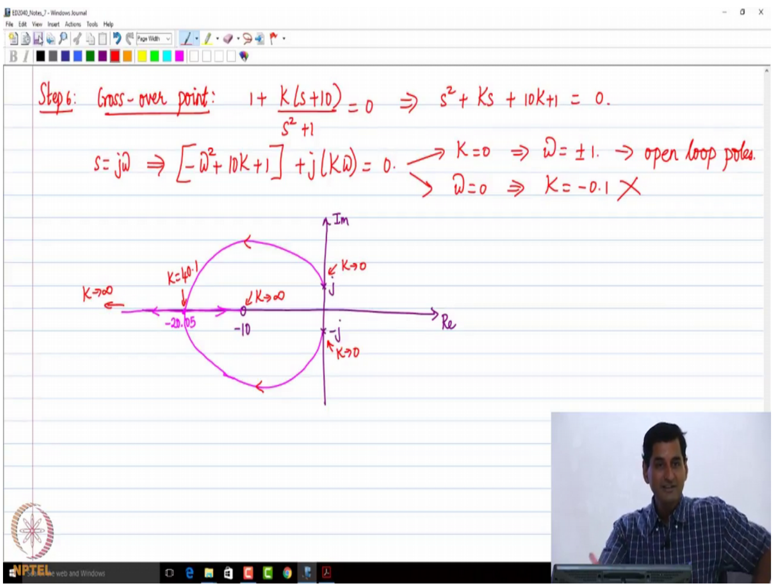

Now, omega equals 0 implies K is going to be equal to minus 0.1 which is not to say under consideration because, we do not we are not constant K to be positive. So, anyway these are the open loop poles right plus or minus j. So, in essence there are no further cross-over points rather than the open loop poles. So, the root locus starts from the open loop poles and it remains in the left half plane always; for all K greater than 0 ok. So, that is what we can figure out ok. So, I will quickly draw the root locus like you can go back and think as to how it came in this shape. And we will start our discussion from this point in the next class ok.

So, let me quickly finish there and then we will stop ok. So, the open loop poles are at plus j and minus j and we have an open loop zero at minus 10. So, what is going to happen to the root locus is that we are going to have two branches ok; which are going to come like this of course, they are mirror images of each other ok. They intersect at minus 20.05 then I have one branch going like this, another branch going like this ok. So, here K tends to 0 are the two open loop poles. The value of K at the break-in point was 40.01. So, the direction is like this here and it comes to an open loop 0 at K as K tends to infinity about this cases ok. So, this is the root locus.

Why should it take this shape? As you as you told you know like with experience you will get it. You know I can also, please note that the closed loop characteristic equation is a second order polynomial right. So, it will take this particular shape ok. So, we are see we can reproduce the exact shape by hand calculation, but it is you know like you draw an approximate shape ok, marking the critical points right ok. So, I will stop here I know like we discussed this once again on in the next class and then we will continue from here right. But please think about the other two questions and come to the class for Monday ok.

Thank you.