Control Systems
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Lecture – 12
BIBO Stability
Part – 2

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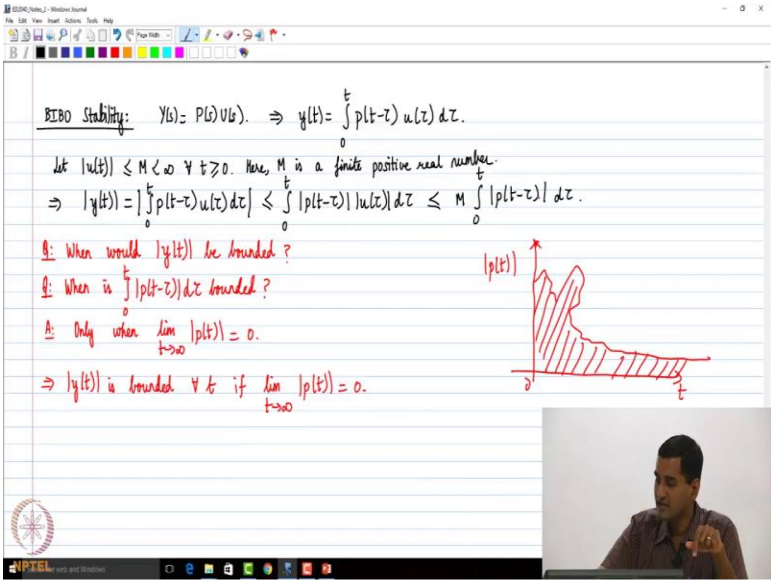

So, let us look at BIBO stability in some detail. Any questions on the examples and the general conclusions which I drew from them? So, I am going to prove one of them ok. So, that that is my intention now; yes any questions? Anyone? Ok, so ok. So, let us let us look at BIBO stability right. So, we know that $Y(s) = P(s)U(s)$ right. So, you give me any plant transfer function, the output to an input U is going to be P(s) times U(s).

This implies that $y(t) = \int_0^t p(t-\tau)u(\tau)d\tau$. So, I am using the convolution integral ok. So, let $|u(t)| \leq M < \infty \; \forall \; t \geq 0$ .Then what will happen?

$$|y(t)| = \left| \int_0^t p(t-\tau)u(\tau)d\tau \right| \leq \int_0^t |p(t-\tau)u(\tau)|d\tau$$

How did I get this? It is a property of the integral right. So just to give an analogy if you have, let us say two scalars A and B and you add A + B right.

Let us say A and B are real numbers right. So, $|A + B| \leq |A| + |B|$, what we call as triangle inequality right, so a similar thing right. So, because you have an integral and you have the integrand, first on the left hand side I am taking the absolute value of the entire integral right and that is going to be less than or equal to the integral of the absolute value of the integrand ok. So, that is what we are doing right ok.

Now, this is going to be

$$|y(t)| = \left|\int_0^t p(t-\tau)u(\tau)d\tau\right| \leq \int_0^t |p(t-\tau)u(\tau)|d\tau \leq M\int_0^t |p(t-\tau)|d\tau$$

Student: (Refer Time: 03:07).

Yes.

Student: M is greater than 0.

Yeah M is greater than 0 right.

Student: (Refer Time: 03:13).

Sorry.

Student: M is greater than 0, is not (Refer Time: 03:17).

Yeah this is a magnitude right. Here M is a finite positive real number. Anyway I am taking the absolute value right; so, absolute value of u(t) right. So, it is going to be a positive number right. So I hope it is clear.

So, now, please remember what was p(t)? It was what is called as the impulse response function right, if you recall the physical meaning of p(t) right. So, question becomes, the question that we would ask ourselves is that, when would the magnitude of y(t) be bounded, given this expression?

So, given this expression when do you think the magnitude of y(t) will be bounded? See the magnitude of y(t) is bounded by this integral $M\int_0^t |p(t-\tau)|d\tau$. Now suppose let us say this integral is bounded right, certainly the magnitude of y(t) is bounded. Don't you agree?

See because $|y(t)| \leq M \int_0^t |p(t-\tau)|d\tau$, the follow up question that we would ask is that when is $\int_0^t |p(t-\tau)|d\tau$ bounded? Correct?

So, when do you think, the integral of a non negative integrand would be bounded? Why am I saying nonnegative integrand? Because I am taking the absolute value of $p(t-\tau)$ right; so, $p(t-\tau)$ anyway can be a real valued function right. It can take positive or negative values, but then once I take the absolute value it is going to be nonnegative.

Why am I saying non negative? It can be greater than 0 or equal to 0 also right. So, $p(t-\tau)$ the absolute value of it is going to be a nonnegative function. So, I am taking the integral of a nonnegative integrand. When do you think this integral will be bounded?

Student: (Refer Time: 06:12).

Only when $\lim_{t\to\infty}|p(t)| = 0$. See otherwise if this limit does not go to 0; See integration is like sum right if p(t) does not go to 0 right what do you have? What happens if you keep on integrating the number? Is only going to increase right.

So, for example, let me just give you a simple illustration right. So, let us say the magnitude of p(t) goes something like this and then like it settles it down to a nonnegative value what is going to happen? Integral is just the area, right.

So, what will happen is, although the magnitude of p(t) is bounded, the integral is going to become unbounded as t tends to infinity, what is the only scenario when the integral will be bounded, when the area will be bounded? Only when the function itself goes to 0 as t tends to.

Student: Infinity.

Infinity right. So, this is the condition. So, we can immediately see that this implies a magnitude of y(t) is bounded for all time, if $\lim_{t\to\infty}|p(t)| = 0$, ok. So, this is the sufficient condition for BIBO stability right. It says that, look you give me the impulse response and if the impulse response decays to 0 as time t tends to infinity then my system is BIBO stable ok, but then this is a good concept, but still we are not there right.

So, because if I want to test it in test a system using impulse response, we know the difficulty that we would face right, because it is very difficult to generate an ideal impulse right. So, then how do I get the impulse response and how do I test the system for its stability ok.

So, that is going to be a question right. So, can there be an equivalent criteria for this? Yes there is and it is based on poles ok. So, that is what we are going to look at. So, let us see how we get an equivalent criteria ok. We are only done with half the derivation right, I hope it is clear till this point right.

We want to figure out when the system will be BIBO stable for a bounded input right. For any bounded input when will the output be bounded? We figure out that the output is bounded if the, what to say, the limit of the magnitude of the impulse response goes to 0 as t tends to infinity ok. Now, let us go further alright ok.

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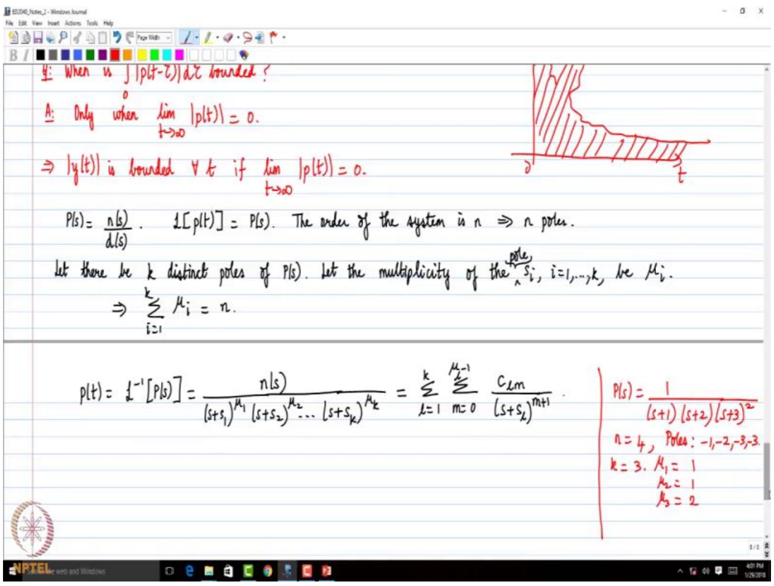

We already know that $P(s) = \frac{n(s)}{d(s)}$ right, this is the plant transfer function alright and the Laplace of p(t) is going to be equal to P(s), this is also something which we know right. So, let there be k distinct poles of P(s).

Let the multiplicity of the pole $s_i, i = 1, \dots, k\ be\ \mu_i$ ok. So, the system order is n ok. So, the order of the system is anyway n right; that is a general notation, anyway we are following right the order of system is n. So, this implies that there are going to be n poles right of the transfer function; that is something which we already know right, we are dealing with a generic case, nth order system ok.

So, let us say out of these n poles, let k be distinct ok. I am doing a generic case. Of course, all n can be distinct that is also right. So, in a few examples we would have seen that the poles were distinct, there are no repeating poles, but there can be repeating poles also right; we are just doing the generic case.

I am saying like let that be k distinct poles k ≤ n right ok, and let the multiplicity of the pole be μ_i ok. What is meant by multiplicity? The number of times it repeats; if it is non repeating multiplicity is 1. So, this implies $\sum_{i=1}^{k} \mu_i = n$, alright.

Let us say if I have a 10th order system and let us say I have 9 non repeating poles with the ninth pole repeating twice alright, summation of the multiplicity should be 10 right, so that I cannot violate right. So, essentially the summation of the multiplicity should be equal to the order of the system, which is the total number of poles correct. Now immediately we see that P(s) can be written in this particular form ok.

$$P(s) = \frac{n(s)}{(s+s_1)^{\mu_1}(s+s_2)^{\mu_2} \dots (s+s_k)^{\mu_k}}$$

So, that is how I can split the denominator polynomial right, I can factorize the denominator polynomial. Now I need to take the partial fraction expansion right. How would the partial fraction expansion look in general? We did a few examples last time right, so if you had a repeating pole what did we do? Let us say if we had a factor like 1 by $(s+1)^2$ what did we do? We did like $\frac{A}{s+1} + \frac{B}{(s+1)^2}$ right. So, you will have two terms, isn't it?. So, what is going to happen is that, if when we do the partial fraction expansion we are going to get something like this.

$$\frac{n(s)}{(s+s_1)^{\mu_1}(s+s_2)^{\mu_2} \dots (s+s_k)^{\mu_k}} = \sum_{l=1}^{k} \sum_{m=0}^{\mu_l - 1} \frac{c_{lm}}{(s+s_i)^{m+1}}$$

So, for example, I hope I did it correctly, let us double check ok. See for example, let us say I have a system whose transfer function is this $\frac{1}{(s+1)(s+2)(s+3)^2}$. What is the order of the system?

Student: (Refer Time: 14:47).

What do you think is the order of the system?

Student: (Refer Time: 14:53)

See the order of the system is, the order of the denominator polynomial of the transfer function right. What is the order of the denominator polynomial?.

Student: 4.

So, n is going to be equal to 4 right, and what are the poles? So, its going to be -1, -2, -3 and -3 right, there are four poles, but minus 3 repeats twice. So, what is the number of unique non repeating poles? k= 3. So, then what, what are the values of μ_1 μ_2 μ_3 which are the multiplicities of the 3 non repeating poles?

$\mu_1=1$, $\mu_2=1$, $\mu_3=2$

If you add $\mu_{1,}$ $\mu_{2,}$ μ_3 what do you get? You get 4 which is n right.

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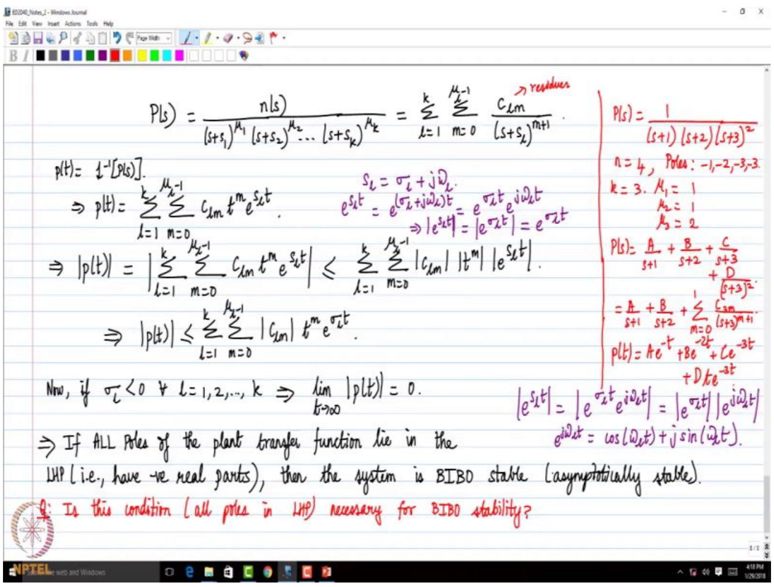

Now, if you want to do the partial fraction expansion what we will do?

$$P(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

That is what we would do right, that is the same thing I have done in compact form right using the summation.

$$P(s) = \frac{A}{s+1} + \frac{B}{s+2} + \sum_{m=0}^{1} \frac{C_{3m}}{(s+3)^{m+1}}$$

So, the C_{lm} are the residues ok, I am just putting two indices, because I have two summations right that is it, what you called as A B C D I am just calling it c subscript l m ok, that is just the change in the notation, because I have two indices l and m. I hope it is clear how we got it right, yeah.

Now we take the Laplace inverse. So, now, in this particular problem, if I take Laplace inverse what will I get? Let us continue with this problem right. If I take p(t) I am going to get $Ae^{-t} + Be^{-2t} + Ce^{-3t} + Dte^{-3t}$.

So, now, I am going to generalize from here right. So, this implies that

$$p(t) = \sum_{l=1}^{k} \sum_{m=0}^{\mu_l - 1} c_{lm} t^m e^{s_l t}$$

I have just generalized from there ok, because you see that when there is no multiplicity there is no t term ok, because anyway you will not have the summation with respect to m. You will have a summation with respect to m, only if you have multiplicity greater than 1. So, then you will have t^m. So, for m = 0 which is the case when you do not have any multiplicity. See for example, for the pole at minus 1, what is μ_1? It is 1.

So, you just use m = 0 that is it right. So, you won't get a t power term at, all, do you agree?

Student: Yeah.

So, but then like for any case where the multiplicity is 2 and above you would get a corresponding t term right, so that is what you, we are going to have. Do you agree? Is everyone in agreement with what I have written? I am just parallely doing as example and then doing the general case so that we can map them yeah.

Now once I get this, so this means that

$$|p(t)| = \left| \sum_{l=1}^{k} \sum_{m=0}^{\mu_l - 1} c_{lm} t^m e^{s_l t} \right| \leq \sum_{l=1}^{k} \sum_{m=0}^{\mu_l - 1} |c_{lm}| |t^m| |e^{s_l t}|$$

Do you agree? That is what we will have right ok. Now let us let us look at this further

$$s_l = \sigma_l + j\omega_l$$

$$e^{s_l t} = e^{(\sigma_l + j\omega_l)t}$$

$$|e^{s_l t}| = |e^{\sigma_l t}| = e^{\sigma_l t}$$

So, we are almost there ok; of course, this was a long derivation right. So, see by and large, I think this is the only proof I am going to do ok. So, in future we will make use of results right, that is going to be the focus for this particular course, but at least I wanted to show you ok, so that you understand how the inner details of whatever you are learning works right.

So, essentially this shows that

$$|p(t)| \leq \sum_{l=1}^{k} \sum_{m=0}^{\mu_l - 1} |c_{lm}| t^m e^{\sigma_l t}$$

Now, if $\sigma_l < 0 \; \forall \; l = 1,2, \ldots, k$, what can you say about the magnitude of p(t) as t tends to infinity.

Student: 0.

It will go to 0 right, because please see that, please note that the left hand side is less than or equal to the right hand side. In the limit t tending to infinity if the right hand side goes to 0; obviously, the left hand side has to go to 0, because the left hand side cannot take negative values right magnitude of p of t is nonnegative ok. So, the only choice for that is, is that like magnitude of p of t should go to 0.

Student: 0.

So this implies that $\lim_{t \to \infty} |p(t)| = 0$. Now immediately one may ask the question; hey what about the t^m term? Alright. It so happens that the exponential term with the negative exponent will dominate the t^m term ok. So, the product will go to zero anyway right. So, it so turns out that the sufficient condition for the impulse response to asymptotically go to zero is that I need to have all poles to have negative real parts ok.

Let us say k there are k distinct poles, k can be less than or equal to n right. So, whatever poles are there, all poles must have negative real parts or in other words we should have all poles lying in the left of complex plane yeah, yes please.

Student: How can we take that magnitude of e^{slp} is nothing, but..

Ah ok. So I just skipped one step there. So, if I take

$$|e^{s_l t}| = |e^{\sigma_l t} e^{j\omega_l t}| = |e^{\sigma_l t}||e^{j\omega_l t}|$$

$$e^{j\omega_l t} = \cos(\omega_l t) + jsin(\omega_l t)$$

So, to summarize we see that this implies that if all poles of the plant transfer function lie in the LHP; that is have negative real parts then the system is BIBO stable ok. Some people will say it is asymptotically stable ok, we are going to shortly see what that is.

So, that is that is what it is. So the important condition here is that I find the system transfer function if all poles are in the left of complex plane, left of the imaginary axis right left of the imaginary axis in the s plane then we have asymptotic stability or what we call as BIBO stability. So, is this also necessary?

So, what do I mean by necessary? We have only shown what is called a sufficiency condition right. So, what we have shown ok, if all poles lie in the left of plane we have shown that system is BIBO stable. Now the opposite question arises, if a system has to be BIBO stable, do all poles need to lie in the left of complex plane?

See I hope all of us understand necessary conditions sufficient condition and necessary and sufficient condition. See if A is necessary for B to happen right; that means, that A needs to be true for B to happen that need not imply that only A is required for B to happen right.

So, if we say A is sufficient for B to happen, if A happens; that means, that even B will happen right, but then B can happen due to other reasons also right. So, we do not know right. So, essentially what we have shown is there, we have only shown that the sufficient condition for BIBO stability is that all poles should lie in the left of complex plane. Now what is the other way round ok; that is if a system, what to say, is BIBO stable then all the poles lie in the left of complex plane right, is that true?

So, is the, what to say the is the condition of all poles lying the left of complex plane a necessary condition for BIBO stability. Why is necessary condition important? Only then we can talk about a corollary right, because if I say for BIBO stability all poles being in the left of complex plane is necessary, then I can say that even a one pole lies in the right of plane it is unstable. With this statement I cannot show right, because let us let us look at this derivation right.

Now, if you tell me I have a system where one pole is in the right of complex plane what will happen to the, what to say this derivation, what conclusion can I draw from this derivation? Nothing, why? Let us say one pole is in the right of complex plane right, 1 sigma l is greater than 0. So, what will that imply?

Student: (Refer Time: 31:15) infinity

Ah I will the, I will only be able to conclude in the magnitude of p of t is less than or equal to infinity that does not help me, that does not tell me it is unstable right is it. See 0 is also less than infinity you know like any other positive number is also less than or equal to infinity right, less than infinity you know. So, it really does not help me alright. So, what we have shown is only sufficiency ok, what about necessary condition ok.

I leave it to as homework you, it is not, and once again it is not necessary as far as this course is concerned ok, but for those who are mathematically inclined please go on look at the proof ok. So, it is available in many standard textbooks on controls. So, I hope my point is clear right, what we have shown is only sufficiency it. So, turns out that it is also a necessary condition.

So, let me what to say pose that question, is this condition necessary. What do I mean by is this condition? All poles in LHP necessary for BIBO stability, the answer is yes, and that is why if we do not, if we even have one pole lying the RHP or on the imaginary axis the system is not BIBO stable that we have shown through examples right.

See to disprove BIBO stability I need to figure out only one bounded input for which the output is unbounded we have already shown it ok, whether we call it critically stable marginally stable or whatever it is, but then we found that it violates BIBO stability definition. And if you have a pole on the pole in the right half plane, you give me any bounded input the output will be unbounded that is that is sure.

If you give repeating poles on the imaginary axis, any bounded input will lead to unbounded output, if you have what are called like non repeating poles on the imaginary axis, you can always find one bounded input for which the output will be unbounded right. So, that is why, this is also the, how do we conclude, because this is also necessary. See I went from examples to a generic conclusion, the best way to do it is also always to prove the generic one and then like do the examples ok, but since this is the first course on controls you know like I wanted to do examples to motivate our understanding and then like we did the general derivation right.

So, this I leave it to you as homework, the answer is yes, you need, it is also necessary; that is why we are able to conclude that, even if one pole lies in the right of plane, system is not BIBO stable ok. And if you have non repeating poles on the imaginary axis once again, you know like to me it is BIBO unstable right, because like I can always find one bounded input for which the system or to it is unbounded right.

So, although many textbooks will call it as critically stable or marginally stable; that is because of another notion of stability ok. So, what I will do is that maybe this is a good place to stop ok, so that you can digest all this information right