# Control Systems

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## Lecture – 11
## BIBO Stability
## Part – 1

Let us solve some problems and make some observations and do a general derivation as far as BIBO stability is concerned. Let us start with problem number 5.

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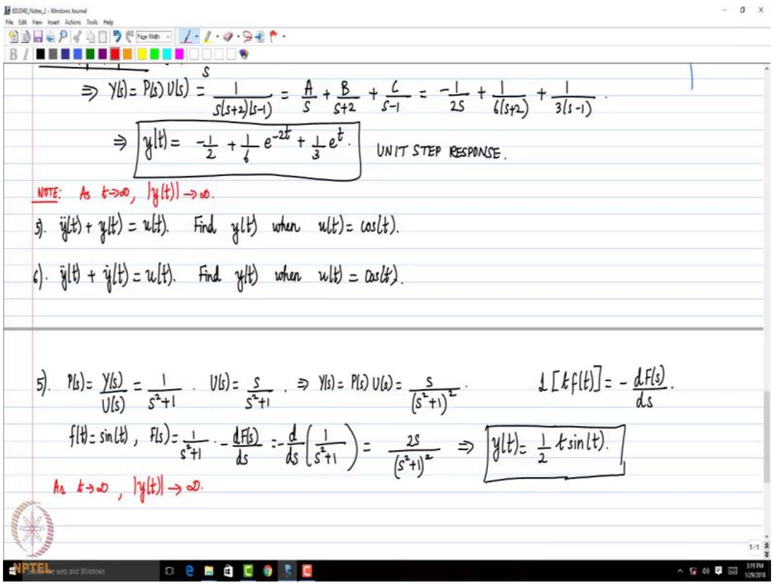


From the previous discussion, we know that the plant transfer function is

$$P(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 1}.$$

System poles at $\pm j$, they are purely imaginary poles. And we saw that when it was subjected to a unit step input, the output was bounded in magnitude.

Now, let us subject it to a cosine input of a very specific angular frequency 1 rad/s.

In general $\cos \omega t$ means $\omega$ is the angular frequency in radians per second. $\cos t$ means $\omega = 1$ rad/s. If $u(t) = \cos t$, then $U(s) = \frac{s}{s^2+1}$.

$$Y(s) = P(s)U(s) = \frac{1}{s^2+1}\frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}$$

Now, we use a property of Laplace transform, the complex differentiation theorem.

$$L[tf(t)] = -\frac{dF(s)}{ds}$$

Let us take $f(t) = \sin t$,

$$F(s) = \frac{1}{s^2+1}$$

$$-\frac{dF(s)}{ds} = \frac{2s}{(s^2+1)^2},$$

$$y(t) = \frac{1}{2}t\sin t$$

We can observe that, as $t \to \infty, y(t) \to \infty$.

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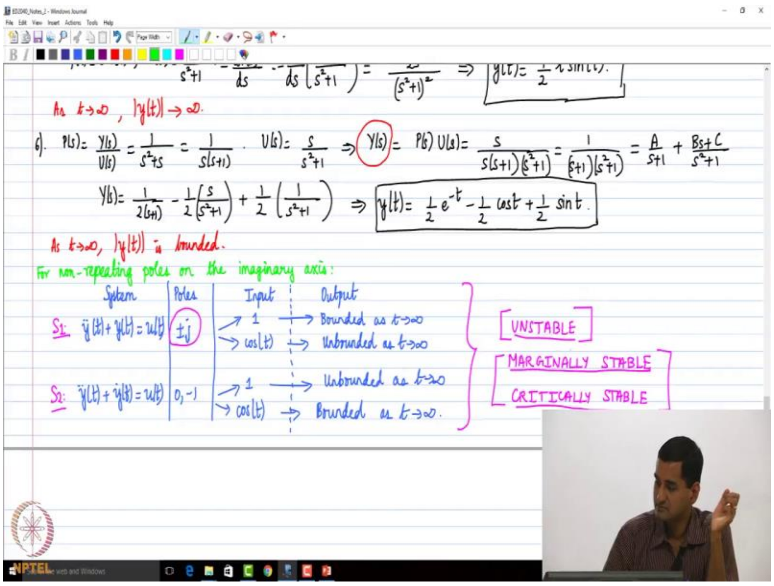


Now, let us solve problem number 6. We already know that the plant transfer function is

$$P(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+s} = \frac{1}{s(s+1)}.$$

$u(t) = \cos t$, then $U(s) = \frac{s}{s^2+1}$.

$$Y(s) = P(s)U(s) = \frac{s}{s(s+1)(s^2+1)} = \frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

Here the cancellation of $s$ in the numerator and denominator was a mathematical operation which came in due to the specific nature of the input that we have, but in general we have a more subtle idea which is called as pole zero cancellation. We will discuss that when we go deeper into control system design.

Solving the partial fractions, we get $A = \frac{1}{2}$, $B = -\frac{1}{2}$ and $C = \frac{1}{2}$.

$$Y(s) = \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}.$$

Taking inverse Laplace, we have

$$y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t.$$

We can observe that, as $t \to \infty, y(t)$ is bounded.

Earlier we saw that, for the same system when we give the unit step response, $y(t) \to \infty$. Now we gave another bounded input $\cos t$, we are getting a bounded output. Let us construct a simple table.

<table>
<tr><th>System</th><th>Poles</th><th>Input</th><th>Output</th></tr>
<tr><td rowspan="2">S1<br>$\ddot{y}(t) + y(t) = u(t)$</td><td rowspan="2">$\pm j$</td><td>1</td><td>Bounded as $t \to \infty$</td></tr>
<tr><td>$\cos t$</td><td>Unbounded as $t \to \infty$</td></tr>
<tr><td rowspan="2">S2<br>$\ddot{y}(t) + \dot{y}(t) = u(t)$</td><td rowspan="2">0,-1</td><td>1</td><td>Unbounded as $t \to \infty$</td></tr>
<tr><td>$\cos t$</td><td>Bounded as $t \to \infty$</td></tr>
</table>

It so turns out that if we use the definition of BIBO stability which states that a system is BIBO stable if its output is bounded for all possible bounded inputs, we should brand these two systems as unstable.

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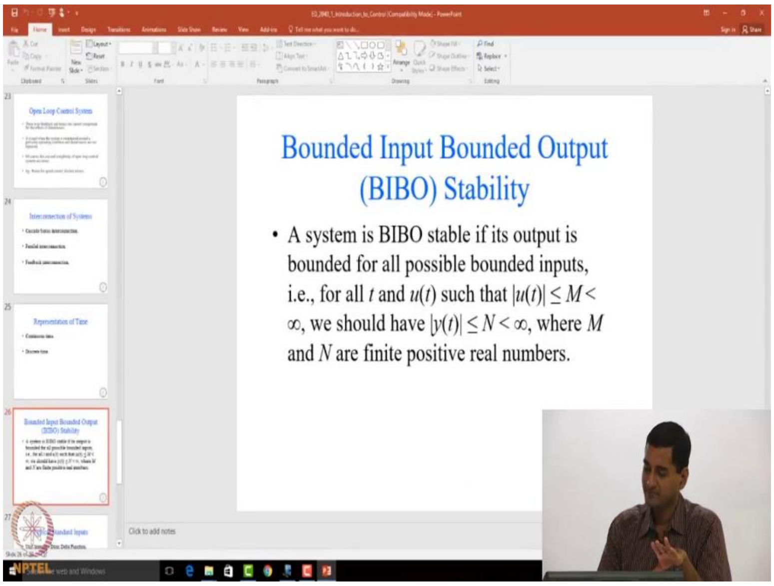


But, for the first system (S1) the output is unbounded, only if we give a $\sin t$ or $\cos t$ as the input. For unit step input or any other bounded input like a sin or a cosine of any angular frequency other than 1 rad/s, we can show that the output is going to be bounded. It so turns out that only when the input becomes a sinusoid whose angular frequency is equal to the magnitude of the poles on the imaginary axis, the output becomes unbounded. For S1 the magnitude of poles on imaginary axis is 1. Some people will call this class of systems as marginally stable or critically stable; that is the output is bounded for all possible bounded inputs except one set. In strict parlance the system is unstable if we use the definition as it is. We do not want to design such systems in practice. If we encounter such systems we want to stabilize them.

If we have a mass spring damper system where there is no damper, we just have a mass and the spring, the equation of motion will become $M\ddot{x} + kx = f$, the moment we give a step input, the output will become unbounded.

Now, on the other hand if you look at system S2, it has a pole at the origin. If we have a non-repeating pole at the origin you give a step input, the output is unbounded. But for all other sinusoidal inputs, the output is bounded. That is the property of S2. We can see that the output is unbounded only for one set of inputs. This result is true only if we have for non repeating poles on the imaginary axis.

If you have a repeating pole on the imaginary axis. For example.

$$P(s) = \frac{1}{s^2(s+1)}$$

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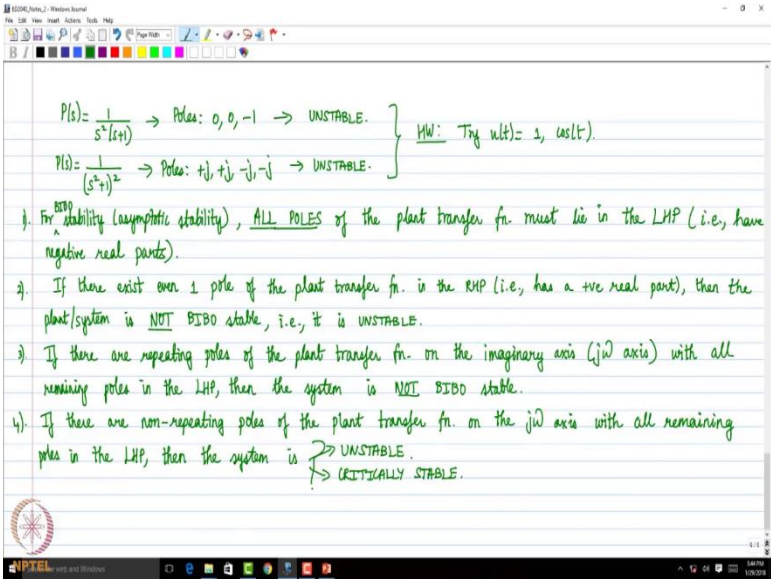


The poles are at 0,0,-1, with the repeating poles this system is unstable.

Similarly, if you have another system where the transfer function is

$$P(s) = \frac{1}{(s^2+1)^2}$$

The poles are $+j, +j, -j, -j$, this is unstable. The above concept of marginal stability or critical stability applies, only if we have non repeating poles on the imaginary axis.

1. For BIBO stability, all poles of the plant transfer function must lie in the LHP or in other words have negative real parts.
2. If there exists even one pole of the plant transfer function in the right half plane, that is have has a positive real part, then the plant or the system is not BIBO stable.

3. If there are repeating poles of the plant transfer function on the imaginary axis (j omega axis) with all remaining poles in the LHP, then system is not BIBO stable.
4. If there are non repeating poles of the plant transfer function on the imaginary axis with all remaining poles in the LHP then the system is unstable or critically stable.

Most textbook we will say it is critically or marginally stable, but if we strictly apply the definition of BIBO stability it is unstable. If we are strict to that extent we should call this as unstable, but if we can say the output is bounded only for one small group of bounded inputs so we call it as critically stable or marginally stable.

That is basically some perspectives of how to interpret system stability when you have poles, non-repeating poles on the imaginary axis, including the origin.

In condition number 1, if we have a system, we find the transfer function and figure out that all poles of the system transfer function lie in the left half complex plane, is the system BIBO stable? And answer is yes and we shall prove it later.