Control Systems
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Lecture - 10
System Response
Part - 2

So, let me do the second exercise, ok. So, let me go back and figure out what was the second exercise. So, I gave $\ddot{y}(t) + y(t) = u(t)$ right.

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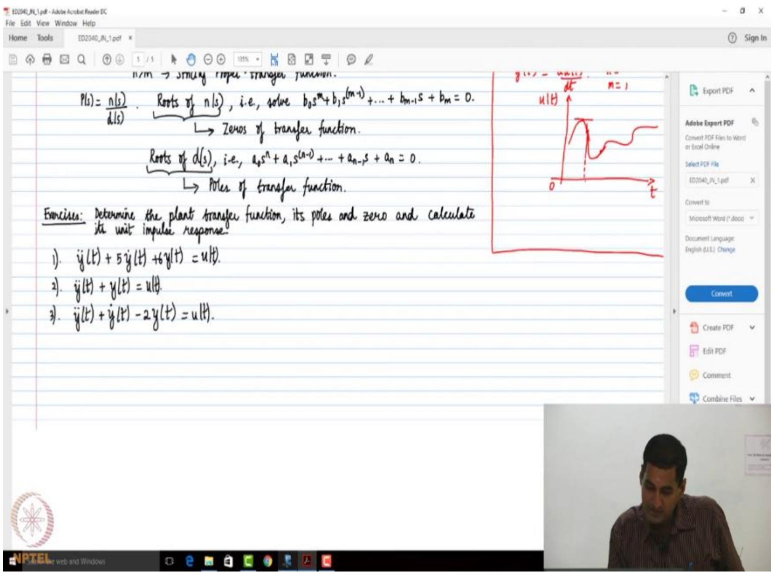

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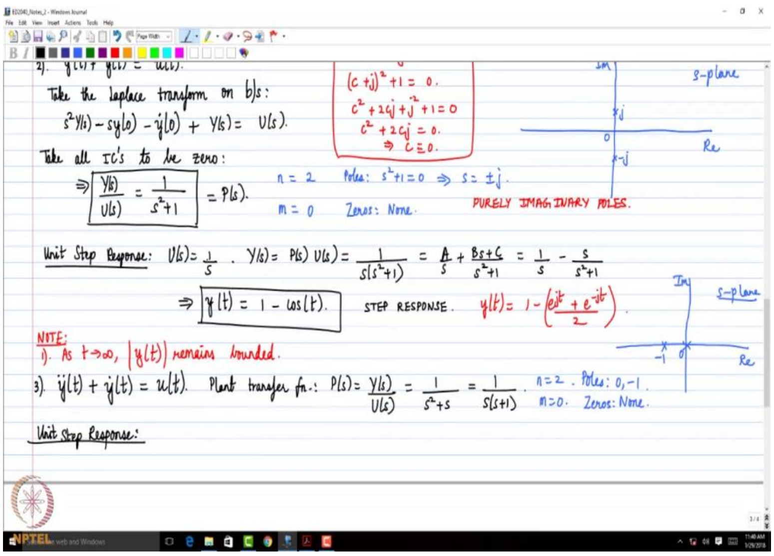

So, let us do this problem $\ddot{y}(t) + y(t) = u(t)$. So, that is the second problem. So, once again you can see that it is a second order system and if you take the Laplace transform and apply zero initial conditions what are we going to get?

So, let us follow the same steps. So, take the Laplace transform on both sides. Of course, I am going to skip a few steps, because by now you know what to do right. Once again please write down the complete transform of the derivatives; do not substitute the initial conditions to be 0 in the first step itself because you never know, ok. In some problem initial conditions may not be zero. So, we need to be a bit careful.

So, always write down the entire expansion. Now in order to find the transfer function take all initial conditions to be 0 right, because that is a definition of the term transfer function, right. So, this implies $\frac{Y(s)}{U(s)} = \frac{1}{s^2+1}$. That is what we will get, if I take all initial conditions to be 0 and I collect terms involving Y(s) and cross multiply, this would be the plant transfer function P(s).

So, now what can I comment, what are the values of n and m? n is 2, second order system, m is 0. What about poles?

Student: (Refer Time: 02:28).

You solve for $s^2 + 1 = 0$. So, this implies that s is going to be equal to of course, we are going to use j, you could use i also, but as I told you we are going to use j in this particular course, ok. So, we will essentially say it is ±j, so zeros, there are none once again. So, if you plot the s plane and mark these two poles, where are they going to be? They are going to be at +j and –j. So, those are the two poles for this particular system, yes.

Student: (Refer Time: 03:18).

Which one?

Student: S plus j a real part may be (Refer Time: 03:23).

Real part is 0. See $s^2 + 1$ means s = ±j. So, your real component of that solution is going to be 0 right, isn't it? See if you substitute any real component if you say C ± j, where C is any real number can you satisfy this equation? Please try ok, you cannot right.

So, that is why the solution is, we take it to be ±j right ok. So, we can we can try it out right, let us see. Since this question has been raised, let me also work it out ok. So, let us say we take it as C ± j, is that your point?

So, C being any real number, let us try it out. So, if I have C ± j, then

$(c + j)^2 + 1 = 0$

$c^2 + 2cj + j^2 + 1 = 0$

$c^2 + 2cj = 0$

Implies, $c = 0$.

So, this will be satisfied only when C is identically equal to 0 correct. So, that is why we are taking plus or minus right. So, the real part is anyway 0 ok. Of course, let me put it in a box. So, that like, whatever I am drawing like this you know like our asides right. So, that is ±j right, those are the two roots for this one right. So, what we are having, is that like we are having what are called as a complex conjugate poles right. We are essentially having a purely imaginary poles here, correct. So, what do we do for the unit step response?

So, we need to get the unit step response. How do we get this? Unit step response means $U(s) = \frac{1}{s}$. So, then $Y(s) = P(s)U(s)$. See this is the advantage; once we find that transfer function we can find the system output for any given input right. So, $Y(s) = \frac{1}{s(s^2+1)}$. How can I simplify this?.

Student: (Refer Time: 06:12) p by s; P s by (Refer Time: 06:15).

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}$$

So, immediately we see A is going to be 1 right, and if you solve for B and C what are you going to get?

Student: (Refer Time: 06:37) minus 1.

Minus 1 and 0; so we will have essentially $\frac{1}{s} - \frac{s}{s^2+1}$. I am sure by now if we have done a few examples, by now I am sure all of you will be familiar with how to calculate these residues right. I think last week we did a few cases right, where I showed you how to do even for second order terms and so on right.

So, once we get this, this implies that y(t)=1-cos(t). Do you agree? Ok. So, that is my step response. So, but then like if we want to make the same observations as before right. So, what happens here, let us say I want to make the same observations as before right. So, what happens as $t \rightarrow \infty$?

Student: (Refer Time: 07:59).

Of course, essentially it oscillates, but is it bounded?

Student: (Refer Time: 08:07).

So the magnitude of y(t) remains bounded, right. So, that is something which we can see. So, using this; See once again step is a bounded input right is it not? So we are getting a bounded output; so from this would you conclude that the system is BIBO stable.

Student: No.

No, why not.

Student: (Refer Time: 08:36).

We do not know about all inputs right, so that is something which we need to check out. And by the way, where are the exponential terms here?

Student: (Refer Time: 08:48).

You could use Euler's relation. So, please note that the general viewpoint even still works out right. So, see I wrote that real part of the poles would appear ok.

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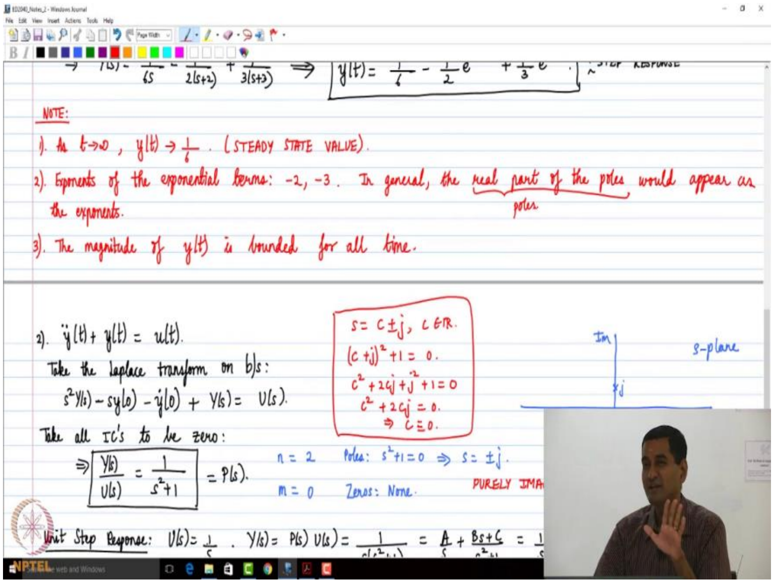

Some people will write the poles would appear. So, that is why I just wanted to make a distinction ok, because see if I write poles would appear, yes that is also correct, but then you see that in this particular problem the real part is 0, so you do not need to necessarily have exponential solutions appearing all the time, its implicit, its contained within the cos and the sine. So, we could say that exponential terms will have the real part as the exponent. If you have complex conjugate you will get cos and sine. You could also write this as $y(t) = 1 - (\frac{e^{jt}+e^{-jt}}{2})$

So, here omega is 1. So, you see that now the poles are appearing as, once again we are going to have two exponential terms corresponding to two poles right, the exponents are the poles themselves, right. So both interpretations I am just writing this way right, both interpretations could be derived out of this exercise, ok. So, you see that the poles are there right as the exponential terms.

So, but if you have complex exponentials you know you can write as cos and sin, right using the Euler's relation. So, that is why I told in general we could also look at for our purpose, we could look at the real parts being the exponents, right. In this particular term 1 - cos t, you do not have an explicit exponential term, because the real parts of the poles are 0 ok, so that is why you have this. Is it clear?

Now what I want you to do, remaining on this problem, we will go to the next problem shortly, I am just renumbering, so please do this. So, I just changed only one thing right. So, instead of $\ddot{y}(t) + y(t) = u(t)$, I just changed the transfer function to be $\ddot{y}(t) + \dot{y}(t) = u(t)$; that is the difference between problems 2 and 3, alright.

So, now, can you repeat the same process? So, what would be the plant transfer function? Please take the Laplace transform on both sides, take initial conditions to be 0 get the ratio of Y(s) and U(s), what would you get? You would get $\frac{1}{s^2+s}$. So, this I can rewrite as $\frac{1}{s(s+1)}$.

So, similarly I can say n = 2, m = 0. What are the poles? The poles are at 0 and - 1 correct and there are no zeros, ok. So, consequently what will happen, I will have this ok. So, where are my poles here in this example? In the s plane I have a pole of the origin, I have a pole at minus 1 ok.

So, these are my two poles. So, please find the unit step response and let me know what do you get?

Student: Sir.

Yes.

Student: (Refer Time: 13:02).

Is it?

Student: It should be because it, (Refer Time: 13:14).

So, to answer that I am going to generalize by doing this problem ok, let us please hold on, ok. The question was that like I neither said that its BIBO stable nor said its unstable right, I only said we need to do further analysis.

So, but to give you the answer it is a tricky question; it depends on whom you talk to that is why I am doing things carefully step by step ok. So, unit step response please, so, unit step response means $U(s) = \frac{1}{s}$.

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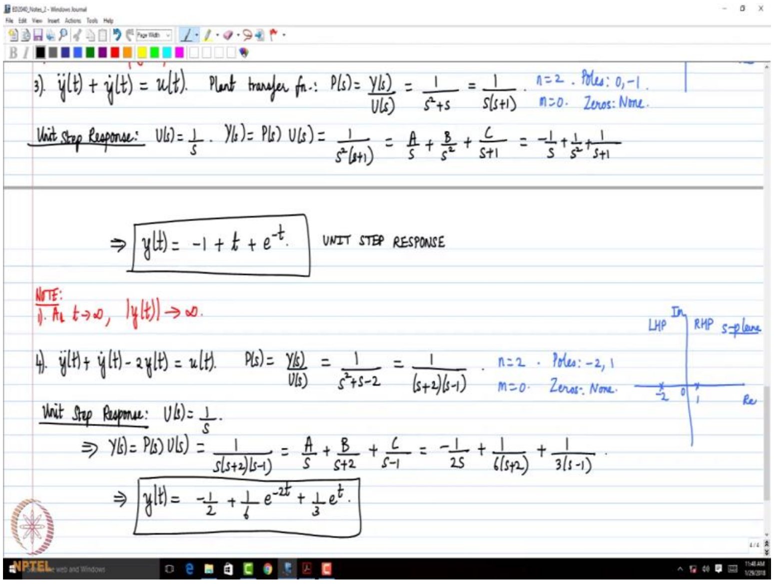

So,

$$Y(s) = P(s)U(s) = \frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

So, what are A, B and C?

Student: (Refer Time: 14:28).

C, I can get as 1 right, B I can get once again as 1, am I correct? So, A you can substitute any other value you want and if you simplify what will you get? Minus 1, is that correct. So $y(t) = -1 + t + e^{-t}$.

So, this is the unit step response ok. Previously also there is a unit step response right. So, now, what can we observe?. So, what do you think happens to the magnitude of y(t) as t tends to infinity, what happens to its magnitude?

Student: (Refer Time: 15:52).

It tends to infinity why right, because I have a t term right. So, once again you see that, the solution has exponential terms the poles are the exponents right. So, the poles are at 0 and minus 1. So, what is e^0?

Student: 1.

It is just like what to say 1 right, so that is why you have a term like that and then you also have $e^{-t} - 1$ also comes in right, but in addition you have a t term ok, which essentially tells you that the magnitude of the output is going to go to infinity right, as the time tends to infinity. So, would you call the system as BIBO stable? Yes or no.

Student: (Refer Time: 16:38).

See what is BIBO stable right?

Student: (Refer Time: 16:40).

So, let me recap the definition very carefully ok, each and every word is very important. A system is said to be BIBO stable, if the output remains bounded in magnitude for all time given any bounded input, ok, the key word is any or all bounded inputs right. So, here if you want to show something is not BIBO stable, you just need to find one bounded input for which the output is unbounded, is it not. See in fact, in life showing that something does not work or is not true is easier right, because you just need to have only one case, where it does not hold true right and for this particular system have we not found one bounded input for which the output is unbounded, we have right.

So, for this particular system we have found a step input for which the output is unbounded ok. Now the question is would you call this as unstable or something else? We will see what that 'something else' is in the afternoon ok, so after doing a few more examples ok. So, just hold on ok. So, we would see that you know like, there are some more examples that we need to do before we can generalize ok.

So, I hope the flow is clear right. So, as the fourth example let us do this problem ok. So, $\ddot{y}(t) + \dot{y}(t) - 2y(t) = u(t)$, alright.

So, what is the plant transfer function P(s) here?

Student: (Refer Time: 18:29).

$$P(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + s - 2} = \frac{1}{(s+2)(s-1)}$$

I am assuming that all of you can follow with me. Please stop me if anything is unclear right. So, this I can rewrite it as 1 divided by s plus 2 times s minus 1. Am I correct? Is that correct? So, once again we can see that n = 2, m = 0, the poles are at -2 and +1. Zeros, there are none.

So, if I draw the s plane for this particular example the poles are going to be at minus 2 and plus 1 ok. So, this is the left of complex plane and this is the right of plane ok. So, what is the step response, unit step response?

$$Y(s) = P(s)U(s) = \frac{1}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

$$A = \frac{-1}{2}, B = \frac{1}{6}, C = \frac{1}{3}$$

So, those are the 3 partial fractions. So, this implies that $y(t) = \frac{-1}{2} + \frac{1}{6}e^{-2t} + \frac{1}{3}e^{t}$

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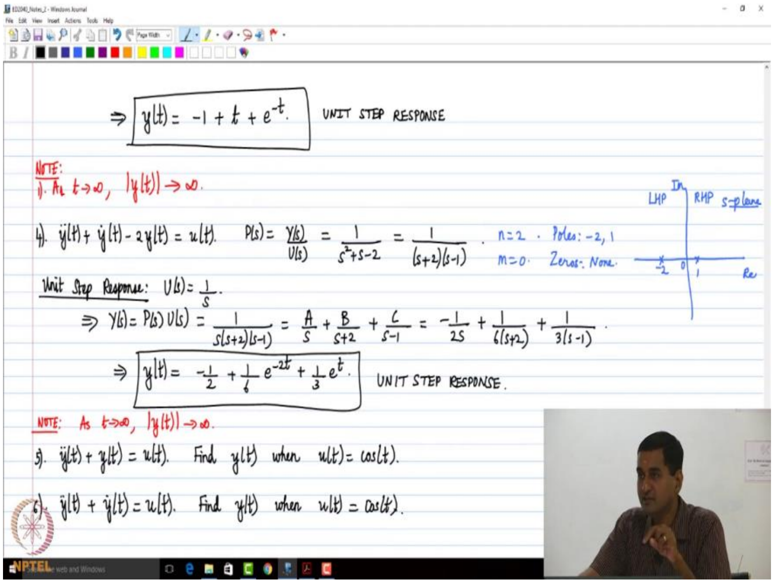

So, now we immediately note that as t tends to infinity what happens to the output y(t).

Student: (Refer Time: 21:35).

We can immediately see that once again there are two exponential terms: the poles appear as exponent or the real parts appear as an exponent, you know because anyway the poles are real here.

So, you could see that -2 and + 1 are the two poles, you could see them appearing as exponents, but then you see that as $t \rightarrow \infty$, the magnitude tends to infinity why, because we have e^t all right, the magnitude of e^t tends to infinity ok, so that is why this blows off right to infinity. Now the question is given this unit step response, you know like with this system be BIBO stable or unstable right.

So, once again you know like, we would figure out the answer right. So, what we are essentially getting at is the following. So, before you come to afternoon's class, you know like anyway you have time right. So, I want you to do this example ok. Take problem number 2, $\ddot{y}(t) + y(t) = u(t)$, please take that problem ok.

So, as problem 5 consider $\ddot{y}(t) + y(t) = u(t)$ ok; now find *y(t)* when *u(t)=cos(t).*

Similarly the 6th problem I am going to give you this, take the third problem $\ddot{y}(t) + \dot{y}(t) = u(t)$, find *y(t)* once again when *u(t)* = *cos(t)*, ok.

So, please do these problems, and you come and tell me what you observed and then we would continue our discussions from here in the next class right.

So, I hope it is clear what I want you to do right. Take problems 2 and 3. For both problems we figured out the unit step response right. Now I want you to figure out what is the response of the system or the output of the system when we give a cosine as the input to the particular system. Do the same exercise, take inverse Laplace transform, get y(t) and then we will discuss, what are the consequences thereafter.