Vehicle Dynamics
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Lecture – 30
Vertical Dynamics - Quarter Car Model

In the last class, we were looking at what is called as a half car model.

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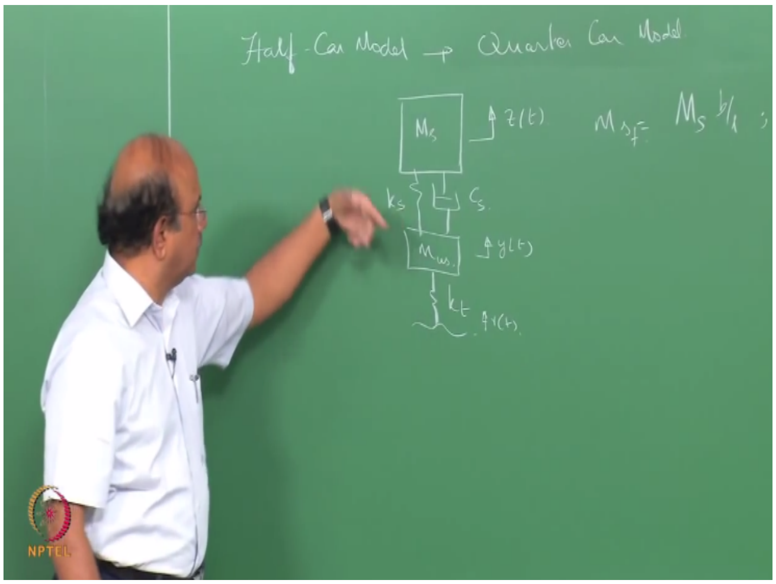

And then we reduced it to what is called as quarter car model. Okay, we reduced this to what is called quarter car model. Now let us understand this, there been lot of equations on it, let us understand what we are trying to do now with the quarter car model. As I told you in last class that you can have a full car model but to quickly understand the effect of suspension system, okay, and 2 of the very important frequencies that go with this system of a suspension sprung mass and unsprung mass as well as the tire of course, okay.

We are looking at this quarter car model. In actuality, we have to consider the road which gives the input, how road is represented, how that goes as an input, all those things are important and we are going to see that from the next class onwards. Okay, next 4 to 5 classes, we will understand the road, the statistical nature of the road, how road is represented, how that affects the performance of the vehicle, all those things will do that from next class.

So, we quickly finish some of the things that we want to do in the quarter car model and then we will go to half car model and finish that part as well, hope we will be able complete it or else we will, maybe it will spill over. So what is a quarter car model, that is the, what is called as the M sprung mass, that is unsprung mass, right if I change anything just tell me, I hope I used the same notation, if there is any difference in the notation, tell me okay, this one of problems like and so on.

Right, so this is the road input, r of t. This we called it as y and that we called it as z, right, I think that is why we are left. Now let us write the equations for this, as I said this is cut into 2 halves, the front and the rear and we had put Ms to be distributed mass between the front and the rear, from the total sprung mass which we called as b/l, this is the front and for the rear, we said that it is Ms*a/l right.

So the system was the same, you have to now rewrite or replace Ms by the corresponding front or rear springs by front and rear and so on, okay. In other words, in the quarter car model, we are delinking both, right. Sometimes we will analyse it with complete Ms, okay as one vehicle and the suspension and so on as one whole vehicle as well, okay.

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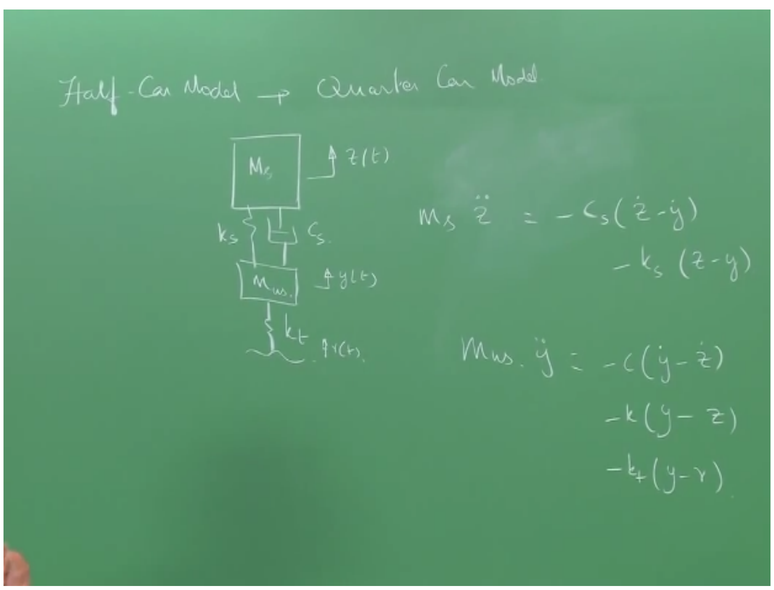

Now let me write down the 2 governing equations, because why 2, because I have 2 degrees of freedom which is y and z, so Ms*z double dot is the first one, okay, which is we had already discussed this. Cs*z dot-y dot – ks*z-y. Same fashion, again write down, Mus*y double dot, okay, so write down, yesterday we did that, write down the forces that are acting, so it will be – C*y-z-k*y-z.

There are the 2 forces that are acting okay and then the tire force, -kt*y-r. Right, these are the 2 forces that are acting. Now we can do 2 things, as I told you we are going to look at, that is the input of course, look at it as linear time in variant system, okay, r can be expressed in terms of the sinusoidal and exponential function and so on and so forth, right.

So what we are going to do to follow this kind of system approach, we can find out the eigenvalues and eigenvectors and then just express them, express the result as a sum of eigenvalues and the eigenvectors. So that is what we are going to do, I think, we did that already yesterday, yeah, we did that right and did we right down z/r, that we also wrote down, okay fine, I think, z/r and y/r, both of them we wrote down, sorry, yes, so we wrote down both of them and so.

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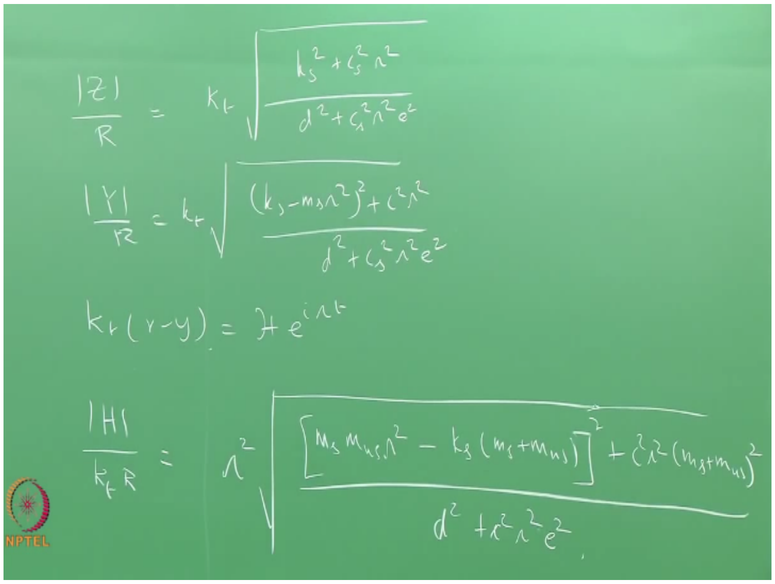

Z=magnitude=kt root of ks squared+cs squared omega squared/d square+c square omega square e square, okay and y/r=kt*root of ks-ms omega square whole square+c square omega square/d square+c square omega square e square okay. I think this is where we stopped right. Like d and e we had written than already, right. So the other one, this is the displacement, the ratio of the displacements, z and y due to r, right. The other one is actually what is called road holding.

Okay road holding is the force that the road exerts or the vehicle exerts onto the road, so how do I express the road holding, what is road holding here, what is the force, kt*r-y, okay so that is the force that supply it. Please note that all of them are perturbations about the static equilibrium, okay in sense that it is the equilibrium due to the weights and so on, it is about that that we are oscillating.

So that I substitute it, rearrange it and so on, so let me call that, how I called it, so let me call that as h*e power i omega t, that is the force, so that h/kt*r, okay = omega square root of ms mus omega square –ks*ms+mus whole square+c square omega square ms okay. The techniques are simple, the only thing is that the problem is you have huge I would say equations are very long that is because of the number of terms that are involved, so apart from that they are not very difficult to understand.

I suggest that you derive it because of the lack of time, I am not going to do that, please derive it and check how this equation are obtained, okay. So most of the questions which I am going to express now, I am not going to derive it, because all of them are just algebra, bring it to the left hand side, bring it to the right-hand side, put them together, divide it, you know those are the things, they are not very conceptually very difficult to understand, right.

"Professor - student conversation starts" In the model why we are not consideration the dampers of the tire. Yeah, the damping usually, the damper effect is not very high for the tire (()) (10:28) absolutely, absolutely but those effects to into rolling assistance, those effects are for rolling assistance, okay. **"Professor - student conversation ends."**

It is a very difficult topic, see sometimes what we do in engineering is that when it is very difficult to handle, we make an assumption that it does not exist, okay so damping itself since you asked this question I want to comment on this, damping is actually not a very easy topic to handle, to understand specially in the first course like this. Damping can be classified into what is called as proportional damping and non-proportional damping, okay.

And the proportional damping in the sense that the damping I am sure you had the background on vibration, so you talked about mode ships, mode superposition, we are going to see that again in the next course<so where we had very nice mattresses okay, when I looked at the mode ships okay and they are orthogonal and so on, right. We actually we did what we call as coordinator transformation and then we had those nice diagonal terms, we uncoupled the differential equations and so on, right.

So that this is the beauty of a proportional damping, where c = alpha time m+beta times k. When I write like that my c matrix behaves nicely and then I am able to get a very similar expressions because c is now split into m and k okay, I will take some of them to m and some of the them to k and then I get a nice matrix okay. If the damping is not proportional, in other words, if I am not able to write c = alpha m+beta k, which is the case actually with tires, they are nonproportional damping, then the way I have to handle becomes difficult.

There is a way of handling it, it is quite an evolved ideas but not here in this course. So people feel that the best way to handle this is, what differences does it make, if it is not much different, then I will remove it. In fact, you would notice that in the next step, I am going to do, I am going to look at the natural frequencies and even in that case I am going to remove the damping in order to calculate the natural frequencies of an umdamped system, okay.

Does it mean, especially you know, the more important damping is this, shock absorber, does it mean that the shock absorber does not have any effect okay, when I remove it fortunately the difference between the natural frequencies of an undamped system and the damped system, the differences are very, very small. Okay. Because though in the whole of mechanical engineering, this system is very highly, one of the very highly damp, okay “under damped system.”

It is a very highly damp system but still it is an under damped system, okay. So the difference between, we are going to see that, difference between the natural frequency of an undamped system and a damped system will be small, so when I want to calculate the natural frequencies, okay, which looks like every mechanical engineer has such a good hold of it, understands it, so is very easy to now calculate the natural frequency if I do not have an undamped system.

But, I cannot say the same thing with respect to the motion or mode shifts, for this motion rather, the ratio of z versus y, I cannot say that, because once I put this damping here okay, the amount of motion for example in a typical car if I do not have a damper, the ratio of motion will be 80 times for, say for example, for omega 2, but if I put a damper, then the ratio of motion comes down to nearly 10, in another words it is nearly an order of magnitude difference in the ratio of, that is, z/y, okay the ratio of the displacement or ratio of motion will be huge.

So, I want to calculate simple only natural frequencies that is fine, I can do that but I do not want to calculate natural frequencies alone, I want look at other things then damping will have an effect, okay and so on and that is what we are going to see now. But in the case of tire, okay, it has been a tradition to remove it, so there will be, in fact, there had been very interesting papers which are now question, for example, tire noise, what are the material parameters which have an effect on tire noise, this is one of the say issues.

Stiffness has an effect, G double prime has an effect and so on. Now people are re-looking at it, the people said that initially in the 90s, there were papers which said that the materials do not have an effect but now there are papers which say that material has an effect on noise and so on, so in simple words, this is a very traditional approach.

In fact, tire damping here itself, our friend here, his research is on the tire damping and it is not a very easy thing to calculate, what is that you are getting 2%, right, so it is a Ph.D topic, he is looking at how tire damping can be calculated, how tire damping can be measured you know in fact he is doing all that, okay, it is a topic which is of interest, which I would say is not at this level of vehicle damage, this is an introductory level, okay fine.

Let me get back, so we are now looking at so that are the 2 things, and now we will look at a situation where I have 2 now, given the sinusoidal input, I know how to that that, but I have a situation where I have to actually say tune the suspension, how do I optimise the suspension, so in order to optimise the suspension, okay I have to understand how the system behaves with respect to omega, or in other words excitation.

Okay, so I have know how the system behaves with respect to excitation, why is it, because the road consists of a number of frequencies, so ultimately when I use whatever be the technique that we will discuss it later, to split this up into a number of frequencies of excitation. Then I have to understand that the c value effects are.

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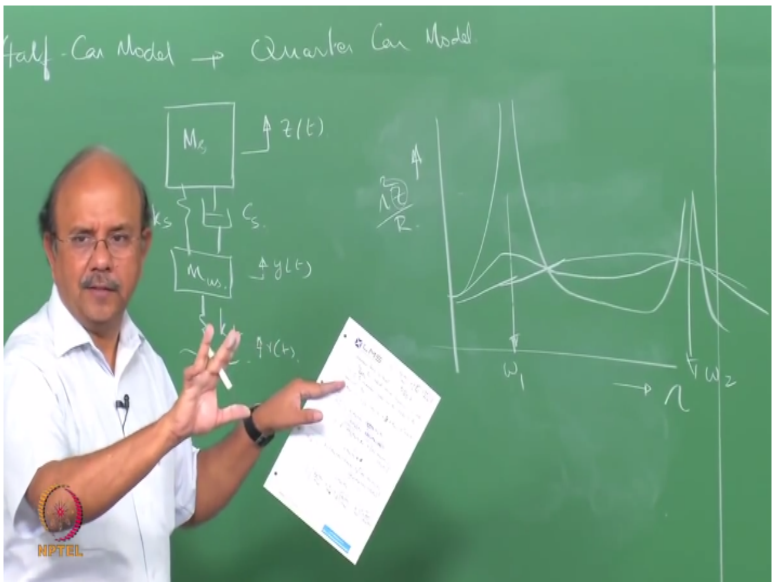

The c value effects are not the same at every frequency, this you already know from elementary vibration classes, okay, I am going to exploit that, and you will see right now that that is not going to be very straightforward technique. Before deriving, okay, let us look at the result, this is not the right way to do it, but this is just to motivate, okay. Now why am I deriving it, that is why I am plotting this here.

Suppose I now plot, I do the derivation of free vibration system, where I find out what would be the z (()) (18:59). Suppose I plot omega, the excitation frequency versus a normalised z, which I would let me plot it as, how do I plot it, omega square zr, okay that is normalised plot, okay. When it plotted like this, then my plot is going to look like this, for c=0 okay, at 2 places you are going to have difficulties and those 2 places are the 2 natural frequencies which is omega 1 and omega 2, okay, those are the 2 places where you going to have trouble.

Now when I introduce damping, then this curve is going to change, obviously all of you know that this would not be infinity, okay and it will come down right. Now, so for example at one level, it will come down like that and it will be like this, right, when I now increase damping it may come out like this, okay and then pass through the same point, may increase, okay, pass through the same point and it can be like that.

In other words without damping, I have 2 peaks, and with various values of damping, I have curves where it will be bad, it will be good and so on, in other words, depending upon the frequency my z value okay would now vary, r is actually the amplitude of the road excitation. Now in other words, a good damper in one frequency you cannot say that it is good or little dampen out in other frequency, okay and so on. Lowe damping is good in 1, bad in another.

Higher damping is good in one, bad in another and so on. Under these conditions, damping optimisation becomes difficult, how do I dampen out, okay, number one, that is why you have techniques where the damping varies with the frequency of excitation and that is in the realm of controls, okay but every car does not have it, it is still lot of things are still in the what you cal as experimental stage, so given a choice what will you do you then.

You do not know, I mean what frequencies are going to be excited, may be one of them, so what would be our choice. The best choice would be that I have c as flat as possible, okay, is it a correct choice? No, but as a first cut you can look at c as one which is almost flat, which is not that it is high at one place, lot at other place and so on. So in other words optimising the suspension specially the shock absorber, is not a task which is very obvious (()) (23:22) because of this kind of graph where it is behaviour is not going to be the same, one compromise results in an increase at other place and vice versa.

Okay, in order to understand, this is clear? You are now going to see how we get this graph, that is going to be my derivation, right. So with a that background, let us quickly run through this, this stuff which you know already, so I am not going to spend a lot of time on this. You have done this already for say single degree of freedom system, right, where you know that there is a crossover point okay, where the damping characteristics are different, on either side of point and so on. So, same thing happens here, okay, hence, we would run through this whole thing.

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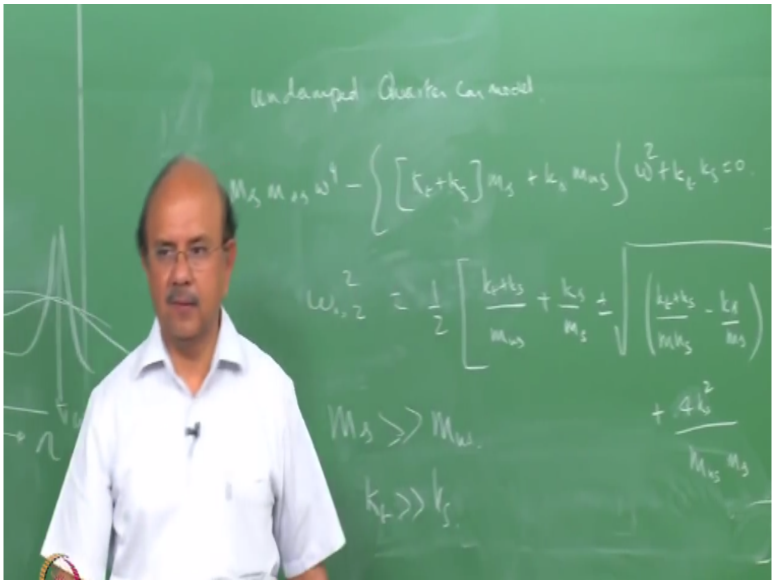

First let us look at, we had finished this part, let us first look at undamped quarter car model, and is very simple to arrive at what we call as the natural frequencies, okay. Follow the same thing, you have the equation, remove c, get the determinant, expand the determinant, all those things are very straightforward you can do that, so you will get that ultimately. If you have any doubts, I will answer it but I think that is a known thing, hence, so the characteristic equation for the undamped system, go back to that equation and then get this, right.

We solved this equation of course for omega and call that as omega 1 square and omega 2 square. Okay and this result in, okay that is expression for omega 1 square and omega 2 square, I am not very happy with expression, it is too long okay. Let us do some simple tricks, immediately get the 2 natural frequencies, without much, as I said keep on assuming, conscious of the fact that my assumption will not affect my results, okay.

This is the standard technique fine, but can someone suggest how else can we do it, one is to do a mathematical (()) (27:49). For example the first thing we observe is that the sprung mass is far, far greater than unsprung mass and the tire stiffness is far, far greater than the ks. These 2 assumptions are, not assumptions, they are facts of flare, yeah. This should be ks, I will just check that, I think that is ks, okay, I will just check that, next class, I will get this thing.

So, these 2 are well known, right. Not, let us get some facts clearly. So there are now 2 natural frequencies, okay, one would say corresponding to the body and the other corresponding to that of the unsprung mass. Okay not that they actually participate, but if you look at the ratios, it is the ratio of excitation, okay we can assume as if that once fixed and the other is exciting and so on, right.

Typical frequencies, we will see in a minute, can be calculated by a very simple assumption, here, so what the assumption be, what do you think of this assumption, a very simple assumption, here. The first is with respect to body okay since it is an undamped system, I am going to remove this and that this mass is greater than this, okay I can neglect it.

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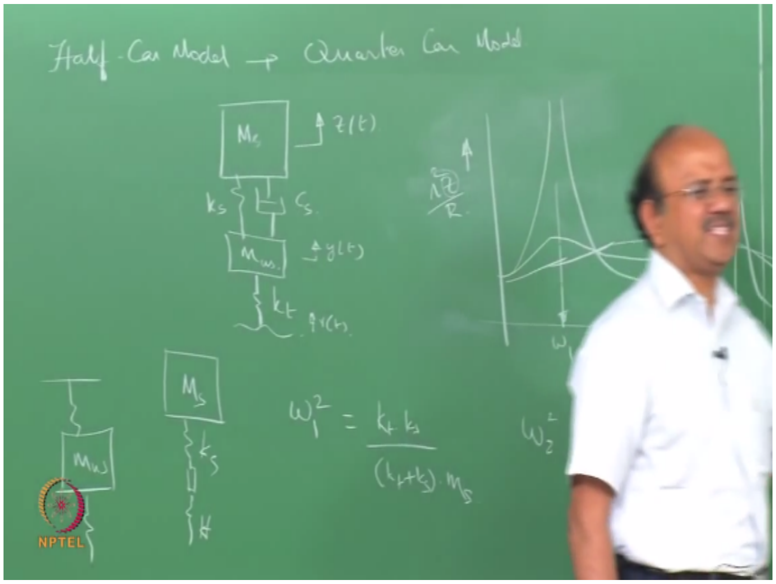

So I can actually look at it as if there is small mass here, okay as if it is going to be like that and 2 springs in series with a mass Ms so that omega 1 square is written as k/M, okay, so this is k spring, k tire, k tire*k spring/k tire+k spring*Ms, okay. So, this is very straightforward, in fact, you can achieve that by an approximate, say for example you can do a Taylor series approximation here and you will get a very similar result right.

So the other omega 2 is arrived at by assuming that the other end is fixed and you have this mass Mus, M unsprung mass, like that. Okay, this is not in series because both of them are going to take the loads, so they are in parallel, so that omega 2 square can be written in terms of k/M which is k in this case is kt+ks/Mus, right. So that gives me the very simple one minute answer for undamped natural frequency with all the approximations that are put in place, clear, okay, yes. We will look at some typical results, okay as to just understand these factors.

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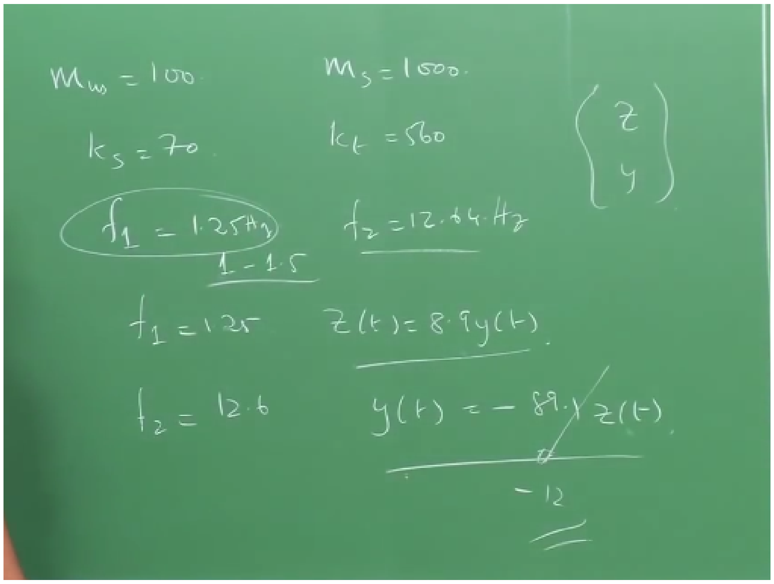

For example for typical a car, which you can take from one of the text book, so let us say that unsprung mass is 100 kg, sprung mass is 1000, ks=70, kt=560, if I calculate now, omega 1 square, so in terms of cycles per second hertz, so f1 happens to be 1.25 and f2 happens to be 12.64 and at f1=1.25 and at f2, okay. These are typical values, this omega 1 which is hertz of course, 1.25 is typical value and usually is between 1-1.5 for all the cars, is between 1 to 1.5 for all the vehicles, that is what we call as the body frequency, right.

And omega 2 which results in frequency of 12.64 hertz, it a typical value and is called as wheel hop frequency of the typical value for the unsprung mass. So, this is a typical wheel hop frequency, okay, so look at that results. Yt for the second case, look at that, 89 or 90 times zt. Which means that the z is almost you know negligible displacement and hence in other words, you can assume as if this is stationary and this guy is jumping, but this is for an undamped system, this is what I was telling, if I damp the system, okay this would come down to say -10 or -12 and so on, so this may be not vary much, this would vary -10 or -12.

“Professor - student conversation starts” Sir why is there is a -sign. Yeah, why is there is -sign, because these are mode shapes, so obviously, the way this, you know, they oscillate, it is oscillating like this, okay, so that is how this, the mode shapes are for this, okay. **“Professor - student conversation ends.”**

Okay, fine, we had already derived the expression for the other cases and we had, that is the approximately the omega 1 and omega 2 what we just now saw, okay you can draw this graph from whatever we know. As I said, optimization of the suspension system is not a very straightforward problem, but one of the earliest attempts to do optimization in the 1950 is by just saying that look at this graph for various c, find out that c for which the curve, let us say this is a point A, all these graphs passes through point B, point C and so on, okay.

It so happens that looking at the graph, people said that if the slope of the graph, the slope of this curve happens to be flat at A, another words slope is 0, okay, or dy/dx=0 at that point, then this curve is flatter almost throughout the range of omega. So this is the first condition, you know very early, okay first condition that they had put.

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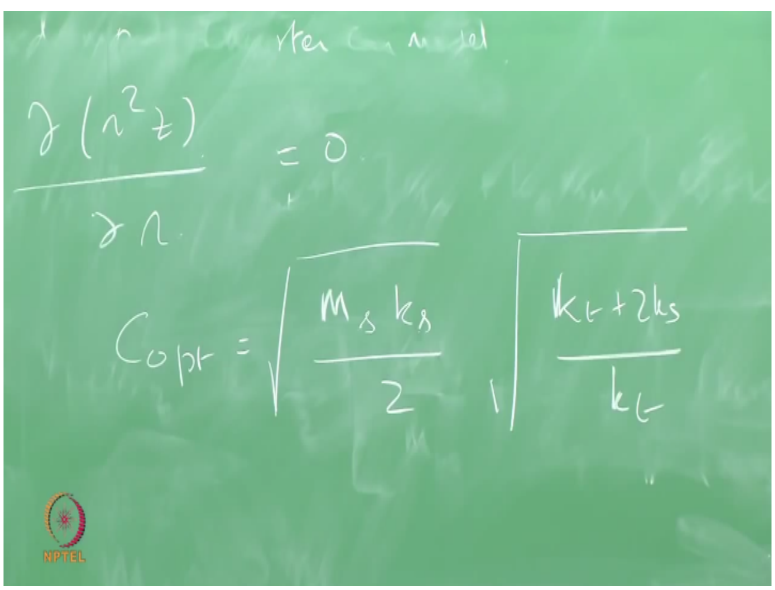

In other words, when they duo of omega 2 z, of course r is what is a given as a constant this=0 okay. This is the first cut optimized c. so the result is that c optimized = root of Ms*ks/2*root of kt+2ks/kt.

“Professor - student conversation starts” (()) (38:01), no, no these are natural frequencies, these are natural frequencies and that frequencies for which the mode shapes are calculated, okay, these are 2 mode shapes for that frequency, that frequency of oscillation, what is the motion, movement of the unsprung mass and the sprung mass that is what we are expressing it here (()) (38:27) no, no we are talking about 0, right answer=0.

This is the free vibration case, if it were to vibrate in that frequency, okay, then what would be the relative motion, this is relative motion, look at that carefully, this it he relative motion between the sprung mass and the unsprung mass, clear, so there is no input here, no, yeah, see more shapes are usually expressed as a vector, okay you would say z, y or z theta and so on, right, as a vector for a particular frequency say omega 1, right.

I am expressing that as a ratio, z/y, normalizing it and expressing it as a ratio if z happens to be 1y equals to this and z=1, y is this and so on, right, so it just as a ratio I am expressing the what we call as a mode ships, clear. So this is an optimized one, I am not very happy with it, since we have to move further away from this, okay. **“Professor - student conversation ends.”**

So in other words, I have to look at the statistical nature of the road and see whether some optimization can be obtained from that perspective. As I said, first cut, this I fine. Now, I am going to shift gears, I am going to quickly run though a case where I am going to back to my half car model, again make some assumptions and look at from the quarter car, I will go over to the half car model, so what are lessons learned in the quarter care mode.

The lessons learnt is that 2 of the important frequencies, okay which are not going to change, in fact, we have done extensive testing with many cars in India, okay, all these cars, the natural frequency, when I put an accelerometer at the action position, the body position and all that, all these cars you can very clearly see this z and y, okay, have natural frequencies between 1.2 to 1.5, right, so that is the natural frequencies which we have and the natural frequencies or the wheel hop frequencies is between 11 to about 13.5, 14.

You know, this I the range in which we have got these frequencies, so these are 2 important frequencies, wheel hop and the body frequencies. We will see it is important again in the in the next class and let us look at what is called as the half car model. I am going to simply this half car model.

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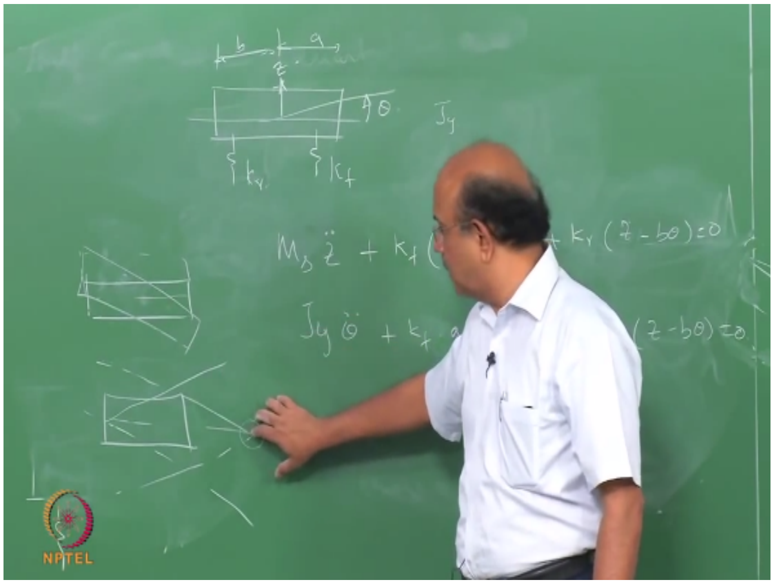

Okay I am not going to derive it again completely because I hardly have 4-5 classes, I want to shift to that statistical nature of the road, when in doubt simplify, that is all. I am going to, look at the way I am going to look at the half car model. I am going to have, that is all my model, okay, I am going to remove the unsprung mass, okay, this is going to give me some important inputs, remove the unsprung mass, okay, put this whole spring in the front, the tire as well as the suspension and call that as kf and kr, right, and then just consider the displacement here as well as the theta okay.

And call that Jy. In other words, what essentially I have done is to compress some of them and expressed this in a very straight forward fashion, okay. Now, what I am going to do is very standard, I am going to do how many equations I am going to get, I am going to get 2 differential equations, okay, we will call that as mass M, okay we have removed or we have neglected the Mus, so that is Ms, so we will call that as Ms, z double prime+kf*z+a theta, a of course you know, what is a, that is b+kr*z-b theta=0.

Okay, that is my first expression, my second expression is Jy*theta double dot+kf*a*z+a theta, these are the 2 equations. I am now looking at natural frequencies, undamped natural frequencies, okay. Because what happens is that for the proportional damping, this x is not that much affected, we will see that in a minute, so I am removing the damping and we are looking at the undamped case.

How far it is correct, lot of questions, okay first cut is fine, right. Now we do not have time to complete this, but what are we trying to do here, we are trying to find out, see there are 2 modes in this, let us understand the physics, equations follow. There are 2 modes to it, what are the 2 modes, one is what is called as the bounce mode and the other is the pitch mode, okay. A bounce mode and a pitch mode, so those bounce and the pitch modes are given by these 2 things, okay.

Now, because of the fact, look at that equation, because of the fact that the equations are coupled unless I have a special cut, they are coupled, rewrite it, you will see that both these equations are couples equations, okay, so what is meant by coupled equation, simply means that z will have an effect on theta and theta will have an effect and so on. There is only one condition into which I can uncouple it, yes I uncouple it, let us know look at before we go into the details, look at the solution for coupled and uncoupled equations.

What do I mean physically by coupled and uncoupled equations. Suppose, I have a method of uncoupling, the condition is very simple, we will see that in a minute, okay. So in other words, you expand it, I have to make the first equation, theta=0 that will be the condition so what will be condition, kf*a=kr*b, it becomes an uncoupled equation, that is what you will see everywhere, right. Here, also you will see the same thing, okay, I think –b or, may this is minus, just check that, we will come to that in a minute.

So that both the equations will become uncoupled, yes minus there. What is meant by uncoupled equation, so when there is a bounce mode, if this is the my vehicle, where there is a bounce mode, there were will be pure bounce at a particular frequency when it gets excited, it will be a pure bounce and when it is a pitch mode, then it will be a pure pitch, okay, this is called as the uncoupled vibration, they are not coupled.

On the other hand, when I have both of them coupled, how it would vibrate, I have Ms, okay, it would vibrate in 2 modes, omega 1 and omega 2, okay. In one case, it will vibrate like this, with respect to a particular point, it will vibrate, so in other words, there will be displacement as well as there is a rotation, which will take it to this, so it will vibrate like this, okay. In another mode, it will vibrate like that.

Okay, both of them are now coupled, this is uncoupled and this is coupled, okay, point number 1, this is clear? Point number 2, what is the significance, why we are doing this, yes I understand the assumption, but why we are doing this? So what is that I want from this, I want minimum vibration, okay or minimum disturbance when I sit in the sprung mass. So where I am going to get the disturbance, when the vehicle goes, say for example a front wheel goes over a bump, okay, now what happens in this case?

When the front wheel goes over a bump, wherever you are sitting, you are going to oscillate. Okay, because that is the mode, so rear wheel goes over a bump, again you are going to oscillate and where there is a bounce, whether you are sitting in the front or rear, you are going to oscillate, so would you prefer this, no you are going to oscillate all the time.

Okay, let us look at this situation, suppose this point happens to be in the front seat, let us say that your front seat which is very close to the suspension and this happens to be the rear, okay. Now, one of them oscillate, when it oscillates because at the place where you give the input, when you would see that the front bump would not affect the rear, rear bump would not affect the front and so on.

This is a much more optimum from point of view of the vibration characteristics when compared to this, right, so in other words, how best can we get this kind of behaviour, or what are the conditions under which we get this behaviour when compared to this is going to the aim of the derivation, right, the first thing is very clear, I have to avoid coupling, right.

The second thing is how do I place, what is this called as the node, okay, in the front, how do I place this node at the rear, this is my next question, okay, how I am going to answer this questions, by looking at the mode shape for omega 1 and omega 2 and that is what we are going to derive from this, we will do that in the next class.