Course: Introduction to Graph Algorithms

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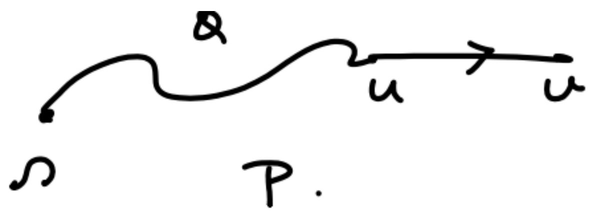
Lecture 09 Properties of shortest path distances Part 2

so here is the next important theorem that we are going to show: if G has no negative cycles, there is an edge uv such that delta v equals delta u plus wuv.

$$\delta(v) = \delta(u) + w(u, v)$$

The sentence is already grammatically correct. Again, the proof is very typical of the kind of proofs we encounter. Let P be the shortest path from s to v, and let u v be its last edge. Therefore, from s, this is the shortest path to v.



Let Q be the part of P from s to u. This part is Q from s to u; therefore, P equals Q plus uv. By adding the edge uv, you obtain P. Consequently, the weight of P equals the weight of Q plus the weight of uv.

$$P = Q + (u, v)$$

$$w(P) = w(Q) + w(u, v)$$

If we prove that Q is the shortest path from s to v, suppose you prove Q is the shortest path, weight of P is equal to the weight of Q plus the weight of uv, if we prove that Q is a shortest path from s to u, if we can demonstrate that it is indeed the shortest path from s to u. The weight of Q is equal to delta u and delta v, which is also equal to the weight of P because P is the shortest path. This is equal to wQ plus the weight of uv, which is delta u plus the weight of uv.

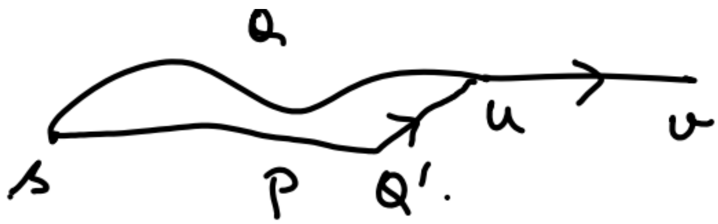
$$w(Q) = \delta(u)$$

$$\delta(v) = w(P)$$

$$= w(Q) + w(u, v)$$

$$= \delta(u) + w(u, v)$$

We would have proven our theorem if we had proven that Q is a shortest path, okay? We prove that Q is a shortest path by contradiction, assuming that Q is not the shortest path. What will be the contradiction we get? This is very typical of what is known as a cut-and-paste argument. Let us say this is P, u, v, and this is Q. If q is not the shortest, let us consider q dash. Let Q be the shortest path from s to v;



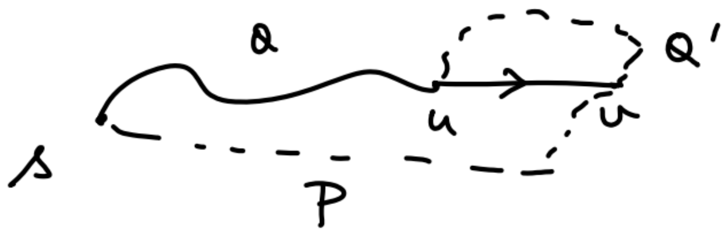
if Q is not the shortest path, let Q dash be a shortest path from s to u. Therefore, the weight of Q dash is less than the weight of Q.

$$w(Q') < w(Q)$$

Now consider the following cases: Q dash does not contain v. Then, Q dash plus uv forms a path from s to v. You can see in the figure that Q dash goes from s to u, and since u to v is added, it constitutes a path from s to v, and the weight of Q dash plus uv is equal to the weight of Q dash plus the weight of uv, but the weight of Q dash is also equal to the weight of Q plus the weight of uv, which is equal to the weight of P and to delta v.

$$\begin{aligned} \omega(Q' + (u,v)) &= \omega(Q') + \omega(u,v) \\ &< \omega(Q) + \omega(u,v) \\ &= \omega(P) \\ &= \delta(v). \end{aligned}$$

Therefore, I have a path from s to v; this path from s to v, its weight is smaller than delta v, meaning that Q dash plus uv is a path from s to v with weight is smaller than the weight of P. P is the shortest path; you cannot have a path shorter than that. This is a contradiction. Thus, Q dash must pass through v; therefore, the scenario is as follows: I have s. This is Q, this is uv, and the whole thing is P. I have the shortest path, Q dash, which must pass through v. so, Q dash looks like this; this is how Q dash is represented.



Q dash is passing through v now. If Q dash does not pass through v, we have a contradiction, which forces us to consider that Q dash must pass through v. Therefore, Q dash plus uv is a walk from s to v, since v is already included. And again, you are visiting vertex v; therefore, vertex v occurs twice in Q plus uv. Thus, it is not a path; it is a walk, right?

However, in any walk, we can remove the cycles and reduce the weight because G has no negative cycles. You can see that Q plus uv is a walk; sorry, I meant Q dash plus uv is a walk. The weight of Q dash plus the weight of uv is less than the weight Q plus weight of uv that is equal to the weight of P, but Q dash plus uv is a walk. This walk from v has a path to u and a cycle from v to u and back to v.

$$\begin{aligned} w(Q') + w(u,v) &< w(Q) + w(u,v) \\ &= w(P) \end{aligned}$$

Therefore, if you remove the weight, you eliminate that cycle. When we remove the cycle v from Q dash uv, we obtain a path s to v, which is part of Q dash . Now you write Q dash as Q dash plus uv, where Q dash goes from s to v plus a cycle C. What is that cycle? It is Q dash from v to u plus the edge uv. This constitutes a cycle, okay?

$$Q' + (u,v) = Q'[s,v] + \big(Q'[v,u] + (u,v) \big)$$

Q dash is written in two parts as it passes through v. Let us express this: Q dash is passing through v, so we can split Q dash at v. Therefore, Q dash is equal to the segment from s to v plus the segment from v to u. There are two parts: Q dash goes from s to u. since v is a part of the path, when I split at v, I obtain two segments. Therefore, Q dash plus the edge uv is equal to Q dash sv plus Q dash vu plus the edge uv.

$$Q' = Q'[s,v] + Q'[v,u]$$

$$Q' + (u,v) = Q'[s,v] + Q'[v,u] + (u,v)$$

This forms a cycle that consists of Q dash from s to u, Q dash from v to u, and u to v, completing the cycle. This cycle is located here. I can drop this because all cycles have positive weights; therefore, the weight of Q dash plus uv is greater than or equal to the weight of Q dash sv, since the cycles have non-negative weights.

$$w\left(Q' + (u,v)\right) \geq Q'(s,v)$$

There is no negative cycle, so any cycle will have either a weight of 0 or a positive weight. You can drop that, and it becomes an inequality. This is a path from s to v, right? Indeed, this is a path from s to v. Q dash is less than or equal to w Q dash plus uv, which is less than wP. This means I have a path with a weight smaller than Q dash.

The weight of Q dash sv is less than or equal to the weight of Q dash uv, which is less than wP.

$$\begin{aligned} w\left(Q'(s,v)\right) &\leq w\left(Q' + (u,v)\right) \\ &< w(P) \end{aligned}$$

This is, again, a contradiction. You cannot have a path with a smaller weight; therefore, this contradicts the minimality of P. P is the shortest path, but I have a segment of Q dash from s to v that has a smaller weight, which is impossible, correct? so this clean proof shows that you cannot have a path shorter than Q; this implies that such a path Q dash, from s to u, with a weight smaller than that of Q, cannot exist.

This implies that Q is the shortest path from s to u, which in turn implies that the weight of Q is equal to delta u, which is equal to delta v. This is also equal to the weight of P, which is equal to the weight of q plus uv, or the weight of q plus the weight of uv. Therefore, this is equal to delta u plus the weight of uv.

$$w(Q) = \delta(u).$$

$$\begin{aligned} \delta(v) &= w(P) \\ &= w\left(Q + (u,v)\right) \\ &= w(Q) + w(u,v) \\ &= \delta(u) + w(u,v) \end{aligned}$$

We have proven the theorem. Here is the corrected version of your sentence: so, what is the conclusion? We obtain the contrapositive of Theorem 1.

This theorem implies that delta v is equal to the minimum of delta u plus the weight of edge uv, for every edge uv in E. If there is an incoming edge to v, consider delta u plus w(uv), which is the same as the minimum over u belonging to V such that uv belongs to E. For each neighbor, you compute delta u plus w uv and take the minimum to obtain delta v. Alright, it is a beautiful equation. We will do a little cleanup here and make it look even simpler.

$$\delta(v) = \min_{(u,v)\in E} \{\delta(u) + w(u,v)\}$$

$$= \min_{\substack{u \in V \\ (u,v)\in E}} \{\delta(u) + w(u,v)\}$$

Suppose we extend the weight function; if an edge is not present, we set w u v is eqal to infinity. w(u, v) is equal to infinity if u and v do not belong to E and u not equal to v . We will also set w(u, u) is equal to 0 for all u ; this defines a weight function. Your input graph has edges, and for each edge, there is an associated weight.

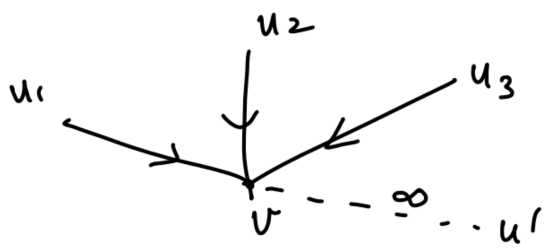
$$w(u,v) = \infty \text{ if } (u,v) \notin E, \; (u \neq v).$$

$$w(u,u) = 0 \;\; \forall u.$$

We extend it mathematically, if an edge is not present for that pair, we associate a weight of infinity with every vertex u . Of course, the graph does not have any self-loops, so wuu is set to 0, and wuv is equal to infinity, under this definition, this holds true for any u. Because if (uv) is not present, wuv is infinite; any quantity will be less than infinity, in any case. We can write this as delta v equal to the minimum of delta u over u, plus the weight, of course, as the minimum.

$$\delta(v) = \min_{\substack{u \\ u \neq v}} \{\delta(u) + w(u,v)\}$$

so, you compute all of this, do you? Not equal to v because v is on the left-hand side. For every vertex v, compute for all u1, u2, and u3, first, calculate delta u1 plus the weight of u1v; then compute delta u2 plus the weight of u2v, delta u3 plus the weight of u3v, and so on.



$$
\begin{gathered}
\delta(u_1) + \omega(u, v), \\
\delta(u_2) + \omega(u_2, v), \\
\delta(u_3) + \omega(u_3, v) \\
\vdots
\end{gathered}
$$

Is that okay? If an edge is not present from vertex u dash, it is assigned a value of infinity. Every quantity is less than infinity; therefore, in your actual computation, you do not have to use it, simplify the notation by introducing vu dash is infinity,

$$
\delta(u') + \omega(u', v) = \infty
$$

Anyway, the quantities delta u dash plus the weight of u dash v will be infinite, so, if a u dash is not adjacent, this quantity automatically becomes infinity and does not affect the minimum. In any case, please calculate these quantities and determine the minimum. Then we obtain delta v. Therefore, we have a very compact and clean formula for the vertices in a graph, delta v. Is that clear?

Now, if the graph has got no negative cycle, what is delta s, from s, the shortest path to s. Essentially, it is a closed path, which means it forms a cycle. But all cycles got 0 or positive, so, why go out of s? or "so, why leave s? stay in s. To reach s from s, do not move. To avoid any calculations, we set delta s equal to 0 since all cycles are non-negative and there are no negative cycles. Therefore set delta s equal to 0, there in sum set delta s equal to 0, delta v is equal to the minimum over u not equal to v of delta plus weight of uv,

$$\delta(s) = 0 \,;$$
$$\delta(v) = \min_{u \neq v} \{ \delta(u) + w(u,v) \}$$

and this is an interesting relation between input and output values, input are the values of the weight functions. The output values we are looking for are the delta values. There is a functional equation established between the input values and the outputs you are seeking. These equations are known as the Bellman equations for the weights of the shortest path weights.

The shortest path weights satisfy this equation. Therefore, if you define a variable xv as one associated with the vertex v, you can formulate the equation. What is the equation you have to form: xs is equal to 0 and xv is equal to min over u notequql to v, xu plus weight of uv. These equations are abstract form of Bellman's equations, which is called the Bellman equations for n variables.

Given a graph, you can always write down these equations because the graph has a weight uv for each edge, and there is a weight for each vertex represented by a mathematical variable. With n variables, if you write down these equations, you will obtain the Bellman equation.

A solution to the Bellman equation is the shortest path distance, We have shown that xv is equal to delta v, is a solution for the Bellman equation.

We have shown that this is the solution for the Bellman equation. That’s a good news.

We have a mathematical equation whose solutions are the values we are looking for, as we want to calculate the shortest path distance. To compute this, we can set up an equation and then attempt to solve it. The solution is that the Bellman equations represent the shortest-path distances. Okay, things look promising; however, we are not there yet.

We still need to show one very important result. We may form the Bellman equation, but what if that equation has non unique solutions? Delta v may be one solution, but if there are other values that are also solutions, after solving the equation, you will obtain several values, yet you will not know what they represent. The shortest path distance or any arbitrary number is provided. We are going to add one more very simple condition that will ensure the sequence has a unique solution: namely, the shortest-path distance. Now we can attempt to solve the equation because, if we succeed, the solution will be unique and will represent the shortest path distance, meaning we will obtain the shortest path distance. so, I state a theorem that guarantees a unique solution for the Bellman equation; it is a very simple extension, if G has no negative cycles. Or no zero cycles then Bellman

Equation have a UNIQUE solution. We shall continue our discussion on these aspects in next session. Thank you.