Course: Introduction to Graph Algorithms

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Lecture 8: Properties of shortest path distances

We have been looking at the properties specifically the mathematical properties of the shortest path distances. Let G is a graph with vertex set E, edge set E, weight function w and a source vertex s.

$$G = (V, E, w, s)$$

Let us assume that this is the instance of a single source shortest path problem and let delta (v) is the weight of shortest path from s to v. These notations we have introduced already. And suppose we have an edge (u, v) you have the value delta (u) associated with this and a delta (v) associated with this.

We have seen that, if delta (u) plus weight of (u, v) is less than delta (v),

$$\delta(u) + w(u, v) < \delta(v)$$

if this is the case then G has a negative cycle. If delta (u) plus w(u, v) is less than delta (v) then G has negative cycle. We are going to look into a kind of a converse for this, which will be our next theorem.

If G has a negative cycle then, G has an edge (u, v) such that delta (u) plus weight of (u, v) is less than delta (v).

$$\delta(u) + w(u, v) < \delta(v)$$

So if the graph has a negative cycle then there is an edge satisfying this property, if there is an edge satisfying this property then G has a negative cycle. So, this is a kind of a characterization, okay.

So, how do we prove this? It is a kind of a converse, this is an important theorem and the proof technique is also pretty canonical in the sense these kind of arguments we will use often in the course of our discussion of proofs for other theorems, okay. The proof is by contradiction, this is by contradiction. Suppose for all edges, right, what is the negative of there exist an edge? It is a negative is for all edge, proof by contradiction, suppose for all edge (u, v) we have the following delta (u) plus weight of (u, v) is greater than or equal to delta (v).

$$\delta(u) + w(u, v) \geq \delta(v)$$

I am assuming this for the sake of contradiction, you will see that. Since G has a negative cycle let C be a negative cycle of size k, so let us say it starts with v0, v1, v2and so on, vk, where vk equal to v0, it is a negative cycle.

$$\langle v_0, v_1, v_2 \ldots, v_k \rangle . \quad v_k = v_0$$

I have given the list of vertices of the cycle v0, v1, v2, vk-2, vk-1, next is vk the same thing is vk it is a cycle, end to end connection

So the same vertex has got 2 notations v0 and vk because it is an end to end connection we have vk is same as v0. If it is an open path the starting vertex and ending vertex would be different, because it is a cycle by our notation we said vk equal to v0 and this is how the scenario is. Note that this is a negative cycle. Since G has a negative cycle, we have taken a negative cycle c and these are the edges of the cycle. We know that for each edge, the condition that we have stated, look at the condition, delta (u) plus w(u, v) is greater than or equal to delta (v) for an edge (u, v).

So, for (v0, v1), delta (v0) plus weight of (v0 v1) is greater than or equal to delta (v1).

$$\delta(v_0) + w(v_0, v_1) \geq \delta(v_1)$$

For the edge (v0 v1) this is true. For the edge (v1 v2), delta (v1) plus w(v1 v2) is greater than or equal to delta (v2) and so on.

$$\delta(v_1) + w(v_1, v_2) \geq \delta(v_2)$$

Finally the last one delta (vk-1) plus   w(vk-1 vk) is greater than or equal to delta (vk)

$$\delta(v_{k-1}) + w(v_{k-1}, v_k) \geq \delta(v_k)$$

Because for each edge this is true, for the edges in the cycle this is true. So, we have k inequalities, adding all these inequalities we get the following, this is sigma i equal to 0 to k-1 because v0, v1 is there, delta (vi) plus, (v0, vi) is an edge of the cycle, (v1, v2) is an edge of the cycle. So these are all the k edges of the cycle, the cycle has got k edges and the weights of this, therefore this is weight of the cycle c. This is greater than or equal to sigma i equal to 1 to k of delta (vi) because here you have v1, v2, vk, i goes from 1 to k.

$$\sum_{i=0}^{k-1} \delta(v_i) + w(c) \geq \sum_{i=1}^{k} \delta(v_i)$$

So adding all these inequalities we get another inequality. Since vk equal to v0, these two are equal, actually this expression and this expression are equal, okay. Why? Let us see.

delta (v0) plus delta (v1) plus delta (vk-1)

$$\delta(v_0) + \delta(v_1) + \cdots + \delta(v_{k-1})$$

is equal, to v0 is same as vk, therefore this is delta (vk) plus delta (v1) plus delta (vk-1), delta (v0) plus delta (v1) plus … plus delta (vk-1)

$$= \delta(v_k) + \delta(v_1) + \cdots + \delta(v_{k-1})$$

Since, v0 is equal to vk delta (vk) plus delta (v1) plus … plus delta (vk-1), move the vk term to the end, you will see that this is delta (v1) plus delta (v2) plus delta (vk-1) plus delta (vk), alright. delta (v1) plus delta (v2) plus … plus delta (vk-1) plus delta (vk)

$$= \delta(v_1) + \delta(v_2) + \cdots + \delta(v_{k-1}) + \delta(v_k)$$

Therefore, these two sums are equal, alright, this implies sigma i equal to 0 to k-1 of delta (vi) is same as sigma i equal to 1 to k of delta (vi) ,

Summation i is equal to 0 to k-1 delat vi is equal to summation i is equal to k delta (vi)

$$\Rightarrow \sum_{i=0}^{k-1} \delta(v_i) = \sum_{i=1}^{k} \delta(v_i)$$

because these two are equal we cancel this. Equal terms can be cancelled and you get this implies w(c) is greater than or equal to 0,

$$\Rightarrow \quad w(c) \geq 0$$

Why this is greater than or equal to 0? I cancel these two equal terms okay, so this term, the term on the left and the term on the right are cancelled. But this is a contradiction weight of the c is greater than or equal to 0 is a contradiction. This contradicts the fact that c is a negative cycle, c is a negative cycle. This implies not all edges will have this property, it is negation, this implies there exist an edge (u v) such that delta (u) plus weight of (u v) is less than delta (v).

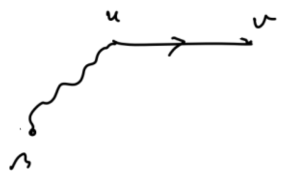
$$\delta(u) + w(u, v) < \delta(v)$$

We have proved by contradiction the statement of the theorem, okay, very clean and simple proof.
So theorem 1 and 2 imply the following:

G has a negative cycle if and only if there exist an edge (u, v) such that delta (u) plus weight of (u, v) is less than delta (v).

$$\exists \; (u, v) \ni \\ \delta(u) + w(u, v) < \delta(v)$$

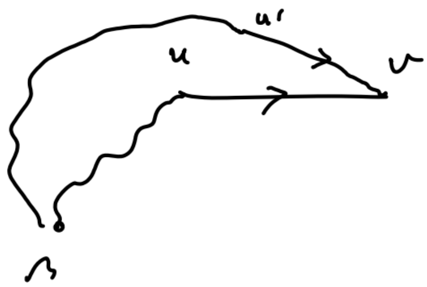
This is the first mathematical property and what is the implication of this mathematical property? If G has a negative cycle then the delta (u) and delta (v) and all are not related by any equation, okay. There is no specific relation, one as an upper bound, some of the neighbors may have larger value, some of them have a smaller value and it could be anything. So we are not able to say precisely anything about the values delta (v). What would you generally expect? Just a common sense requirement. If (u v) is an edge, if s is a vertex suppose this is a shortest path.



If this is a shortest path and if this is a last edge how would you have reached u. Just a common sense thinking, you will not wander around and reach u. The way in which you would reach u from s must be a shortest one, ideally this is what you would expect, so you would expect a kind of a relation where you have something like delta (u) plus weight of (u, v) is equal to delta (v) kind of a relation.

$$\delta(u) + w(u, v) = \delta(v)$$

And if this is a shortest one if you reach any other way, right, suppose you reach from some other vertex, some u dash,



even if you have taken a shortest path to u dash, that path is not shortest path, it is longer than that, therefore you would expect delta (u dash) plus w (u, v) is greater than or equal to delta (v), right.

$$\delta(u') + w(u,v) \geq \delta(v)$$

If you reach v in any other way that will define a longer path, but if you reach via u that is going to define the shortest path. Any other way is going to be longer, therefore this will be the shortest. It is a kind of intuition. This is what we would want, right. In other words we would want something like, intuitively we want something like delta (u) plus weight of (u, v) is greater than or equal to delta (v) for all (u, v) in , any last edge.

$$\delta(u) + w(u,v) \geq \delta(v)$$

for all (u,v) belongs to E

$$\forall\ (u,v) \in E$$

And there exist an edge (u, v) such that

$$\exists\ (u,v) \ni$$

delta (u) plus weight of (u, v) is exactly delta (v),

$$\delta(u) + w(u,v) = \delta(v)$$

This what we would intuitively want. If this is the case then we can write something like delta (v) equal to minimum over (u, v) belong to E such that delta (u) plus weight of (u, v).

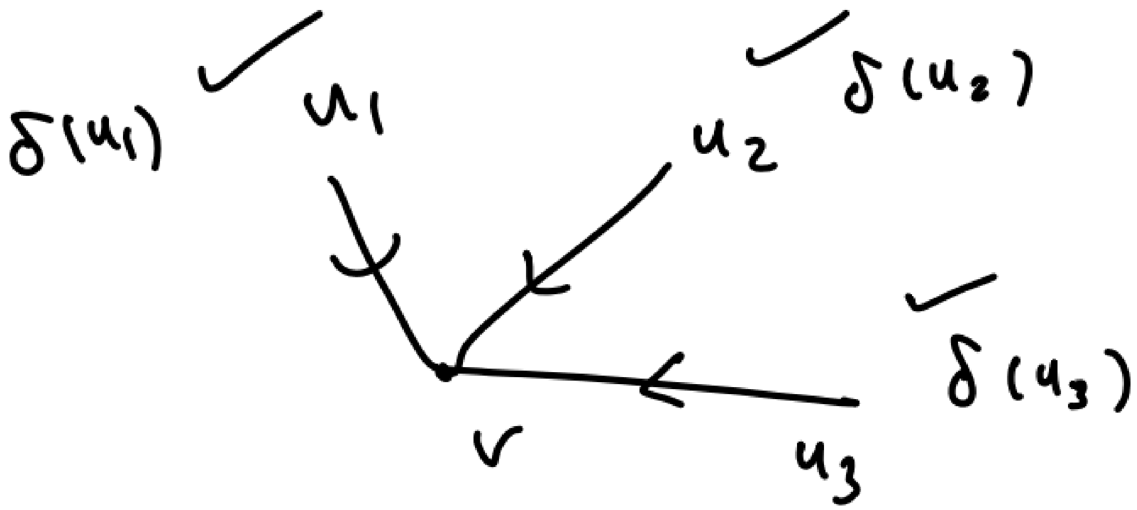
$$\delta(v) = \min_{(u,v) \in E} \left\{ \underline{\delta(u) + w(u,v)} \right\}$$

Calculate all these values, take the minimum you will get delta (v).

Pictorially suppose this is v and from u1 there is an edge, from u2 there is an edge, from u3 there is an edge.

For u1 calculate this, for u2 you calculate this and for u3 you, for every incoming edge. Assuming that I have the information related to the shortest path, the distance to the neighbors, then I can find the distance, the shortest, weight of the shortest path to v because it is just going to be the minimum. This is intuition, right. This is what ideally we would like to have. I mean if it is there then we can kind of proceed but that is not going to happen if G a negative cycle. If G has a negative cycle you can see that there are edges for which this kind of inequalities do not hold good, right.

Therefore, we are going to work only with graph that do not have negative cycle, okay. If G has a negative cycle then even if I have the information about all the shortest paths to the neighbor, that is not enough to find the shortest path to v. In other words I have v, I have some u1, I have information delta (u1), I have u2 and I have delta (u2) this information.



Like this I may have several neighbors and I may have information about them still that is not enough to find delta (v). A shortest path from s to v must have a last edge ending in v, it should be one of these edges (u1, v) or (u2, v) or (u3,v) one of the edges must be there the last edge. However, none of these edges are helpful in finding the shortest path. Because if G has a negative edge we do not have any inequalities, we do not have a specific relation, okay. Therefore, if G has a negative cycle we are unable to determine the weight of the shortest path in terms of the weight of the shortest path of its neighbors, okay. So, we assume that G has no negative cycles. In fact, we can show that no efficient algorithm is likely to exist if G has a negative cycle. If G has negative cycle then finding the shortest paths and shortest path distances is NP complete, it is a very hard problem alright. We have no way of solving them efficiently. NP complete means there is no hope that we will be able to find a polynomial time algorithm for it.

Therefore the assumption that G has no negative cycle is becoming important even from the complexity theory point of view, okay. So, what are the immediate consequences of assuming that G has no negative cycle? If G has no negative cycles, one, contrapositive of theorem 1 is true. That means for every edge, for all (u v) in E, delta (u), plus weight of (u v) is greater than or equal to delta (v).

$$\forall (u,v) \in E$$

$$\delta(u) + w(u,v) \geq \delta(v)$$

A property that we were looking for is true, okay. The second most important property that, if G has no negative cycle, there exists an edge (u v) such that delta (v) is in fact equal to delta (u) plus weight of (u v).

$$\exists \text{ an edge } (u,v) \ni$$
$$\delta(v) = \delta(u) + w(u,v)$$

This property alone is not enough, they all provide the upper bound, delta (v) is smaller than these quantities, but what is the exact one? It is smaller than all these quantities, one of them is equal, therefore it is the minimum of these kind of quantities, right. That is the conclusion we get the moment we prove this. So this is the next important theorem, if G has no negative cycles then there exist an edge (u v) such that, delta (v) equal to delta (u) plus, is a statement of the theorem.

So if G has no negative cycles, the contrapositive implies these inequalities but they provide only an upper bound, this does not give still a formula. In order to have a formula we need one more result. In other words if G has no negative cycle we are going to show that there exist an edge (u v) such that delta (v) equal to delta (u) plus w (u v).

We will see the proof of this theorem in our next session thank you.