Course: Introduction to Graph Algorithms

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Lecture 7: Single source shortest path problem

Namaskara, we have already seen in a directed edge weighted graph, the concepts like walks, paths, shortest walks, shortest paths and some of their properties. These are all the things that we have seen. It turns out that finding a shortest path between a pair of vertices is as hard or as easy as finding the shortest path from one fixed vertex to all other vertices. So instead of finding the shortest path between u and v, I choose a fixed vertex say s and I will try to find the shortest path from s to all other vertices. Both problems have got the same complexity, same issues to be addressed and so on. So, henceforth we will focus on single source shortest path problem. The vertex $s,$ the fixed vertex s that we have chosen is called the source.

So, you are given a graph G which is V E weight function and a fixed vertex s belong to $V,$ that is also given.

$$G = (V, E, w, s) \qquad s \in V$$

Now we have to compute $\delta(s, v)$ the shortest path distances from s to all other vertices.

$$v \in V - \{s\}$$

So if V has n vertices, other than s you have $n - 1$ vertices. Therefore $n - 1$ weights, shortest path weight. We have to find not only that we want physically the path. Let P_v denote shortest path from s to v. We will make one simplification in the notations, since s is fixed we denote $\delta(s, v)$ by $\delta(v)$. So in the discussions related to the single source shortest paths, s is always fixed. We are interested in finding $\delta(v)$. $\delta(v)$ is nothing but $\delta(s, v)$; from s to v the weight of the shortest path, okay. From s to v the weight of the shortest path. So, given G equal to (V, E, w, s), output delta v and P_v for all v belong to $V\ minus\ \{s\}$

Given,

$$G = (V, E, w, s) \qquad s \in V$$

Output,

$$\delta(v),\;\; P_v \; \forall\; v \in V - \{s\}$$

this is called single source shortest path ($SSSP$) single source shortest path problem ($SSSP$) problem.

The vertex s is called the source. We are using only one source and from that source to all other vertices we have to find out the shortest paths. So we need the shortest path weights and the shortest paths. This is what we want. It is quite typical in Mathematics when we have to find or determine certain unknown values, we set up certain functional equations. We derive certain bounds. We do the mathematical analysis of the quantities that we are interested in and find the equations that are satisfied by the values and so on. If we have certain equations that would have their solutions, right, that would have these values as their solution, then we first formulate that equation then attempt to solve, because the solution of those equations are the values we are looking for.

Therefore, we would derive several mathematical properties of $\delta(v)$'s. We first focus on the delta v values and then construct certain equations whose solutions are this $\delta(v)$. We will mathematically prove that there are certain equations and the solutions of this those equations are this $\delta(v)$ values. Once we have mathematically proved that, our algorithm would involve or our computational process would involve formulating those equations and attempting to solve them. Once we have the solution we have the values we are looking for. That is the master plan.

So how to find equations whose solutions are $\delta(v)$ values? How to construct equations whose solutions are $\delta(v)$ values? Once we know how to do this mathematically our algorithm would begin with the formulation of these equations and it will go about solving those equations. The output for that is the values we look for, okay. So in order to derive the equations that would satisfy them we have to look at various properties.

So mathematical foundations of shortest path, mathematical foundations or mathematical properties of shortest path weights, okay. Here is the first and one of the most important theorems related to shortest path distances. Let G equal to (V, E) be a directed graph with weight function w and a source vertex s, okay.

$$G = (V, E, w, s)$$

Let $\delta(v)$ be the weight of shortest path from s to v, okay. If there exists an edge (u, v), if there exists an edge (u, v) such that $\delta(u)$ plus weight of (u, v) is less than $\delta(v)$, strictly less than $\delta(v)$, we call this as equation one, we will refer this very often.

$$\delta(u) + w(u, v) < \delta(v) \quad \ldots \ldots \ldots \; \|1\|$$

So, such that $\delta(u)$ plus $w\ (u,v)$ is less than $\delta(v)$, then G has a negative cycle, okay. So some structural properties, if the values have certain properties the graph should have certain property and so on.

So it is purely mathematical thinking looking into properties, these kind of properties are crucial in arriving at the equations at a later point okay. Very important theorem. We give a proof for this, let P be a shortest path from s to u, that is because you can see that on the left hand side you have $\delta(u)$ and weight of (u,v). So let P be a shortest path from s to u, you can visualize like this s to u you have P and the edge (u,v) okay and the edge (u,v).

Case 1, P does not contain v, that is what I have drawn already I have drawn P, v is not in the path, so v is outside so (u,v) is going to extend a path and it will be a path. So in P does not contain v, in this case, in this case, P + (u,v) that is P added with that edge, extended with an edge is a path, in this case P + (u,v) is a path and weight of P + (u,v), weight of this path, is equal to weight of the path P + weight of (u,v).

$$w\big(P + (u,v)\big) = w(P) + w(u,v)$$

Weight of the path P, P is the shortest path from s to u therefore this is $\delta(u)$ plus weight of (u,v).

$$w\big(P + (u,v)\big) = \delta(u) + w(u,v)$$

But $\delta(u)$ plus weight of (u,v) is less than $\delta(v)$ by equation, by inequality 1,

$$\delta(u) + w(u,v)\ <\ \delta(v) \quad by\ \|1\|$$

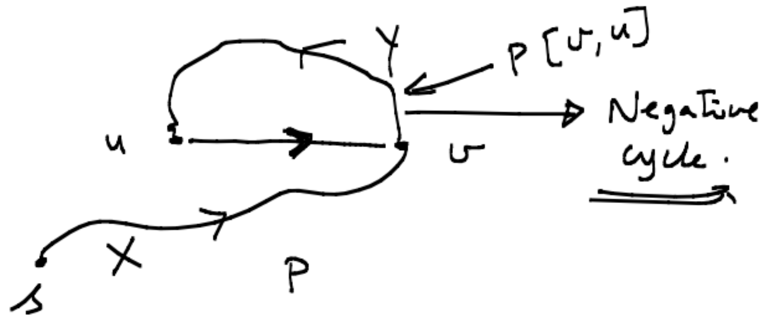
therefore weight of P plus (u,v) is less than delta v, this is impossible, right.

$$w\big(P + (u,v)\big) < \delta(v)$$

Why this is impossible? $\delta(v)$is the weight of the shortest path from s to v. Can we have any other path from s to v whose weight is smaller? It is not possible because that is the weight of the shortest path. No path can have weight smaller than $\delta(v)$. But here is a path, P plus (u,v), is a path from s to v, that is shown to have a weight smaller than $\delta(v)$, this is impossible. Why this is impossible? A path from s to v with weight smaller than $\delta(v)$

is not possible. A path from s to v with weight smaller than $\delta(v)$ is not possible or you can write cannot exist, such a path cannot exist, and therefore this case is impossible.

The other option, what is the other option, P containing v, hence P must contain $v. P$ not containing v is not possible. Therefore P must contain v. That means the shortest path from s to u must go through v, must contain v, must contain v, as shown below s to v and then it goes to u.



This is the structure of P, P contains v, (u, v) is the edge. P now contains, P must contain v, so somewhere in P, v is there, so from s to u you have a path and v is a part of that path. Since v is a part of the path, I can break P into two parts. P from s to u can be written as P from s to v plus the same path, portion of the path, part of the path from s to v and part of the path from v to u.

$$P[s, u] = p[s, v] + p[v, u]$$

So P is broken into two parts with an intermediate vertex. There is a path, there is an intermediate vertex, you split into two parts, okay.

Call the weight of this part X and call the weight of this part Y, that is let weight of P $[s, v]$ equal to X and weight of P $[v, u]$ to be Y.

$$w(P[s, v]) = X$$

$$w(P[v, u]) = Y$$

In this picture, let me write X here and Y here. X and Y represent their weights, I just written them there to show that that part has weight X the other part has weight Y. P is a shortest path, so its weight is $\delta(u)$. P is the shortest path from s to u its weight is $\delta(u)$, so we get $\delta(u)$ equal to X plus Y, call this as equation 2.

$$\delta(u) = X + Y \quad \ldots\ldots\ldots \quad ||2||$$

Now in the graph s to v, the portion of the path $P[s,v]$, is some path, it has a weight, that weight must be larger than the weight of the shortest path, right? Since $P[s,v]$ is a path from s to v, weight of $P\ [s,v]$ is greater than or equal to $\delta(v)$,

$$w(P[s,v]) \geq \delta(v)$$

$\delta(v)$ is the weight of the shortest path, this is some path, so its weight must be larger than or equal to the weight of the shortest path. But weight of $P[s,v]$, we have, that means, this implies X is greater than or equal to $\delta(v)$. X is greater than or equal to $\delta(v)$.

$$X \geq \delta(v) \quad \ldots\ldots\ldots. \|3\|$$

Equation or inequality 3.

What we know is the following, right? We have assumed that you can see the equation 1 $\delta(u)$ plus $w(u,v)$ is less than $\delta(v)$. From 1 $\delta(u)$ plus $w\ (u,v)$ is less than $\delta(v)$

From 1,

$$\delta(u) + w(u,v) < \delta(v)$$

but $\delta(u) = X + Y$, that means, this implies X plus Y plus $w\ (u,v)$ is less than delta v,

$$\Longrightarrow X + Y + w(u,v) < \delta(v) \quad by\ \|2\|$$

this is from 2, by 2. Because 2 says $\delta(u)$ equal to X plus Y, therefore X plus Y. But X is larger than $\delta(v)$, this implies X plus Y plus $w\ (u,v)$ is less than X,

$$\Longrightarrow X + Y + w(u,v) < X \qquad by\ \|3\|$$

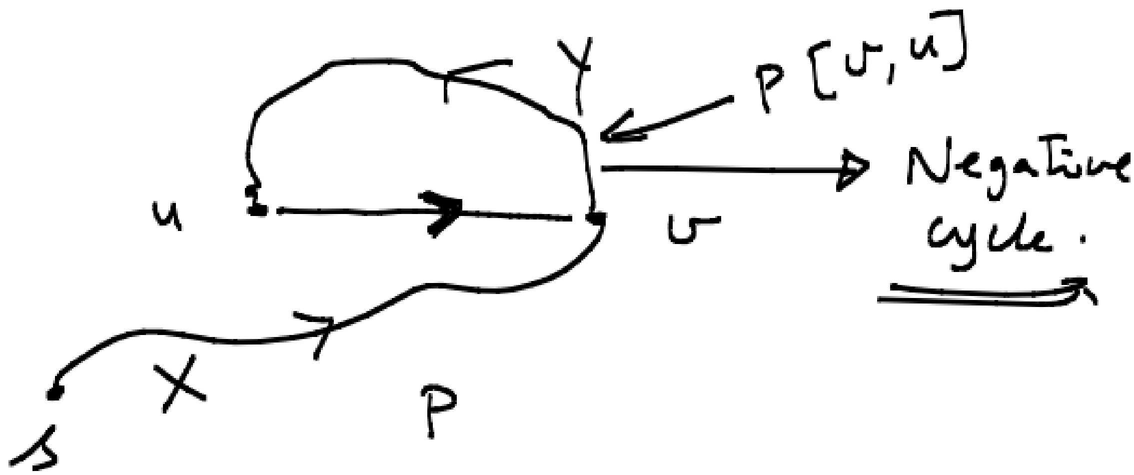
why it is less than X, X is still larger than delta v, X is greater than or equal to delta v, therefore this is by 3.

Cancelling X, we get Y plus $w\ (u,v)$ is less than 0, this implies Y plus $w\ (u,v)$ is less than 0.

$$\Longrightarrow Y + w(u,v) < 0$$

But what is Y plus $w\ (u,v)$? Y is a path from v to u, $w\ (u,v)$ is the weight of the edge (u,v), it is a cycle right. This implies weight of the cycle, what is the cycle $P[v,u]$ plus the edge (u,v), look $P[v,u]$, this is the part of $P[v,u]$ from v to u, part of P, that is a path and from u to v again you have an edge. Therefore we have $P[v,u]$ plus (u,v) is a cycle and what is the weight of that cycle? This implies weight of the cycle $P[v,u]$ plus (u,v), but the weight of the cycle is, weight of $P[v,u]$ is Y plus $w\ (u,v)$ is less than 0, that means this is a negative cycle.

We have shown the negative cycle explicitly, where that negative cycle is even located we have shown. So the graph has a negative cycle. So if, this proves, thus we have shown that if there is an edge (u, v) such that $\delta(u)$ plus weight of (u, v) is less than $\delta(v)$ then G has a negative cycle. The negative cycle is nothing but portion of the shortest path from s to u from v to u, that part, okay, this is the remark. The proof is over it is just as a note of explanation, I am adding this. The portion of P from v to u and the edge (u, v), Let me write like this, the portion of the shortest path from v to u plus the edge (u, v) is the negative cycle. It is just a remark, I am adding so that you can visualize from the previous in this picture.



So this picture should clearly tell you where the negative cycle is.

So we have an important theorem which says that if there is an edge satisfying certain property, then G has a negative cycle. Consider the contra positive, right? Consider the negation or contra positive of this theorem. If G has no negative cycle, that is you have

$p \Rightarrow q$

kind of a statement, if there are some properties, I have some other statement, p implies q; the contrapositive is not q implies not p,

$$\neg q \quad \Rightarrow \neg p$$

is the contrapositive statement. There is an implication, the contrapositive of that implication will have the following form not q implies not p. If G has no negative cycle, there is no edge satisfying this condition, right, because there exist an edge satisfying this condition is negated.

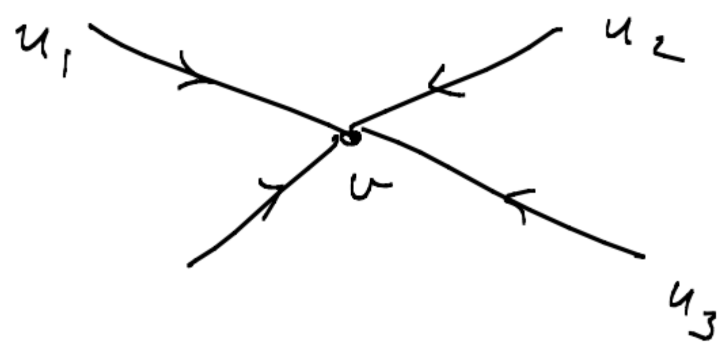
No edge is satisfying this condition, that means if G has no negative cycle, then for every edge (u, v) we will have $\delta(u)$ plus weight of (u, v) greater than or equal to $\delta(v)$.

$$\delta(u) + w(u, v) \geq \delta(v)$$

So I am getting an upper bound for $\delta(v)$, right, what is the other way of writing? $\delta(v)$ is less than or equal to $\delta(u)$ plus $w\ (u, v)$ same thing I have written in this way.

$$\delta(v) \leq \delta(u) + w(u, v)$$

If J, if G has no negative cycle I have an upper bound, okay. If G has negative cycles, there will be lot of issues, lot of problems, we will see that later. For the time being if G has no negative cycle I have some interesting result. How do we understand this? (u, v) is an incoming edge to v, (u, v), if this is v, if this is u, (u, v) is an incoming edge, from u the edge arrives.



So there are a lot of edges that may arrive from u_1, there is one edge from u_2, from u_3, so in this way several edges may arrive. So these are all the neighbors of v right. However delta v is going to be smaller than $\delta(u_{1)}$ plus weight of (u_1, v), it will be smaller $\delta(u_2)$ plus $w\ (u_2, v)$. I know that $\delta(v)$ is smaller than all these quantities, smaller than or equal to, but still I do not know how to find $\delta(v)$. Even if I know $\delta(u)$ for all neighbors, I know that they form the upper bound, but still this is not giving me a formula or an equation for $\delta(v)$. Our next result is towards getting an exact formula, an equation is only inequality, it is an upper bound, we will get an exact expression through another result. The next important theorem we will see in our next session, thank you.