Course: Introduction to Graph Algorithms

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Lecture 06 Algorithms for finding Shortest Path Part 2

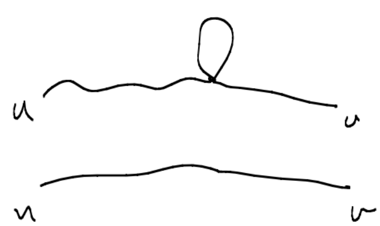
Namaskara, we continue our discussions on the basics of walks and paths and move on to describe the shortest path problem and then derive certain important mathematical properties of the parameters that we are looking for, okay. The shortest walk, the weight of a shortest walk can be reduced to minus infinity if there is a negative cycle in the walk you can move around that negative cycle again and again keep reducing the weight of the walk and you can make it to minus infinity okay. if there is no negative cycle then shortest path and shortest walk are one and the same okay we will make these kind of observations now. First let us note that if W is a walk from u to v then W contains a path from u to v it is a subset of edges what is the definition of a path. No repetition of vertices a walk may have repetition of a vertex. Repetition of a vertex means you start from a vertex you go through various edges and then again come back to that vertex. So, there is a cycle of edges that are defined between two successive occurrence of a repeating vertex remove all of them. when you remove them one repetition is removed because you have removed that cycle of edges it is still a walk the other parts are intact. In other words if I have a cycle because this is a vertex that is repeating this is the first occurrence and then the second occurrence between the two successive occurrences there are lot of edges these edges will form a cycle okay between the first occurrence and the next occurrence remove them so when you remove them you get still a walk. So whenever there is a repetition there is a cycle of edges between the two successive copies of the repeated vertex remove those when you keep removing them all the duplicates are removed. So this process will lead to a path okay.

If W is a walk from u to v then W contains a path how to obtain that path we obtain the path by removing the cycles from the walk. keep removing the cycles that will keep removing the duplicates finally you will have a connection from u to v with no duplicates and that is actually the path right. So this is the important property we must remember this. we also see that now if G has no negative cycles assume that G has no negative cycles. let W be a path from W be a walk, u to v assume that g has no negative cycles and let w be a walk from u to v then W contains a path P from u to v such that, weight of a path is less than or equal to weight of the walk.

$$\omega(P) \leq \omega(W)$$

Why this is so? It is simple the graph has got no negative cycles, so all the cycles that you remove are either 0 cycles or positive cycles removing 0 cycle does not change the weight removing positive cycle reduces the weight. Therefore the path P you have got by the removal of the cycles will have either equal weight or a smaller weight and that is what is stated here. the path that you find in the walk it is a subset of edges it forms a path the weight of that path will be less than or equal to weight of W that is because the cycles that you are removing from the walk to get the path P are all either 0 cycle or positive very important property. So, you can call this as property 1, you can call this as property 2, we will also make another observation interesting property. So a walk if you look into a graph a there has no negative cycles the shortest walk will be a shortest path that is the important property 3. If G has no negative cycle then the shortest walk from u to v will be a shortest path from u to v okay. If G has no negative cycles we will also assume that G has no 0 cycle because 0 cycles the edges of 0 cycles can be removed without changing the weight by removing all the edges of a 0 cycle. the weight is reduced by 0 which means the weight is not altered. So we can assume that no 0 cycle also if G has no negative cycle and no 0 cycle that the shortest path will indeed be a shortest walk.

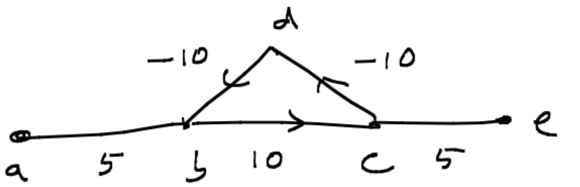
Can the shortest walk have any repetition of vertices in other words can it contain a cycle okay how do you prove this? Prove for property 3 if the shortest walk has any cycles, any cycle, it can be removed and a shorter walk is obtained. If you, if the shortest walk has any cycle I can remove that cycle the cycle has positive weight this is because the weight of the cycle is positive, there is no 0 cycle there is no negative cycle. So, if at all there is a cycle it will have a positive weight. So if I have a walk like this and if I have a walk like this without that cycle it will be smaller in weight alright.



Therefore the shortest walk cannot be shortened further by definition that is a shortest. How can you reduce the if there is a cycle it can be reduced since it is not possible to reduce there are no cycles, there are no repetition okay. This contradiction shows that shortest walk cannot contain any cycle. That means shortest walk is indeed a shortest path, this is possible only right when the graph has got no negative or 0 cycle. If the graph has got no negative or 0 cycles, then the shortest walk will indeed be a shortest path. Hence if G has no negative or 0 cycle then alpha uv equal to delta uv

$$\alpha(u,v) = \delta(u,v)$$

Shortest walk is same as the shortest path it is indeed a path. G has no negative cycle if g has negative cycle well things could become different okay for example if I have a again take this simple example of this. 5 -10 5 -10 the example we have seen already.



There is only one path from a to e, a b c d e that path is a to b to c to e this is the only path is only one path

$$\langle a, b, c, e \rangle$$

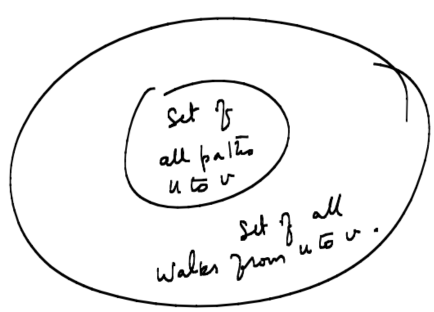
Therefore that is the shortest path, its weight delta ae is 5 plus 10 plus 5 is 20.

$$\delta(a,e) = 5 + 10 + 5 = 20$$

but we have seen that alpha ae is – infinity

$$\alpha(a,e) = -\infty$$

They are different alpha uv equal to delta uv only if G has no negative cycle. If G has a negative cycle this example shows while delta is well defined alpha could become – infinity. Okay. Since every path is a walk set of all paths from u to v. It will be smaller than set of all walks from u to v.



So bigger set every path is a walk but there are walks that are not paths therefore it is a bigger set minimum over bigger set will be smaller. Alpha u v is less than or equal to delta u,v

$$\alpha(u,v) \leq \delta(u,u)$$

If you have a bigger set superset minimum of a superset will be either smaller or equal to the minimum of the subset okay. So these properties are all very useful in the design as well as the proof of correctness of algorithms. With this I conclude this part we will discuss about the mathematical foundations and basic properties of shortest path weights in the next session thank you.