Course: Introduction to Graph Algorithms

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Week: 02

Lecture 05 Algorithms for finding Shortest Path

In this lecture we begin our discussions on algorithms for finding the shortest paths. First we will give certain basic definitions then we will focus on the mathematical foundations or theoretical foundations for the algorithms for shortest paths then we look into the details of the algorithms themselves. The mathematical foundations are very important those mathematical properties and ideas are key for the design of the algorithm, okay.

We are going to work on directed graphs. A directed graph G is given by a pair of sets vertex set and edge set,

$$G = (V, E)$$

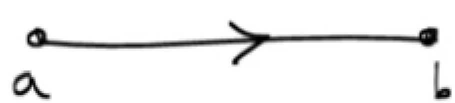
E is a subset of V x V,

$$E \subseteq V \times V$$

a typical element (a,b) belong to E is called directed edge

$$(a, b) \in E$$

and it is denoted by marking two vertices a and b, these are end vertices and drawing an arrowhead so that it indicates the direction therefore the edge leaves a and arrives at b or you can say it is going out from a or an outgoing edge for a and it is coming in to b or an incoming edge to b. Its a and b are adjacent, okay.



The notion of path extends the notion of adjacency, so adjacency defines the neighborhood; paths account for reachability, so when we have a chain of edges they define a path and we need to distinguish between two kinds of reachability, so we have to look at the notion of walk and notion of path, these are the two types of reachabilities available in a directed graph. In the discussions related to shortest path algorithms, we work on what is known as weighted graph, weighted directed graph so in fact these are, the weights are associated with edges when we do that we call them as edge weighted graph. We may associate the weight - a real number or an integer, weight is just a real number or an integer; we may associate a weight with vertices in that case, vertex weighted graph. We are going to look at edge, weighted, directed graphs.

An edge, weighted, directed graph will have a weight function associating weights to each edge, so we will have a weight function w that goes from edge set, it goes to integers, we assume for our purposes, the weights are all integers. Weight of an edge (u,v) when (u,v) belongs to E is denoted by this

$$w : E \rightarrow I .$$

$$w(u,v), \quad (u,v) \in E$$

and weight of (u,v) could be positive, zero or negative. Weight could be any kind of integers, right. Weight of an edge could be positive it could be negative or it could be 0. It is convenient to associate or extend the weight function for the pairs that do not belong to E.

So let us say, the pair (a,b) belongs to V x V but (a,b) does not belong to E, there is no edge connecting, in this case we would set w (a,b) weight of this pair, this is not an edge just a pair so this is called the extended weight function.

$$(a,b) \in V \times V$$

$$\text{but } (a,b) \notin E$$

$$w(a,b) = \infty .$$

we are extending the weight function to all pairs. Weight functions are defined only for edges, if there is an edge we associate an integer for that, okay. So weight function is defined on the edge set E, extended weight function. We use the same is, we use the same notation w, extended weight function is defined on V cross V.

Okay, so if an edge is in, if a pair is in E we call that pair as an edge and the weight will be a finite integer. If the pair is not in E we said that as infinity. Okay, we can first define the notion of walk and path now, okay.

We define, we denote an edge weighted directed graph as V, E and W, this is the weight function, okay.

$$G = (V, E, w)$$

Every edge will have an integer associated with that and we view this association by the function w. This is the general notation we use for a weighted directed graph.

As we have already seen, the concept of a walk or a path extends the notion of adjacency. So we will consider a sequence of edges, a sequence of edges e1, e2, ek,

$$\langle e_1, e_2, \ldots, e_k \rangle$$

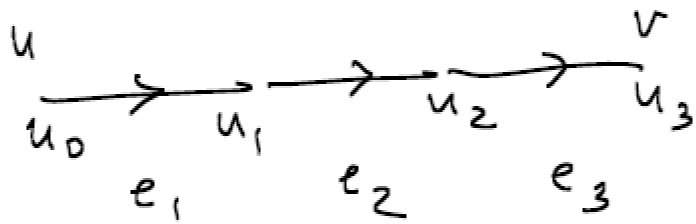
is said to be a walk from vertex u to vertex v if, the sequence of edges is a walk if ei is ui-1, ui and it is an edge, it is of the form ui-1. So, the first edge has got u0 u1 and u0 u1 it starts from u. u0 is u and it ends in uk because ek is (uk-1,uk), uk is v.

$$e_i = (u_{i-1}, u_i) \in E$$

$$u_0 = u$$

$$u_k = v.$$

So it is going to be like this, u0 to u1- this is e1, e1 is u0 u1, e2 is u1 u2, u1 u2 is e2. e3 is u2 u3, u2 u3 this is e3, so in this way we form a sequence of edges, where one edge ends the next edge begins, that is why it is a chain. we imagine as if we are walking from one vertex to another vertex along the edges and since the edges are chained up we can imagine this as a walk from u to v. First vertex is u, u0 is u, the last vertex is v, so this u3 is v. So you are going from u to v in this example using 3 edges e1, e2, e3.



The number of edges in a walk is called the length of the walk, okay. k is the number of edges in the walk, is called its length. okay, so a walk of length 1 means it is simply an edge okay. A walk of length 3, in this example is made up of 3 edges, therefore these 3 edges when you chain up like this, it defines a walk. For any walk, the vertex u and v, the first and last vertex, they are called end vertices. u and v .that is u0 and uk are called end vertices they are called end vertices.

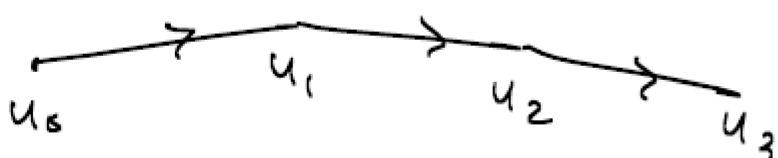
u, v

u_0, u_k

If k is greater than or equal to 2, if there are 2 or more edges, there is only one edge, it is like this u v. If there are more edges, right, from u you go somewhere and then finally come to v.

if $k \geq 2$

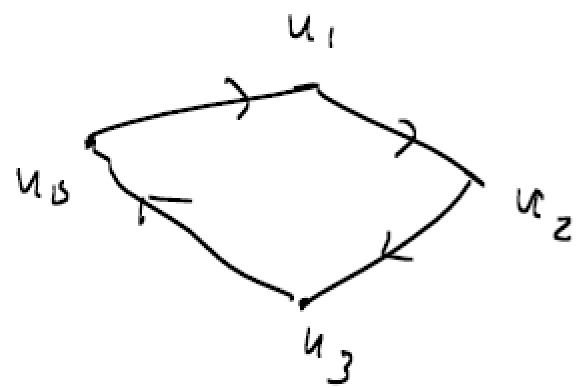
So if k is greater than 2 to all vertices in the walk, other than end vertices, are called intermediate vertices. So if it is u0 to u1, u1 to u2, u2 to u3, u1, u2 are intermediate vertices. An edge does not have any intermediate vertex otherwise if you have a long walk you have intermediate vertices, end vertices and intermediate vertices, okay.



A walk is a closed walk if the end vertices are same, right, we view that as, you start from some vertex go to various edges and come back where you are started. So extending this definition, we can say a closed walk is a walk where u equal to v or end vertices are same.

$$u = v$$

So you can start from u0 you can go to u1 you can go to u2 you can go to u3, this basically u4, from u3 you would have gone to u4 in the normal way, but it is close to walk, therefore that become, this is the case where u4 is actually equal to u0. So we represent the walk by u0, u1, u2, u3, u4 where u4 equal to u0. So after u3 you have come back to u0, therefore a closed walk can be represented in this.



$$\boxed{u_4 = u_0} \qquad \langle u_0, u_1, u_2, u_3, u_4 \rangle$$

$$\text{Where } u_4 = u_0$$

By the way, a walk can be represented by a sequence of edges or you can represent a walk by a sequence of vertices. Okay, here I have used the sequence of vertex notation. So you can represent this walk as u0, u1, u2, u3, with an understanding that ui-1, ui belongs to E. i equal to 1 to 3 i equal to 1 to k in general here i equal to 1 to 3.

$$\langle u_0, u_1, u_2, u_3 \rangle, \quad (u_{i-1}, u_i) \in E$$

So u0,u1 is an edge u1, u2 is an edge u2, u3 is an edge, it is a chain So you can represent this by a chain of edges or a sequence of vertices, both representations are useful.

First notice that there is no limit on the size of a walk, we will come to the distinction after defining the concept of a path okay. A path is a walk in which all vertices are distinct. A path is a walk in which all intermediate vertices are distinct. Of course if end vertices are distinct, you get what is known as a open path, when end vertices are same you get closed path and closed path is called a cycle, okay.

A closed path, is a closed walk in which all vertices are distinct and a closed path is called cycle. This is the key difference between a walk and a path, in a walk intermediate vertices need not be distinct, there can be repetition; in a path there is no such repetition allowed, okay. Therefore maximum length of a path, if cardinality V equal to n if there are n vertices,

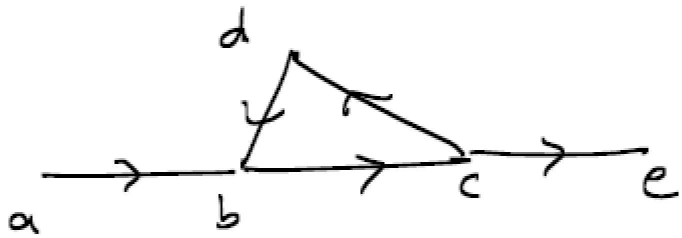
$$|V| = n$$

all the n vertices are here let us say 1, 2, 3, 4, 5, 6, 7. Let us say there are 7 vertices, all the 7 vertices are in the path. That means there is going to be 1, 2, 3, 4, 5, 6 edges, therefore length of any path is less than or equal to cardinality of n-1, you take any path it cannot have more than n-1 edges.

Maximum number of edges possible in any path is only n-1.

$$\text{Length of any path} \leq |V| - 1 = (n-1)$$

Walk is not like that a walk can have arbitrary number of edges, it can be arbitrarily long, okay. There may not be any bound on length of a walk. Any path in a graph cannot have more than n-1 edges because all intermediate vertices are distinct, okay. There may not be any bound on the length of the walk. This is because repetition is allowed. Consider for example a,b,c,d,e; consider this graph, it is a directed graph



The sequence of vertices a, b, c, c I should draw here, a, b, c, e, it is a walk from a you had gone to b, b to c, c to e, it is a walk and it has got 3 edges, its length is 3.

$$\langle a, b, c, e \rangle \quad - \quad 3$$

However you can start from a, you can go to b, you can go to c, you can go to d, again you can go to b, again you can go to c, now you can go to e, it is a walk, that is you have gone round the cycle and then you have continued.

$$\langle a, b, c, d, b, c, e \rangle - 6$$

This is also a walk, sequence of edges where one ends the next one begins you start from a and you end in e. All these are walks from a to e, you started from a you finished at e.

But it is length is 1 2 3 4 5 6 7, 6. You can have even a longer walk, visit that cycle once again, right. Start from a, go to b, go to c, go to d, go to b, go to c, again go to d, go to b, go to c, now you exit. You can see that 1 2 3 4 5 6 7 8 9 10, all these are walks from a to e.

$$\langle a, b, c, d, b, c, d, b, c, e \rangle - 9.$$

You can now imagine a walk from a to e of unbounded length, it can be made arbitrary large visit the cycle again and again and you are going to have a walk of arbitrary length, there is no bound.

For path there is a bound, even if all vertices are present you can have only n minus 1 edges. Now let us see the paths and walks and their weights in a weighted graph. Weight of a walk, W, it is very simple definition, because walks are extending the notion of edges the weight is considered as sum of the weights of the edges, sum of the weights of the edges e belonging to W, that is all. So as you are going along the edges, add their weights, each edge has got a weight associated with it. Add it. That way you get the weight of a walk.

$$w(W) = \sum_{e \in W} w(e)$$

Similarly you can define weight of a path. What is the weight of a path? Weight of a path P is sum of the weights of the edges in the path. Consider those edges which are in the path. Each has a weight add all of them you get.

$$w(P) = \sum_{e \in P} w(e)$$

similarly, weight of a cycle C, w (C) Sigma e belong to C, all the cycle edges, you consider them and then add you get the weight of the cycle.

$$w(C) = \sum_{e \in C} w(e)$$

A cycle is said to be positive cycle if weight of C is greater than 0, is said to be a 0 cycle if weight of C is 0, is said to be negative if weight of cycle is less than 0, okay.

$$\begin{array}{ll} \text{positive} & w(C) > 0 \\ \text{zero} & w(C) = 0 \\ \text{negative} & w(C) < 0 \end{array}$$

A cycle, is a positive cycle or a negative cycle or a 0 cycle depending on the value of the weight of the cycle. If the value is positive the cycle is called positive cycle and so on, natural way to define So once you have a weight associated with an entity, you can compare which path is longer, which path is shorter, compare the weights. So each path has got a number associated with it, you can use it to find out which is shorter which is, okay.

We now define shortest path weight delta (u,v).

$$\delta(u,v)$$

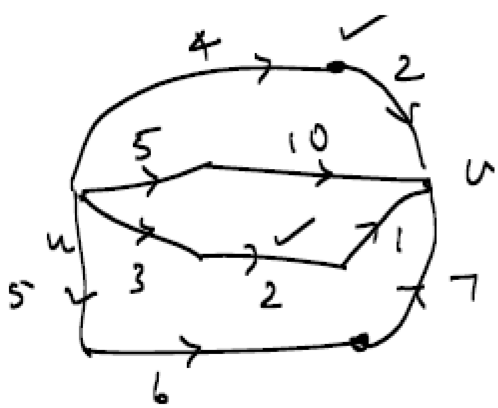
Delta is used for shortest path, shortest path weight delta (u,v) is defined as: delta (u,v) equal to minimum over all the weights of path such that p is a path from u to v.

$$\delta(u,v) = \text{Min}\{ w(P) : P \text{ is a path from } u \text{ to } v \}$$

So from u to v there may be several paths, each path has got a weight associated with it, the minimum weight, that is called the shortest path weight, okay. This is well defined because number of paths is finite, okay. Since number of paths from u to v is finite, delta (u,v) is well defined and finite. If there is a path it has got a weight, if there are several paths each path has a weight and find a minimum; that number is called the shortest path weight. Any path whose weight is delta (u,v) is called the shortest path from u to v.

$$w(P) = \delta(u,v)$$

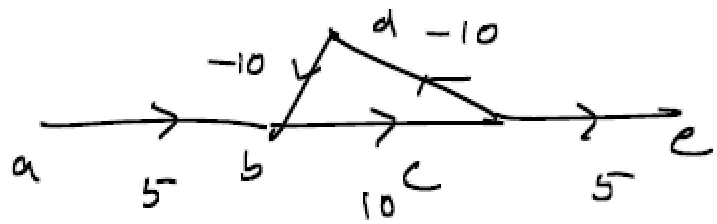
This is shortest path weight. Any path P with weight of P equal to delta (u, v) is called a shortest path from u to v. It is called the shortest path from u to v. It is a very simple example, this is 5 this is 10. So from u to v in this picture there are four paths, each path has got its own weight.



This one path has got weight 13, another path has got weight 3 plus 2 plus 1 6, another path has got weight 15, another path has got weight 6, so delta (u,v) is 6.

$$\delta(u,v) = 6$$

So therefore this is a shortest path and this also a shortest path, in this graph there are two shortest paths. Both of them have length 6, 6 is the minimum weight path, okay. Any path P with w(P) equal to delta (u,v) is called the shortest path. Now let us see what happens when we examine this concept in the context of a walk. Things become little tricky in this case. That is because walks can have unbounded number of edges, because they can have unbounded number of edges by adding all of them and if the edge weights are negative, it can be made smaller and smaller, okay. In other words weight of the shortest walk could be minus infinity, let us see a simple example again a,b,c,d,e, okay, let us say 5, 10, 5 and this one is minus 10 and minus 10.



Just, I have chosen any number, I could have chosen any number, I am choosing 5 and 10 for ease of quick computation that's it. So, you can have a walk a to b to c to e. (a,b) has weight 5, (b,c) has weight 10, (c,e) has weight 5, that will be 20.

$$\langle a, b, c, e \rangle \; - \; 5 + 10 + 5 = 20.$$

But now consider the following walk, a to b, b to c, c to d, d to b, b to c, c to d. You have gone round the cycle once and then you have. You can see that there is a repetition of vertices, repetition of edges and it is walk. What is its weight a to b is 5, b to c is 10, c to d is minus 10, d to a is minus 10, d to b is minus 10, b to c is 10 and then you have plus 5, c to e is 5 and this is 10.

$$\langle a, b, c, d, b, c, e \rangle \; +5 + 10 - 10 - 10 + 10 + 5 = 10$$

You can see that its weight is reduced by 10, why it is reduced by 10? You have gone round this cycle (b,c,d), but what is the weight of (b,c,d), weight of (b,c,d), (b,c,d,b), weight of (b,c,d,b), 10 minus 10 minus 10 which is equal to -10, therefore this is a negative cycle,

$$\text{wt of } \underline{\langle b, c, d, b \rangle} = 10 - 10 - 10 = -10$$

it is a negative cycle, you go around the negative- 10 is reduced. The weight is reduced by 10. You go around once again, another 10 would be reduced. In this way, it will keep on reducing, because each visit of that cycle, since the weight of the cycle is minus 10, it keeps reducing

Let us say a, b, c, d, b, c, again you go d, b, c and now you come out of e. You can see that its weight is 0. Sorry I have written 10.

$$\langle a, b, c, d, b, c, d, b, c, e \rangle \; - \; 0$$

Further reduced, earlier it was 20, when you go once it became 10 when you go twice, it becomes 0, you go again and again it will become minus 10, minus 20, minus 30. So you can have walks with more and more edges but when you are adding the cycle of edges the weight reduces by 10 therefore we will have the minimum to be minus infinity. Minimum weight walk as minus infinity as its weight and it is also using infinite number of edges. This is the reason why we normally say that the shortest walk is not well defined, it is not finite, okay.

This is one reason why we say shortest walk is not well defined, especially if G has a negative cycle. If G has a negative cycle, okay, it could become minus infinity. In other words if alpha (u,v) is the weight of shortest walk then alpha (u,v) could become minus infinity

$\alpha(u, v)$ could become $-\infty$

while delta (u, v) is always well defined alpha (u, v) may become minus infinity, okay.

while $\delta(u, v)$ is always well-defined

$\alpha(u, v)$ may become $-\infty$.

We will continue our discussions on basic definitions and properties in the next session thank you.