Course: Introduction to Graph algorithms

Professor: c Pandu Rangan

Department: Computer Science and Engineering

Institute: IIsc

Week: 03

Lecture 10 Bellman Equation Part 1

Namaskara, in the previous session, we have seen the mathematical properties that would relate the shortest path distances in an equation called bellman equation.

Let G = (V, E, w, s) and delta v is weight of the shortest path from s to v.

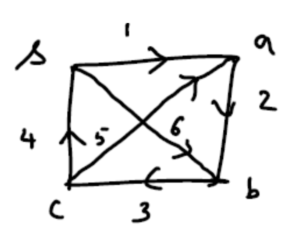
We have formed the following bellman equations when G has no negative cycle. We assume that G has no negative cycles. The bellman equations are given by, we define a variable, x v is a variable associated with vertex v belong to V. and we form the following equation: x of s is equal to 0 and x of v is equal minimum over u not equal to v of x of u plus weight of u v.

$$x_s = 0$$
$$x_v = \min_{u \neq v} \left\{ x_u + w(u,v) \right\}$$

This is for all vertices v other than s, okay. so, these are called bellman equations, we can form the bellman equation and then attempt to solve them. so, let us see an example to understand the subtleties and issues involved in dealing with bellman's equation.

consider the following graph, here is an example to explain the ideas and concepts behind the bellman's equation; the computations to be done with bellman equations.

consider the graph s, a, b, c; it is a directed graph and we have the edges like this. These are all the edges. First I will give the following weight 1, 2, 3, 4, 5 and 6. These are the weights of the edges.



For example, weight of the edge cs is 4. so this is a weighted graph, s is the source vertex. so I can write down the bellman equation. In bellman equation you will have for each edge uv playing a role for the variable xv. xv for any vertex v is minimum of uv into xu plus weight of uv. so, for each incoming edge there will be a term.

$$x_v = \min_{(u,v)} \left\{ x_u + w(u,v) \right\}$$

so, for each incoming edge there will be a term. so, for the vertex v if the in-degree is 10, there will be 10 terms and you have to take the minimum of those 10 terms. Okay. so, number of terms here is same as in-degree of v, because for each uv I will have one term, Okay.

so let us write down this, x of s equal to 0. What can you say about x of a? x of a equal to minimum of: there are 2 edges coming into a each edge will contribute a term; x of s plus weight of sa, x of c plus weight of ca. This is same as x of s is 0, weight of sa is 1, 0 plus 1 is 1, x of c we do not know that is a variable x of c, plus weight of ca weight of ca is 5, x of c plus 5 therefore this is x of a. okay.

$$x_s = 0; \quad x_a = \min \left\{ x_s + w(s,a), \ x_c + w(c,a) \right\} = \min \left\{ 1, x_c + 5 \right\}.$$

x of b, b also has got 2 edges coming into it sb and ab okay. so, let me write down here x of b is minimum of two terms, one term is x of a plus weight of ab which is 2, another term is x of s plus 6 which is same as minimum of; x of s is 0, 0 plus 6 is 6, xa plus 2. This is x of b.

$$x_b = \min \left\{ x_a + 2, \ x_s + 6 \right\} = \min \left\{ 6, x_a + 2 \right\}$$

x of c there is only one term, there is only one edge coming in. so x of c equal to minimum of one value is just the same value. so I write x of c as x of b plus 3, okay.

$$x_c = x_b + 3$$

so let me write down in the next page. x of s equal to 0, x of a equal to minimum of: 1 and x of a is xc plus 5, x of b equal to minimum: of 6 and x of b is minimum of xa plus 2, x of c equal to x of b plus I think it is 3 right x of b plus 3 you are knowing. So this is the bellman equation for the graph.

$$\begin{aligned} x_s &= 0; \\ x_a &= \text{Min}\{1, x_c + 5\} \\ x_b &= \text{Min}\{6, x_a + 2\} \\ x_c &= x_b + 3. \end{aligned}$$

I have to solve this set of equation. I know the value of x of s but I do not know the value of others. so I have to solve it. When, if I am able to solve it and find a solution that will be the shortest path distance, okay, we will check that out.

I am not applying any specific method, just a visual and logical argument, visual inspection based combination and logical argument okay. You can see that x of c is x of b plus 3, so I can substitute x of 3 and I can write like this: x of a equal to minimum of 1 and xb plus 3 plus 5 because x of c is x of b plus, which is equal to minimum of 1 and xb plus 8.

$$\begin{aligned} x_a &= \text{Min}\{1, x_b + 3 + 5\} \\ &= \text{Min}\{1, x_b + 8\} \end{aligned}$$

x of b is minimum of 6 and xa plus 2.

$$x_b = \text{Min}\{6, x_a + 2\}$$

You can see the circularity xa is depending on xb, xb is depending on xa okay. so how do we solve this? so we will look into possibilities. Is it possible that xa equal to xb plus 8, is there a possibility for that? If xa were, if xa equal to xb plus 8, then xb is equal to minimum of 6 and xa plus 2 is xb plus 6 plus 2 sorry 8, xb plus 8 if xa is xb which is equal to minimum of 6 and xb plus 10.

$$\text{if } x_a = x_b + 8,$$
$$x_b = \text{Min}\{6, x_b + 8 + 2\}$$
$$= \text{Min}\{6, x_b + 10\}.$$

Can xb equal to xb plus 10? Not possible, therefore the only option we have here is xb equal to 6. Okay, this forces us, if xa equal to xb plus 8 then xb must be equal to 6, this implies xb equal to 6, xb equal to 6. substituting xb equal to 6 in the first equation, okay xb equal to 6 in xa, since xa equal to xb plus 8, this implies xa equals to 6 plus 8 that is equal to 14.

$$\Rightarrow x_b = 6.$$
$$\text{Since } x_a = x_b + 8$$
$$\Rightarrow x_a = 6 + 8 = 14$$

Can xa be ever 14? xa is minimum of 1 and some number. If that some other number is 10, minimum of 1 and 10 is 1. If that other number is minus 2, minimum of 1 and minus 2 is minus 2. So minimum of 1 and any other number cannot be 14 right. xa equal to minimum of 1 and some other number, so xa will be 1 or some smaller number right. xa equal to minimum of 1 and xb plus 8, xa is less than or equal to 1,

$$\text{Since } x_a = \text{Min}\{1, x_b + 8\}$$
$$x_a \leq 1$$

It could be 1 or some smaller value, right? This contradicts xa equal to 14. It is not possible for xa to have value 14. That is because we have assumed this and when we proceeded to fit in the value it did not work. There are two options, xa is either 1 or xb plus 8, xb plus 8 is ruled out, therefore this implies xa must be equal to 1. This implies xb which is equal to minimum of 6 and 1 plus 2, xa plus 2, xa is 1, which is equal to minimum of 6 and 3 which is 3 and this implies xc equal to xb plus 3, 3 plus 3 that is equal to 6. so the answer is xs equal to 0, xa equal to 1, xb equal to 3,xc equal to 6 okay, xs equal to 0, so we have solved. We got a solution.

$$\Rightarrow x_a \text{ must be } = 1.$$

$$\Rightarrow x_b = \text{Min}\{6, 1+2\}$$

$$= 3$$

$$\Rightarrow x_c = 3+3 = 6.$$

$$x_s = 0,\ x_a = 1,\ x_b = 3,\ x_c = 6$$

You can see that, this is nothing but the shortest path distance from s to a, b and c, okay. You remember 1, 3 and 6. You can see the picture. s to a, s to a, the edge weight is 1. That is the shortest path. Of course, you cannot go from s to a in any other way and therefore that is the, therefore delta a is 1. okay s to b you have 2 choices you can go s to a, a to b or you can go from s to b the minimum is 3, so delta b is 3, delta a is 1. You can see, delta a is 1, delta b is 3. Of course delta s equal to 0, delta a equal to 1, delta, what about delta c? What is the shortest path from s to c? You have 2 ways to reach from s to c, you can go from s to a, a to b, b to c. That would be 1 plus 2 plus 3 which is 6. There is another way that is s to b, b to c that is 6 plus 3 that is 9. Therefore the shortest path is 6 which means delta c equal to 6.

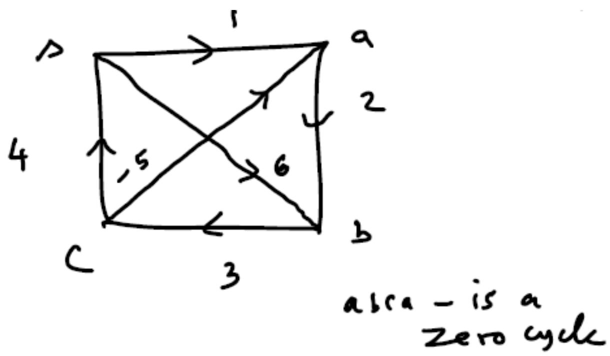
$$\delta(s) = 0;\ \delta(a) = 1,\ \delta(b) = 3,\ \delta(c) = 6$$

So you can see that the solution of bellman equation is xv is given by delta v okay. Delta v okay what is the notation I have used I do not want to, so delta with within parenthesis v not sub so let me clean up this part. Delta s equal to 0, delta a equal to 1, delta b equal to 3, delta c equal to 6 and here we have xv is delta v for all v belonging to V.

$$x_v = \delta(v) \quad \forall v \in V$$

So we have shown that and there is no other solution possible because when we tried other possibility it did not end up in a clean way. Therefore, the only solution is the one that is written down here and this solution is unique because other possibilities did not result in any value for xv. There are two options the second option led to contradiction therefore that possibility is ruled out. The only possibility is xa is 1 and if xa is 1 the only possibility for the xb and then only possibility therefore I have one solution obtained and that is the only solution therefore the solution is unique. So the graph has no negative cycles, in fact all the weights are positive there is no negative cycle at all okay.

Now you see what happens if it has got a 0 cycle okay. so I am going to change the same graph I will take. I will change one weight so that I bring in a 0 cycle, I will write down the bellman equation and then discuss the nature of the solution okay. So let me draw the graph again s, a, b, c; you have 1 2 3 and 4; 2 plus 3 is 5 so I am going to change the weight of this edge to minus 5. Now this is a 0 cycle okay, abca is a 0 cycle. This is there of course, its weight is 6.



The same graph, I have changed one weight to minus 5 that has brought in a 0 cycle. Now let me write down the bellman equation for this graph, again x of s equal to 0, x of a equal to minimum of x of s plus sa which is 1, x of c plus ca, x of c minus 5. this incoming edge, x of c must be added with –5 so x of c and x of b is - there are 2 possibilities - minimum of 6 because x of s is 0, 0 plus 6 is 6, another one is x of a plus 2. x of c is x of b plus 3. x of c equal to x of b plus 3.

$$x_s = 0; \quad x_a = \text{Min}\{1, x_c - 5\}$$
$$x_b = \text{Min}\{6, x_a + 2\}.$$
$$x_c = x_b + 3.$$

Let us proceed in the same way x of s is known, I eliminate x of c, put x of c there x of a equal to minimum of 1, x of c is x of b plus 3 so x of b plus 3 minus 5 that is equal to minimum of 1 and x of b –2, x of b is minimum of 6 and x of a plus 2. x of a is minimum of 1 and xb-2, x of b is minimum of 6 and xa plus 2, okay.

$$x_a = \text{Min}\{1, x_b + 3 - 5\}$$
$$= \text{Min}\{1, x_b - 2\}.$$
$$x_b = \text{Min}\{6, x_a + 2\}.$$

Is it possible that xa to have value 1? It is possible because it does not lead to any logical contradiction. okay that is because if x of a equal to 1, minimum of, this implies x of b equal to 3, because minimum of 6 and 1 plus 2, 3 is 3 and x of a is 3 minus 2 is also 1. okay so it is consistent, it does not cause any contradiction. You put x of b equal to 3 here, x of a equal to minimum of 1 and x of b is 3, 3 minus2 that is equal to minimum of 1 and 1 which is 1 so no contradiction. if x of a equal to 1, x of b equal to 3, x of b equal to 3 implies x of c equal to 6 so I do have one solution okay.

$$x_a = \text{Min}\{1, x_b + 3 - 5\}$$
$$= \text{Min}\{1, x_b - 2\}.$$
$$x_b = \text{Min}\{6, x_a + 2\}.$$
$$\text{if } x_a = 1 \Rightarrow x_b = 3, \quad x_a = \text{Min}\{1, 3-2\}$$
$$\Rightarrow x_c = 6 \checkmark \quad = \text{Min}\{1,1\}$$
$$= 1$$

This solution is same as the shortest path distance. because from s to a there is only one way to go, its weight is 1, therefore delta a equal to 1, delta b equal to 3, delta c equal to 6, all those things do not change, their shortest path weights are same. That is there as a solution of this but the solution is not unique. The solution not going to be unique. So for example, if x of b equal to 2, can x of b equal to 2? There is a possibility, because x of b equal to 2 means, if x of b equal to 2. 2 minus2 is 0, x of a equal to 0, x of a equal to 0 and if x of a equal to 0, x of b which is the minimum of 6, 0 plus 2 is 2 which is equal to 2 this agrees, there is no contradiction here. So if x of b equal to 2, x of a is 0 and x of c equal to 2 plus 3 which is 5 this is also a solution okay.

$$\text{if } x_b = 2 \checkmark, \quad x_a = 0, \quad x_b = \text{Min}\{6, 2\}$$
$$x_c = 5. \qquad\qquad = 2 \checkmark$$

You can see that other than the shortest path distance, some other set of numbers also form a solution alright. one solution is x of s equal to 0, x of a equal to 1, x of b equal to 3, x of c equal to 6, another solution is x of s equal to 0, x of a also equal to 0 integer, x of b equal to 5, no sorry x of b equal to 2, this is what we did x of b equal to 2, x of c equal to 5.

$$x_s = 0, \quad x_a = 1, \quad x_b = 3, \quad x_c = 6.$$

$$x_s = 0, \quad x_a = 0, \quad x_b = 2, \quad x_c = 5$$

This is another solution. Not only this, we can see that, if x of b is set to 1, that will lead to another set of solution, okay. so let me write down that. x of a equal to minimum of 1 and x of b minus 2, x of b equal to minimum of 6 and x of a plus 2.

$$x_a = \text{Min}\{1, x_b - 2\}$$
$$x_b = \text{Min}\{6, x_a + 2\}$$

If x of b equal to 0. If x of b equal to 0, b minus 2, x of b minus 2 is minus 2, minimum of, if x of a is minimum of 1 and minus 2 which is minus 2; If x of a is minus 2. Substituting in the second equation, x of b equal to minimum of 6 and minus 2 plus 2 which is minimum of 6 and 0 which is 0 and this is consistent with the choice

$$\text{if } x_b = 0, \quad x_a = \text{Min}\{1, -2\}$$
$$= -2$$
$$x_b = \text{Min}\{6, -2+2\}$$
$$= \text{Min}\{6, 0\}$$
$$= 0$$

Therefore, another solution is x of s equal to 0, x of a is minus 2, x of b equal to 0, x of c equal to 3 right. This is another solution.

$$x_s = 0, \; x_a = -2, \; x_b = 0, \; x_c = 3$$

In fact you can have infinitely many solutions. okay we can have infinitely many solutions. So let us assume that x of b is set to x of a plus 2 right, x of b is x of a plus 2, then substituting x of a equal to minimum of 1 and xa. I have substituted x of b equal to x of a plus 2, x of a plus 2 minus 2, which is minimum of 1 and xa, will be minimum of 1 and xa if xa is less than or equal to 1.

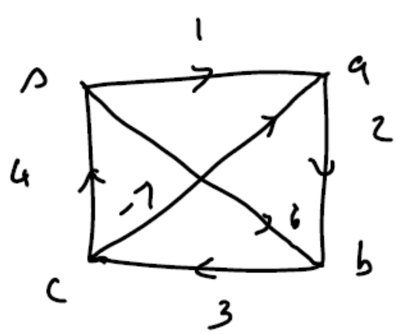
$$x_a = \text{Min}\{1, x_a\}$$
$$\text{if } x_a \leq 1$$

If xa is less than or equal to 1 then minimum of 1 and whatever is that value that will be xa right. This is true xa equal to minimum of 1 and xa if xa is, so any value for xa less than or equal to 1 will work. For example xa equal to minus 7 will work, in this case xb equal to xa plus 2 which is minus 5, of course this is less than 6 therefore xb will be given and xc equal to xb plus 3 which is minus 2, this another solution.

$$x_a = -7$$
$$x_b = -5$$
$$x_c = -2$$

In fact any xa is less than or equal to 1 can be used to construct a solution that means this has infinitely many solutions. You cannot say. When you, even after solving the bellman's equation, you get some numbers, but those numbers may not be the weights of the shortest path. They are some numbers, they satisfy the equation and that is frustrating. You form an equation, you take effort to solve, but the answer you get may or may not be the one you are looking for, right. The values you are looking for is also a solution and in addition to that there are lot of other junk values also satisfy. so when you solve and get some values you would not know whether you got useful value or junk value. The shortest path distances are some other numbers. suppose you solve and then you got these values. These are not shortest path distances right. so if you have a 0 cycle the situation is pretty dicey, the situation is very unpleasant. The solution need not be unique and there could be infinitely many solutions. So solving the equation may not lead to the answers you are looking for.

Let us make the cycle negative, same thing we will take s, a, b, c; 1, 2, 3, 4; this is same 6, now this one, I make it minus 7.



Now abca is a negative cycle, this graph has a negative cycle, You can write down the bellman's equation. Let us see what happens when you write the bellman equation what happens you get x of s equal to 0. x of a equal to minimum of x of s plus sa, let me write, x of s plus weight of sa and x of c plus weight of ca. This is same as minimum of x of s is 0, wsa is 1 minimum of 1, and xc minus 7, because the edge weight of ca is minus 7, xb equal to xs minus - say minimum of, xb equal to minimum of 6 and xa plus 2. x of c equal to x of b plus 3.

$$x_s = 0; \quad x_a = \mathrm{Min}\{x_s + w(s,a), x_c + w(c,a)\}$$

$$= \mathrm{Min}\{1, x_c - 7\}$$

$$x_b = \mathrm{Min}\{6, x_a + 2\}, \quad x_c = x_b + 3$$

So x of c causes no problem, if you could get x of b add 3 to it you get x of c, x of s is 0. so it is a small toy example, you have to determine x of a and x of b. so let us look at the equations for x of a and x of b. x of a is xc minus 7, xc is xb plus 3 therefore xa equal to minimum of 1 and xb plus 3 minus 7 that is equal to minimum of 1 and xb minus 4. Minimum of these 2 numbers is xa. and what about xb? xb is minimum of 6 and xa plus 2.

$$x_a = \mathrm{Min}\{1, x_b + 3 - 7\}$$

$$= \mathrm{Min}\{1, x_b - 4\}.$$

$$x_b = \mathrm{Min}\{6, x_a + 2\}.$$

Can we have a pair of values xa and xb satisfying this condition, xa must be equal to minimum of 1 and xb minus 4, xb must be minimum of 6 and xa plus 2. Will we have some values satisfying this? We will show that no values can satisfy this equation. Solution does not exist.

So let us consider the possibilities, suppose xa equal to 1, if xa equal to 1, if xa equal to 1 why xa is chosen as 1? xa is a minimum of 1 and xb minus 4. 1 happens to be smaller, that is why you have if xa equal to 1 this implies, 1 is a smaller quantity than, 1 is less than or equal to xb minus 4, that is the reason why minimum of these two happens to be 1, which implies xb is greater than or equal to 5, xb is greater. If xa equal to 1, xb is greater than or equal to 5.

$$\text{if } x_a = 1; \Rightarrow 1 \le x_b - 4$$
$$\Rightarrow x_b \geq 5.$$

But when you put xa equal to 1 in the second equation, xb becomes minimum of 6 and xa is 1, 1 plus 2 is 3 which is 3, xa is minimum, xb is minimum of 6 and 3 which is 3, xb is 3 according to second equation but according to the first equation xb must be greater than or equal to 5 contradiction. right. so if xa equal to 1 no consistent xb is possible.

$$x_b = \text{Min}\{6, 3\}$$
$$= 3$$

One equation says xb must be 3 another equation says xb must be greater than or equal to 5. There is no way this is possible ,therefore xa cannot be 1. The other possibility is therefore I have to check whether xa equal to xb minus 4 is possible. Is xa equal to xb minus 4 possible? When will you choose xb minus 4? This you will choose only if x minus 4 is smaller than or equal to 1, this is chosen, that implies xb minus 4 is less than or equal to 1. That is the reason why between these two, xb minus 4 is chosen. This implies xb is less than or equal to 5.

$$\text{if } x_a = x_b - 4\,? \Rightarrow x_b - 4 \le 1$$
$$\Rightarrow x_b \le 5.$$

If xa equal to xb minus 4, the implication is xb must be a value smaller than or equal to 5, but if xa equal to xb minus 4, substituting in the second equation, xb equal to minimum of xa plus 2, is minimum of 6 and xb, xa plus 2 is xb minus 4. Minimum of xb minus 4 plus 2, that is equal to minimum of 6 and xb minus 2, minimum of xb cannot be equal to xb minus 2, a value cannot be equal to minimum of 2 numbers must be one of the numbers, but xb cannot be equal to xb minus 2, that means xb must be equal to 6, this implies xb must be equal to 6, it is a contradiction.

$$x_b = \text{Min}\{6, x_b - 4 + 2\}$$
$$= \text{Min}\{6, x_b - 2\}. \Rightarrow x_b = 6$$

One equation says it must be less than or equal to 5, another equation says it must be 6, how can a value be less than or equal to 5 and will be also equal to 6, no such xb can exist. Therefore, no solution exists, all possibilities I have considered and that has led to contradiction okay, that shows solution does not exist.

You can form the bellman equation, that is writing with variables and values, here you have the bellman equation, but do you have a solution for that bellman equation which you have formed? No, the solution does not exist. so, if G has a negative cycle, bellman equation has no solutions. That is the reason why if the graph has a negative cycle we are not even going to take this approach. We are not going to formulate the bellman equation or any such thing because bellman equation does not make any sense, it may not have a solution at all, then where is the point of using that to find something, that, if G has a negative cycle. If G has a 0 cycle, bellman equation may have infinitely many solutions or solution may not be unique, several solutions are possible. In fact, we have shown an example where infinitely many solutions are possible. In that case, suppose you solve and then you get something, it could be a junk value and even if you know that it is a junk value, you have to solve again to find another solution if that is also junk again you have to solve it takes you nowhere. so if the solution is not unique, right, there is no point in again pursuing this line. That is the reason why we are going to assume that; hence we assume that G has no negative cycle and also G has no zero cycles. It can be shown that, I am not going to give the proof for that, but the proof is reasonably simple, it can be shown that if G has no negative or zero cycles, then the solution for bellman equation is unique and the solution is in fact xv equal to delta v.

$$x_v = \delta(v)$$

Therefore, I am going to assume that the graph has no negative cycle or zero cycle. Therefore, we solve single source shortest path problem on a G that has got no negative cycles and no zero cycles. We are going to solve single source shortest path (SSSP) problem only on such graph. How can we strategize to solve the single source shortest path problem? You again a graph, form the bellman equation attempt to solve, but solving the bellman equation even when the solution exists, even when it is unique, is impossible. and therefore although we have done considerable progress, we are stuck at a point where I have an equation, I know the solution of that equation is what I want but computationally determining that solution, okay, has become very hard. So we need to find suitable strategies to overcome this difficulty. We will see the strategies that are adapted by the ingenious creators of the algorithms for these problems in our next session. Thank you.