Discrete Mathematics
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Lecture -09
Rules of Inferences in Predicate Logic

Hello everyone, welcome to this lecture on rules of inferences in Predicate Logic.

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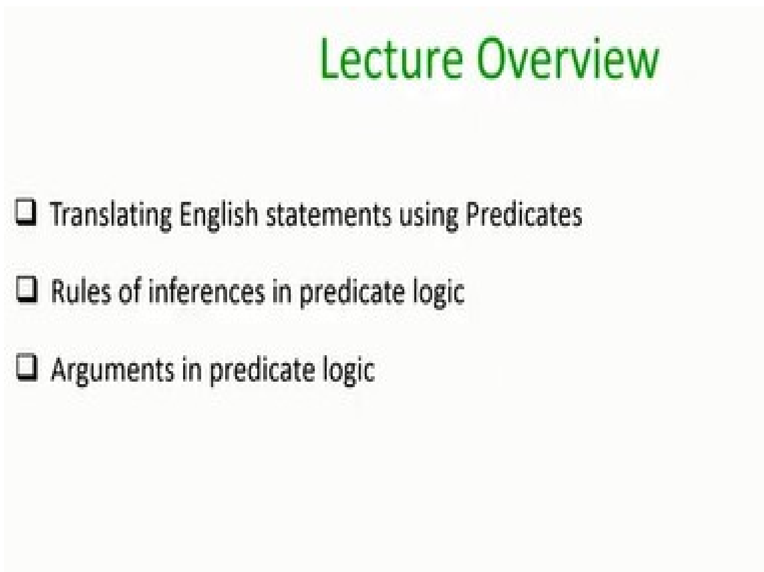

Just to recap in the last lecture we started discussing about predicate logic, the motivation for predicate logic and then we saw two forms of quantifications namely existential quantification and universal quantification. The plan for this lecture is as follows; in this lecture, we will see how to translate English statements using predicates, then we will see rules of inferences in predicate logic and then we will discuss arguments in predicate logic.

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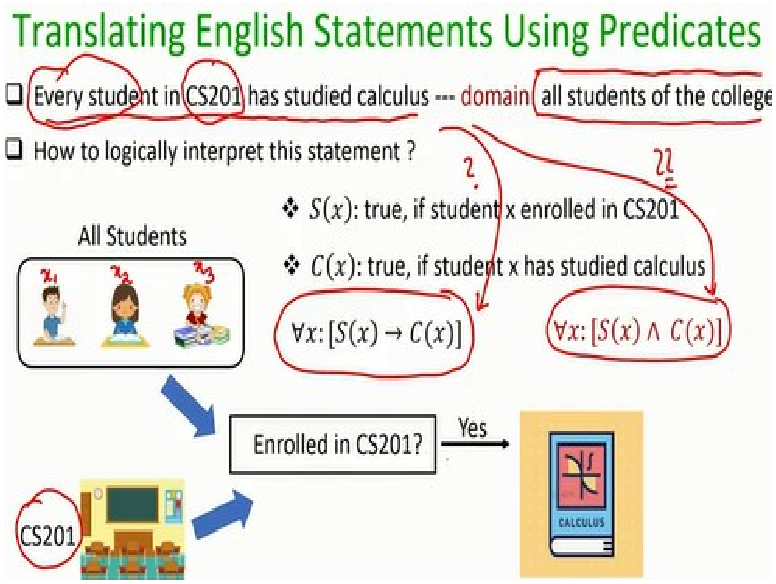

So let us begin with an example where we are given an English statement and we want to represent it using predicates and we will be encountering this situation again and again where we will be given English arguments and then we have to verify whether they are logically correct or not and for that we have to convert those English statements, into the predicate world. So the example that we are considering here is the following.

I want to represent a statement that every student in course number CS201 has studied calculus. If you are wondering what is this CS201 well at my institute IIIT, Bangalore the course number for discrete maths course is CS201 and say my domain is the set of all students in a college. So since I am considering for instance IIIT, Bangalore, my domain is the set of all students in IIIT, Bangalore but it could be any domain.

So I want to represent a statement or assertion that in a college every student in course number CS201 has studied calculus. So, how I am going to represent it using predicates. So the first thing here is that we have to understand how to logically interpret this statement. So for instance imagine you have a domain say consisting of three students, well your domain will be very large but just for simplicity I am assuming here that my domain has three students and say I have class CS201.

So the property that I want to infer or the fact that I want to represent from this logical statement is the following: I want to say that if say x_1, x_2, x_3, x_4 and x_n are my students of the college, then I want to represent here the fact that if x_1 has studied or if x_1 has enrolled in course number CS201 then he has studied calculus. In the same way I want to state that if x_2 has studied or in if x_2 has enrolled for course number CS201, then it has studied calculus.

In the same way I want to represent that, if x_3 has enrolled for CS201 then it has studied calculus. So when I am saying that every student in my domain who is enrolled for CS201 has studied calculus the interpretation of that is that I am making a universal statement, a universally quantified statement where I am saying that all for every student x in my domain, if student x has enrolled for CS201, then student x has studied calculus.

That is what is the logical interpretation of the statement that every student in CS201 has studied calculus, I am making an assertion about every x from my domain, okay? So now I have to introduce some predicates here to represent the statement at every student x in my domain if student x is enrolled for CS201 then it has studied calculus. So, let me first introduce a predicate here S(x) while you can use any predicate variable but I am using S(x) for my convenience.

And, remember in the predicate world we use variables in capital letters for denoting predicate functions. So, S(x) will be true if student x has enrolled for CS201 where as S(x) will be false if student x in your domain has not enrolled for CS201 and let me introduce another predicate here I am denoting it as C(x) and it will be true if student x in your domain has studied calculus else, it will be false.

And, I do need these two predicates here because I want to assert or relate properties of a student x with respect to whether he has studied calculus or not and whether he has enrolled for CS201 or not. So that is why I have introduced two predicate functions here. Now coming to the question how do I represent a statement that every student in CS201 has studied calculus? So I am writing down here two expressions.

One expression is for all x, $S(x) \rightarrow C(x)$ this represents that for every x in the domain here domain is the set of all students in my college, if student x has enrolled for CS201, then he has studied calculus, whereas the other expression the right hand side expression here denotes that every student x in the college has enrolled for CS201 and studied calculus. Now an interesting question here is whether the statement that I want to represent is represented by the first expression or is it represented by the second expression?

Very often students do think that it is the second expression which is representing the statement every student in CS201 has studied calculus but that is not the case.

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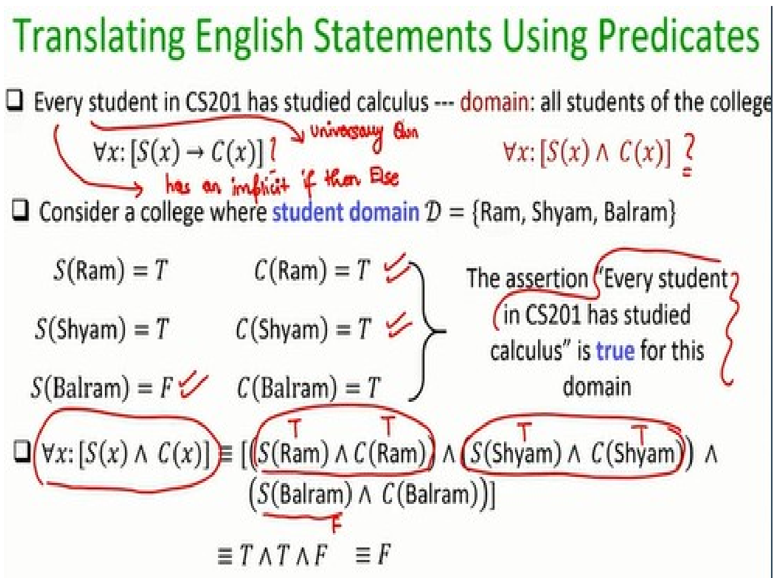

So, let me demonstrate that why the second expression is an incorrect expression and it is the first expression which represents the statement every student has studied calculus in CS201. So consider a college where the student domain has three students Ram, Shyam, and Balram and say in that college, all the students except Balram has enrolled for calculus; so that is why S(Ram) is true, S(Shyam) is true and S(Balram) is false.

So remember S(Ram), S(Shyam) and S(Balram), they are now propositions because I am assigning the values x equal to Ram, x equal to Shyam, x equal to Balram and as soon as I assign concrete values to my predicate variable, the predicate gets converted into a proposition and say in the same domain Ram, Shyam, and Balram all of them have studied calculus that means the proposition C(Ram) is true C(Shyam) is true and C(Balram) is true.

Now you can see here that in this domain indeed the assertion that every student in CS201 has studied calculus is true because you check Ram has studied, Ram has enrolled for CS201 and indeed he has studied calculus. Shyam has enrolled for CS201 and indeed has studied calculus but Balram he is not enrolled for CS201 so I do not care whether he has studied calculus or not. My assertion was that every student in CS201 has definitely studied for calculus or not.

I do not care about the students who are outside CS201; they may or may not have studied calculus that is not conveyed through this statement. Now, let us consider the two expressions our goal is to identify whether it is the expression number one or expression number two which represents the assertion that every student in CS201 has studied calculus.

So if I consider the first expression which is for all x, $S(x) \rightarrow C(x)$ and if I substitute x equal to Ram, x equal to Shyam and x equal to Balram then this universally quantified statement gets converted into conjunction of three propositions. Why conjunction of three propositions because recall from the last lecture a universally quantified statement is true, if it is true for every x in the domain.

And, my x in the domain are Ram, Shyam and Balram and it is an implication statement, so it will be conjunction of three implications. Now, with respect to the truth values that have been assigned to S(Ram), S(Shyam), S(Balram) and C(Ram), C(Shyam) and C(Balram). In my domain it turns out that each of the implications is true. Indeed S(Ram) is true and C(Ram) is true, so true implies true is true.

Now S(Shyam) is true, C(Shyam) is true, so true implies true is also true and S(Balram) is false, so I do not care whether C(Balram) is true or false, false implies anything is true and the conjunction of true, true, true is of course true, so you can see that the expression for all x, $S(x) \rightarrow C(x)$ indeed turns out to be true with respect to this domain where the assertion that every student in CS201 has studied calculus is true.

Whereas consider the expression, second expression, namely for all x, S(x) conjunction C(x). So if I substitute the different values of x, I get conjunction of three propositions here and each individual proposition is conjunction of two propositions namely S and C. If I assigned a truth values, if you check the truth values that we have assigned for the proposition S and proposition C for Ram, Shyam and Balram, it turns out that the first compound proposition here is true because both S(Ram) and C(Ram) are true.

The second conjunction here is also true because S(Shyam) is true and C(Shyam) is true, but S(Balram) is false and C(Balram) I do not care whether this true or false because false conjunction with anything is false and hence the over all expression is false and indeed the second expression here should turn out to be false here because the second expression here denotes the assertion that every student of the college has enrolled for CS201 and he has studied for calculus.

But that is not what we want to assert here, our assertion that we are interested to express is that if at all a student x has enrolled for CS201 then he has studied calculus. So the summary here is that even though there is no explicit "if then" statement given here the statement of the form every student in CS201 has studied calculus has an implicit, it has an implicit, "if then else" form and the second thing here is that this is a universally quantified statement because I am making a statement about every x in my domain.

So even though the statement is not given of the form for all students that word for all is not explicitly given here you have to understand that it is implicitly hidden here and that is why the quantification that we have used in this predicate is for all x.

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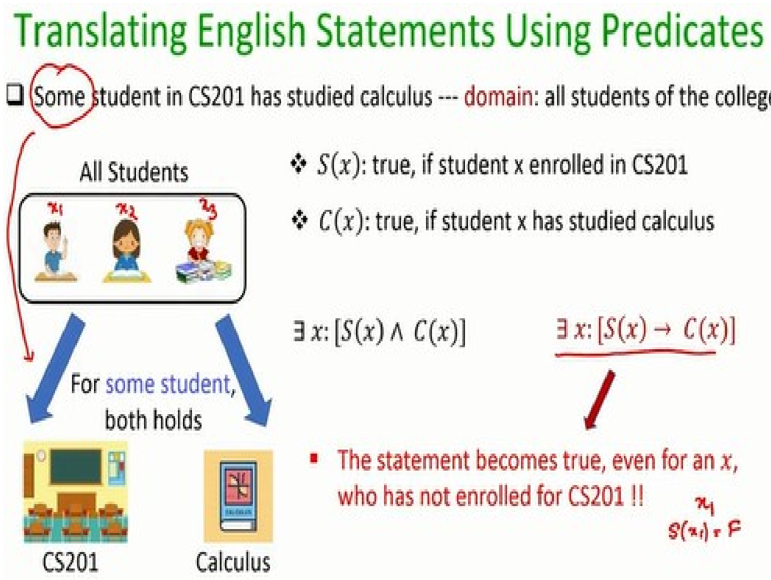

Let us see another example, so my domain is still the students of my college and I want to represent the statement that some student in class CS201 has studied calculus and let me retain the same two predicates S(x) and C(x) from the previous example. So again, we have to understand whether this statement is universally quantified or is it existential quantified whether it involves any kind of "if then" or not and so on.

So if you see here closely, it turns out that this statement some student in CS201 has studied calculus means that I want to represent a fact that for some x in my domain, so I have multiple x values possible from my domain I want to represent the assertion that for some x from my domain the x satisfies two properties simultaneously namely the same x has enrolled for CS201 and the same x as studied calculus.

That means the property that x is enrolled for CS201 and has satisfied calculus hold simultaneously for the same x was from my domain and this is true for at least one x because I am saying here that it is true for some x I am not saying it is true for all x. So it turns out that this statement or this assertion will be represented by this existentially quantified statement namely there exists some x in my domain such that the property S(x) and C(x) are simultaneously true for the same x.

And, I have explicitly put the parenthesis here because this existential quantification it is applicable both over the predicate S as well as C here. If I do not put the parenthesis here then you get ambiguity whether x is within the scope of, where the occurrence of x in both S(x) and C(x) is within the scope of there-exist or not. So that is why to avoid confusion I have explicitly added parenthesis here because I want to represent the fact that it is for the same x that both S(x) and C(x) holds simultaneously.

Now an interesting question here is why cannot we represent this assumption by this second expression there exists x such that S(x) → C(x) might look that this second expression also can represent the same assertion but that is not the case because if you closely see here this second expression, this expression becomes true even for an x who is not enrolled for CS201 that means if you have say some x_1 such that $S(x_1)$ is false.

Then even for such an x_1 this existential quantification becomes true because since $S(x_1)$ is false, it does not matter whether $C(x_1)$ is true or false the overall implication will be true because false implies anything is true.

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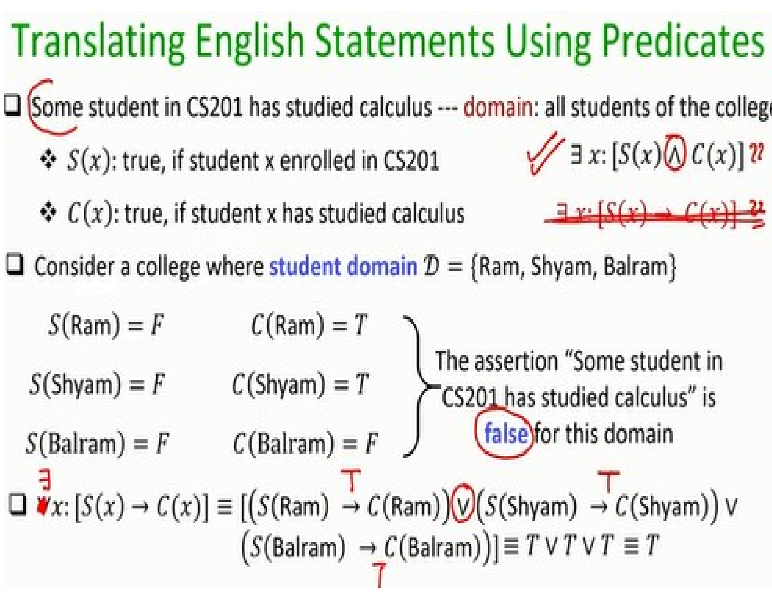

So to make my point more clear, our goal is to identify whether it is the first expression or the whether it is the second expression which represents my assertion that some student in CS201 has studied calculus or not and again consider a college which has three students Ram, Shyam and Balram and say for that college none of the students has enrolled for CS201 and say only

Ram and Shyam has studied calculus while Balram has not studied calculus.

Now you can check here that indeed in this particular college the assertion some student in CS201 has studied calculus is false. For this particular domain because there is no student in CS201 at the first place itself, it does not matter whether they have studied calculus or not. That means if expression one represents my assertion, then that expression should turn out to be false.

Whereas if expression 2 represents my statement; then the second expression should turn out to be false with respect to this domain. Let us check whether it is expression 1 or whether it is expression 2 which turns out to be false with respect to this particular truth assignment, so if I consider expression number 1; the expression number 1 is an existential quantified statement, which has a conjunction involved.

Now if I expand x and give it values Ram, Shyam and Balram I get that this expression is logically equivalent to disjunction of three statements, why disjunction? Because remember an existentially quantified statement is true if it is true for at least one x value in the domain, and now you can check with respect to the truth values that have been assigned to x variable in S propositions and C proposition this expression turns out to be the disjunction of false, false and false which is overall false.

And, that is what we want because indeed in this particular domain the assertion that some student in CS201 has studied calculus is false and that is what expression number one also tells us. But what about expression number two? The expression number two is for all x, sorry for the typo here, it should not be for all x it should be there exist x. The second expression is there exist x.

So, again if I expand this there exist statement since it is an existential quantification, it will be disjunction of three propositions where each proposition is an implication, $S(x) \rightarrow C(x)$ and x can take values Ram, Shyam and Balram. Now you can check here that each of the individual x compound propositions here are true, with respect to the truth values that have been assigned. $S(Ram) \rightarrow C(Ram)$ will be true because S(Ram) is false and false implies anything is true.

S(Shyam) → C(Shyam) will be true because S(Shyam) is false and false implies anything is true. S(Balram) → C(Balram) is also true because false implies false is true and disjunction of truth is always true that means even though the assertion that some student in CS201 has studied calculus is false with respect to this domain, the second expression turns out to be true with respect to this domain. That tells us that it is not the second expression which represents the assertion that we are interested to state here. It is the first expression which is the correct expression, so these two examples are very important, it tells you the significance that where to use implication and where to use conjunction, whenever you have assertions of the form “some” definitely, and some properties are involved here, then you have conjunction involved whereas in the previous example it is a universally quantified statement we are an implicit if then was present.

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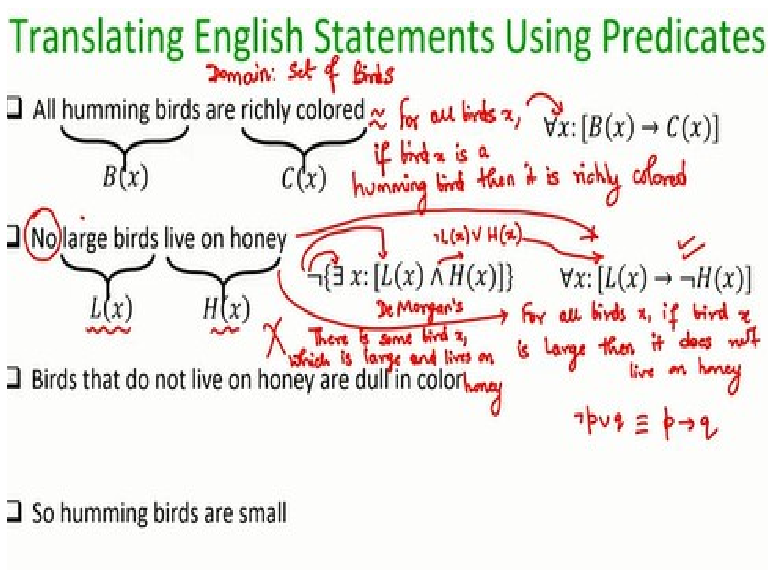

Now, let us take another example to make the concepts more clear here you are given an English argument a set of English statements and you have to convert everything into predicates and your domain here is a set of birds because I am stating several properties about birds here, so my domain is set of birds. So whenever you are given English arguments you have to first identify what is the domain.

The domain may or may not be explicitly given to you here it is not explicitly given but by identifying the statements we find out that we are making statements about birds here, that is why the domain will be set of birds. So the first statement is all hummingbirds are richly coloured. So, let me introduce predicates B(x) and C(x) here. So B(x) will be true if the bird x is a humming bird.

Whereas the predicate C(x) will be true if and only if the bird x is richly coloured that is the definition of my predicates B(x) and C(x) and that is the case and this statement will be represented by for all x, $P(x) \rightarrow C(x)$ because an equivalent form of this statement is for all birds x, if bird x is a hummingbird then it is richly coloured. That is what is the interpretation of this statement.

And, then you can check here that indeed this implication, this universally quantified implication represents this equivalent statement. The second statement is no large birds live on honey. So I have to introduce a predicate L(x); where L(x) will be true if and only if the bird x is a large bird and my predicate H(x) will be true if and only if the bird x lives on honey that is the interpretation of the predicates L(x) and H(x).

Now again, if you closely see here, there is a universal quantification involved, okay? So let us so there are two forms of the same statement, I can represent this English statement either by this first expression as well as by the second expression. So let us see the second expression, why? The second expression is the representation of this English statement. If you see here closely, if you interpret it closely the logical form of this interpretation of this statement is the following.

I want to represent that for all birds x. If bird x is large, then it does not live on honey that is what is the logical interpretation and indeed this expression represents this statement, whereas the second expression is arrived as follows, so for the moment forget about this negation which is present outside. Let us forget about this negation for the moment, let us try to understand what exactly there-exists x, L(x) conjunction, H(x) represent.

This represents that, there is some large bird some bird x which is large and lives on honey. That is what will be the interpretation of this expression but this is not what I want to represent; I want to represent that there is no such bird exist which is simultaneously large as well as lives on honey and that is why I have put a negation outside. If I put a negation outside that means this property is not possible which is indeed what I want to represent, okay?

Now, if you closely see if I apply the rules of equivalence for predicates here if I apply the De Morgan's law for predicates, which I have discussed in the last lecture. Then I can take this negation inside and when I take negation inside they are exists gets converted into "for all" and this negation will also go with L. So, I will get negation of L(x) and this conjunction gets converted into disjunction and now you know that negation p OR q is logically equivalent to p → q.

So I can further rewrite this expression as this and that is how I get the second expression. So you can get the second expression by reinterpreting this statement in the form that for all birds if bird x is large, then it does not live one honey or you can first arrive at this first expression and then apply the De Morgan's law and apply it to get into the second expression. So both the expressions are correct.

You can use either the first expression or the second expression to represent the statement that no large birds live on honey.

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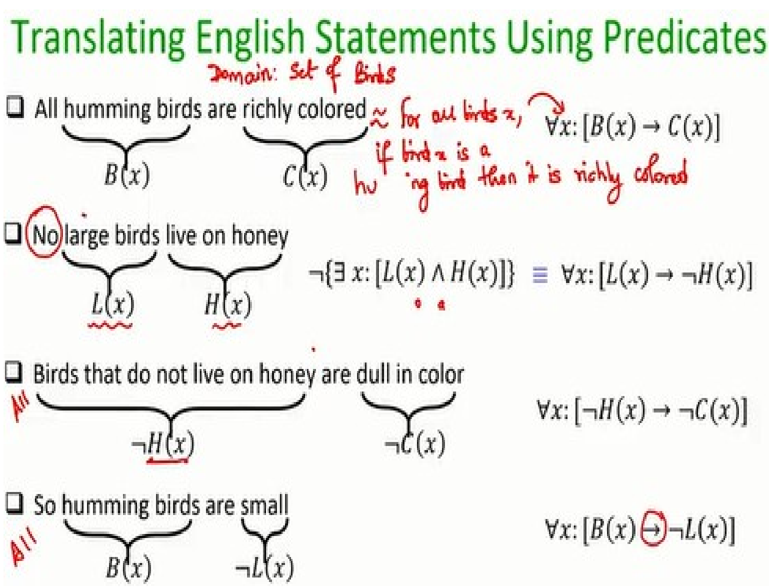

Now what about the third statement? So I do not need to introduce new predicates here because I have already introduced the predicate H(x) over to represent that bird x lives on honey and I have already introduced the predicate C(x) to denote that bird x is richly coloured. So dull in colour will be negation of C(x). Now the question is, is this universal quantified statement or existential quantified statement?

It turns out that it is a universally quantified statement because I am making or asserting this property for all birds, I am not saying it just for some specific bird, right? I am trying, so you can imagine that another way to re-interpret this statement is I am making the statement that for all birds x, if bird x does not live on honey then it is dull in colour. So there is “if then” involved here and it is a universal quantified statement.

And that is why this will be represented by this expression and what is the last statement that hummingbirds are small, again I do not need any new predicate here, hummingbirds is represented by the predicate B(x) and L(x) was used to represent that bird x is large so negation of L(x) will represent that the bird x is small and again this conclusion is about all hummingbirds, it is not about a specific hummingbird, right?

And again this property, another way to reinterpret this English statement is that for all birds x, if bird x is hummingbird then it is a small bird. So that is why there is an implicit “if then” involved here that is why this English statement will be represented by this expression. So, that is how you can convert your English statements into predicates.

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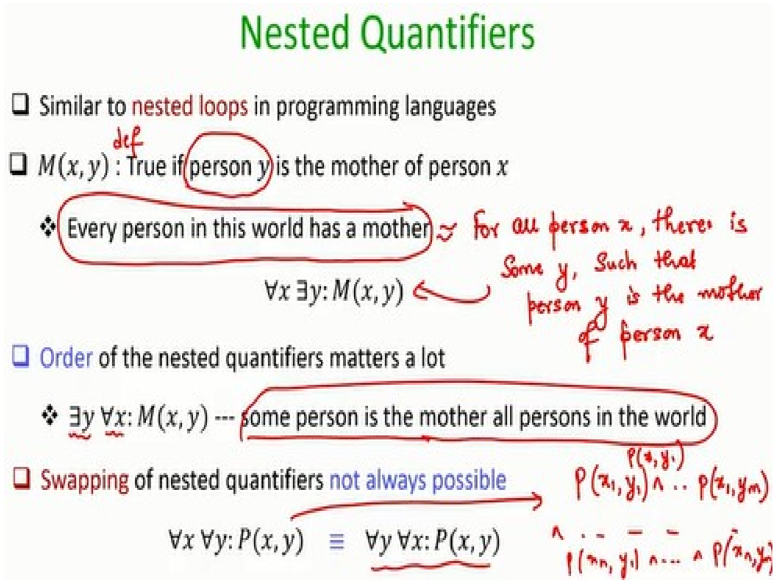

Now let us try to understand Nested Quantifiers, so there is very often we encounter statements where we need to have a nested form of quantification and this is similar to nested loops in programming languages. So let us see an example here, so say the predicate M(x, y) is defined in such a way that it is true if person y is the mother of person x, that is the definition of the predicate M(x, y).

And, I want to represent a statement that every person in this world has a mother. So my claim is that this can be represented by this expression for all x there exist y such that M(x,y) is true and this is an example of nested quantification. You want to say that you fix a value of x, for that fixed value x there exists some y, you change x then for the new x there might be another y, you change x then for the new x you have another y such that this property M(x, y) is true.

And why this is the expression representing every person in this world has a mother; well, this is equivalent to saying that for all person x, there is some y, such that person y is the mother of person x which is indeed what is represented by this expression. Now when you are dealing with nested quantification the order of the quantification matters a lot because if you change the order of the quantification then the logical interpretation of the statement changes completely.

For instance if I write an expression there exist y for all x, M(x,y); where M(x,y) is as defined above, the interpretation of that is you have there exist coming outside first, that means you want to say that there is some person y, such that all the x are related to that y. Namely the same y is the mother of all persons x in the world, that is not what we want to interpret. This statement some person is the mother of all persons in the world and every person in this world has a mother, they are two different logical statements.

And hence they are represented by two different nested quantifications. So that is why swapping of quantifications are not always possible, it is possible only when you have the quantifications of the same type occurring throughout the expression. That means if you have an expression of the form for all x for all y or a sequence of quantifications which are of the same type then it does not matter whether it is y appearing first or whether it is x appearing first.

You can conveniently swap the order of the quantification and both LHS and RHS will be equivalent if you want to check that well for all x for all y can be considered as follows if you expand the for all x and for all y then it will be considered, imagine that x takes values from x_1 to x_n and y takes values from y_1 to y_m, right? I can expand this left hand side in this form and everything is conjunction here.

And, then I can swap and can shuffle around all the $P(y_1)$ first all of so I can shuffle around all the expressions of the form anything P of anything followed by y_1 and take them together and then followed by conjunctions of all P anything of y_2 and so on and that will be equivalent to the second expression right and this shuffling around is possible because everywhere AND is appearing and it satisfies the associative law.

But if you have an expression where you have quantifications of different form, then this kind of swapping may not be possible. The logical interpretations might be completely different.

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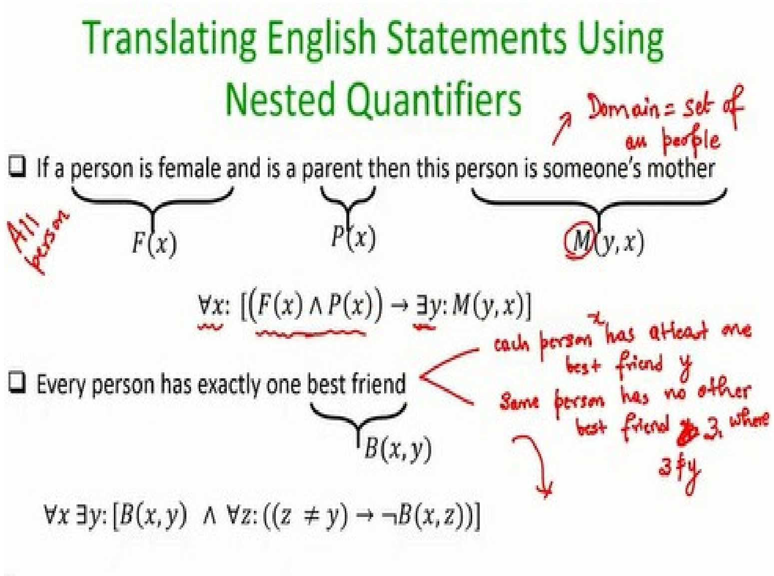

So, now let us see some more examples here how we can start translating statements using the help of nested quantification. So suppose I want to represent a statement that if a person is female and is a parent then this person is someone's mother. So we have to first identify or define the predicates that we have to use here and here again, the domain is not explicitly given but you can imagine here that the domain is the set of all people.

So let me introduce this predicate F(x) which is true if person x is female and I also need a predicate P(x) to represent that person x is parent and I had already introduced a predicate M in the previous slide which I am retaining here. So first of all, this is a universally quantified statement because I am making a statement about all persons here, I am not making a statement about some specific person, I am making a statement about all persons.

So that is why this will be a universally quantified statement and this is an if statement of the form if-then your premise is for all person x in the domain I want to state that if the person x is female and if the person is a parent, so that is why conjunction of F(x) and P(x) then for the same x there exist a y, a person y such that x is the mother of y and you see how carefully I have put the parentheses here.

If I do not put the parentheses then the expression becomes ambiguous it will not be clear that whether it is x which is appearing first and then followed by y and so on. So x is occurring on a higher level and for each x there will be some y. Similarly if I want to represent statements of the form that every person has exactly one best friend, so this statement has two parts.

The first part is that each person has at least one best friend definitely, thats the first part of this statement : is one best friend in fact each person x has at least one best friend y and the second part is the same person x has no other best friend z, where z is different from y and this is true for all x that is what is the logical interpretation of this statement. So let me first introduce the required predicates here, so I introduce a predicate B(x, y) which is true if person y is the best friend of person x that is a definition of my predicate B(x, y).

And, now you can see here that since I have identified the two parts of this English statement, the first part is that for every person x there is some y such that y is the best friend of x and I want to state that for the same x there is no different person z different from y who is also the best friend of x that should not be possible, so that is why the left hand side represents the first part of this expression represents that person x has at least one best friend.

And the second part of the expression represents that person x has the possibility of a second best friend as well I want to avoid that and that is why I put a negation in front of that if I put the negation in front of that then that rules out the possibility that there is no second person z different from y who is also the best friend of x because of the occurrence of this negation.

And then conjunction of both these conditions will represent what I am interested to assert. Of course now, if you do want to apply the De Morgan's law of quantifications, you can take the negation, this negation that is here, and you can take it inside and then conjunctions get converted into disjunctions and so on and then you can apply the rule that negation P OR Q is equivalent to $P \rightarrow Q$ and this is another equivalent form of the same expression.

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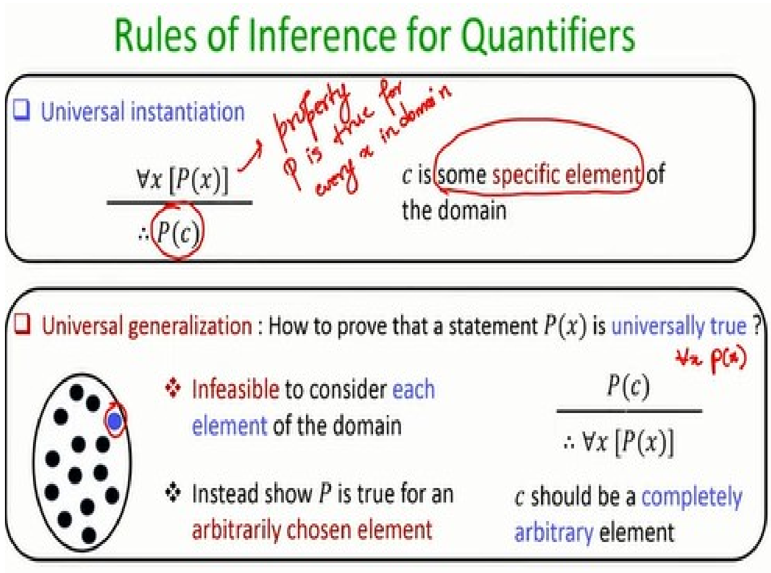

Now let us do some rules of inferences for quantified statements, so which are very important the first rules of inference is universal instantiation and argument form of this universal instantiation is if you are given the premise that for all x, P(x) is true, then you can come to the conclusion that the predicate P is true for some element c in the domain, where c is some specific element that you are interested in that you want to explicitly specify.

And, this is because since the premise for all x, P(x) is true that means property P is true for every x in the domain, property P is true for every x in the domain. So of course, it will be true for the element c as well, ok, whereas universal generalization has a different argument form, so what exactly universal generalization is used for so imagine you want to prove that a property P is true for every x in the domain that means you want to prove or assert that for all x, P(x) is true.

How do you do that? One option could be that you check whether property P is indeed true for x_1 or not, x_2 or not, x_3 or not and so on, where x_1, x_2, x_3 etc are the various values in your domain but this becomes infeasible if your domain is infinitely large. So to prove statements of the form that prove that something is true for every x in the domain where domain is infinitely large, very often we encounter statements of the form that prove some property is true for every integer x.

How do we prove it? We cannot take each and every integer and show that indeed the property is true for every integer that you have chosen. So to prove statements of that form, what we do is we pick some arbitrary element of the domain when I say arbitrarily element of the domain that means there is no specific property of that element, it is just some arbitrary element and show that the property P is true for that arbitrarily chosen element; if it is true for that arbitrarily chosen element, you can come to the conclusion that P is true for any element in the domain because the sample point that you have chosen was arbitrary. So the argument form here is if you show or if you know the premise that property P is true for element c where c is some arbitrarily chosen element then you can come to the conclusion that for all x, P(x) is true. So this is called universal generalization.

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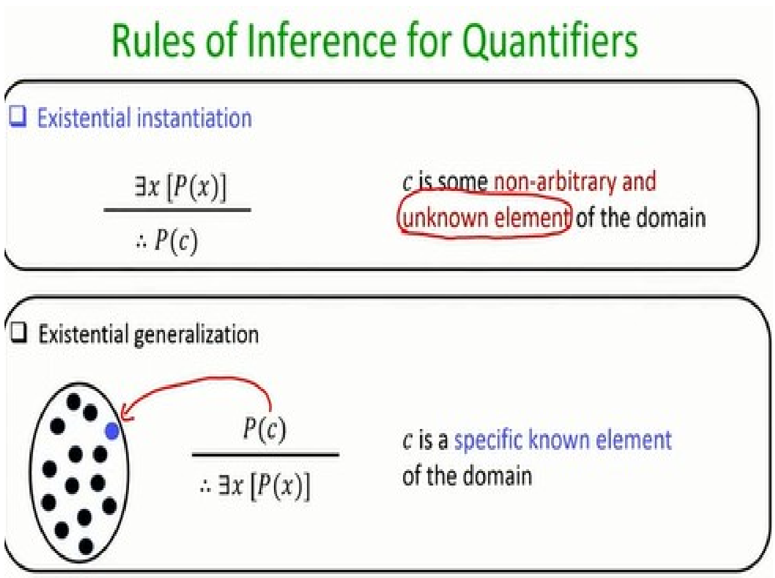

Now the duels of these rules are existential instantiation, which says that if you have the premise there exist P(x) then you can conclude at proposition P(c) is true where c is some non arbitrary but unknown element, I stress here that you may not be knowing what exactly is the element but you will be knowing that since P is true for some x in the domain, let c be the x for which it is true what exactly is that c you may not know that whereas existential generalization says that if you know that property P is true for element c in the domain where c is some fixed element, which you are aware of, that means you have a witness c explicitly for which the property P is true, then you can come to the conclusion that there exist x, P(x) is true.

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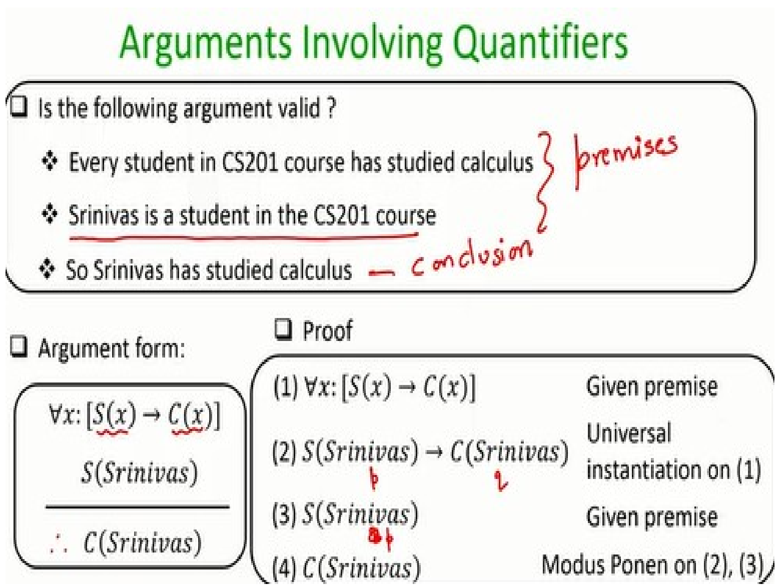

So these are four popular rules of inferences which we use involving which we use while dealing with quantifications. So now let us do an example to verify how to verify whether argument forms are valid or not, in predicate logic. So here you are given two premises and conclusion here. So I am retaining the same predicates S(x) and C(x) that we have defined in some earlier slides.

So the first statement is : every student in CS201 has studied calculus. So that is represented by for all x, S(x) → C(x) that is your first premise and the second premise is Srinivas is a student in the CS201 course that means the property S(x) is true for x equal to Srinivas; that means S(Srinivas) which is now a proposition is true, that is your premise. I said this is now a proposition because you have now assigned a value x equal to Srinivas.

The conclusion you are drawing here is that Srinivas has studied calculus that means you have to show that C(Srinivas) is true. So, let us see whether this argument form is valid or not, so you are given the premise for all x, S(x) → C(x) so what you can do is you can apply the universal instantiation and you can substitute x equal to Srinivas and get the proposition S(Srinivas) → C(Srinivas) to be true.

You are also given the premise S(Srinivas) to be true, now what you can do is you can think that this is now P → Q a proposition and a proposition P both these premises are true so you can apply Modus Ponen and come to the conclusion that C(Srinivas) is true.

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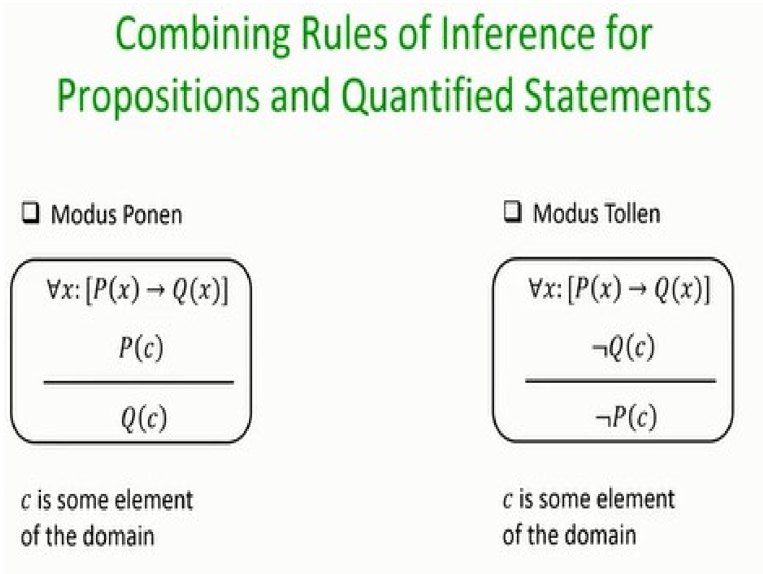

So that leads us to the Modus Ponen and Modus Tollen rules. These are the generalizations of Modus Ponen and Modus Tollen to the predicate world. Modus Ponen says the following if you are given the premises for all x, $P(x) \rightarrow Q(x)$ and if P is true for some element c in the domain then you can come to the conclusion Q(c) and then same way Modus Tollen is generalized.

So that brings me to the end of this lecture. Just to summarize. In this lecture we saw how to convert English statements using predicates and logical connectives, we saw some rules of inferences using predicate logic and we saw how to verify whether a given argument form involving predicates is a valid argument form or not, thank you.