Discrete Mathematics

Prof. Ashish Choudhury
Department of Mathematics and Statistics
International Institute of Information Technology, Bangalore

Lecture -44
Graph Theory Basics

Hello, everyone welcome to this lecture, so the plan for this lecture is as follows.

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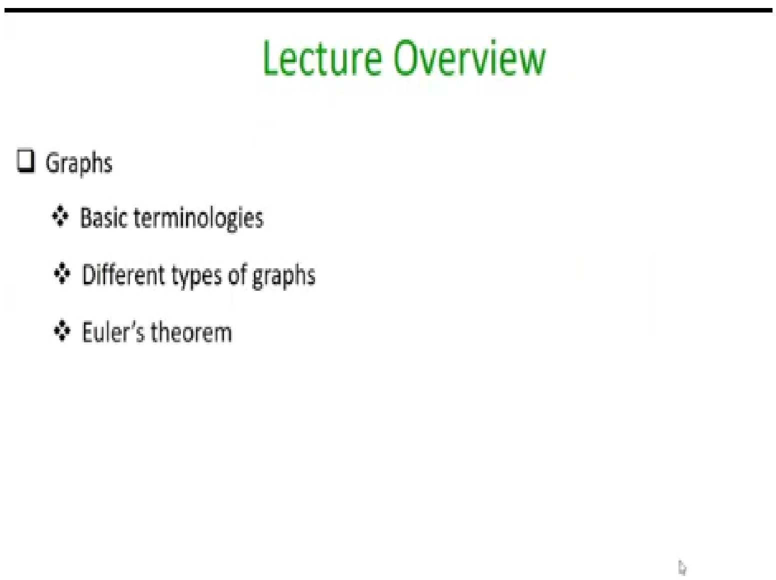

In this lecture we will introduce the basic terminologies related to graph theory. We will discuss about different types of graphs and we will also discuss about Euler's theorem.

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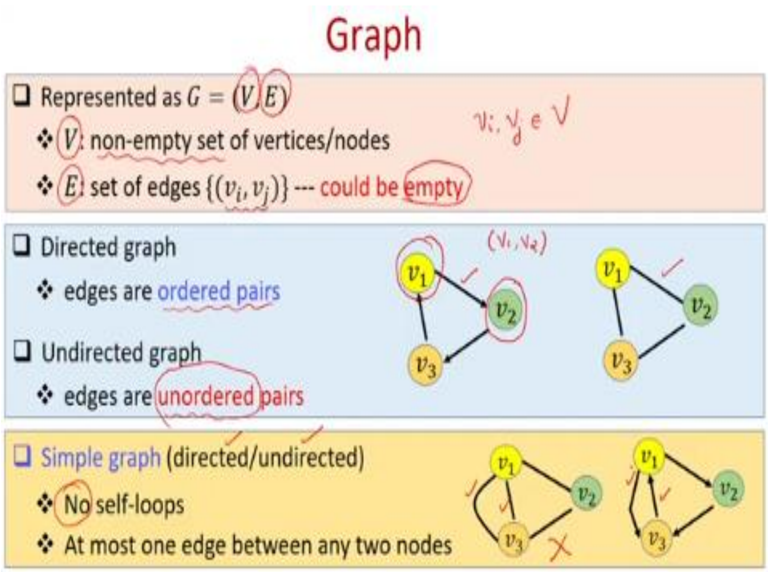

So, what is a graph? It is a collection of two sets, a set of vertices and a set of edges. So, the set of vertices is denoted by the set $V, V \neq \phi$ it is a non-empty set of vertices also called as nodes and we have another set of edges denoted by this notation E, which have edges of the form (v_i, v_j) where both $v_i, v_j \in V$ and this edge set could be empty, $E = \phi$. So, what it means is that in a graph you can have a graph which has no edges but the vertex set cannot be empty. The vertices are always there, you cannot have a graph which is where the vertex set is empty as well as the edge set is empty.

So, we have two types of graphs directed graphs and an undirected graph. When I say directed graphs then pictorially you can imagine that in directed graph the edges have directions associated with it. That means we have the notion of the starting point of an edge and the ending point of an edge. And in terms of set theoretic notations, we can say that a directed graph is a graph where the edges are ordered pairs and as soon as I say that edges are ordered pairs, that means it matters whether the starting endpoint is v_i or the starting end point is v_j, if you are talking about an edge (v_i, v_j). So, for instance if I consider this directed graph, then the edges that we have here are the ordered pairs (v_1, v_2) because the starting point of this edge is v_1 and the end point is v_2 i.e if $(v_1, v_2) \in E$ then $(v_2, v_1) \notin E$. Whereas if we consider undirected graphs then the edges have no direction associated with them which is equivalent to saying that the edges are unordered pairs. That means when I consider this edge from the vertex v_1 to vertex v_2, if $(v_1, v_2) \in E$ then $(v_2, v_1) \in E$.

So, let us now define what we call as a simple graph and this definition is applicable both for the directed graph as well as undirected graph. So, a simple graph is a graph which has no self loops and there can be at most one edge between any two nodes. So, if I consider this undirected graph then this is not a simple graph because between the nodes v_1 and v_3, you have two edges this edge as well as this edge which is not allowed as per the definition of an undirected simple graph.

But if I consider this directed graph, then this is a simple directed graph because even though you have two edges involving the nodes v_1 and v_3, this is one of the edge and this is another edge. They are different directed edges because the ordered pair $(v_1, v_3) \neq (v_3, v_1)$. So, this is an example of a directed simple graph.

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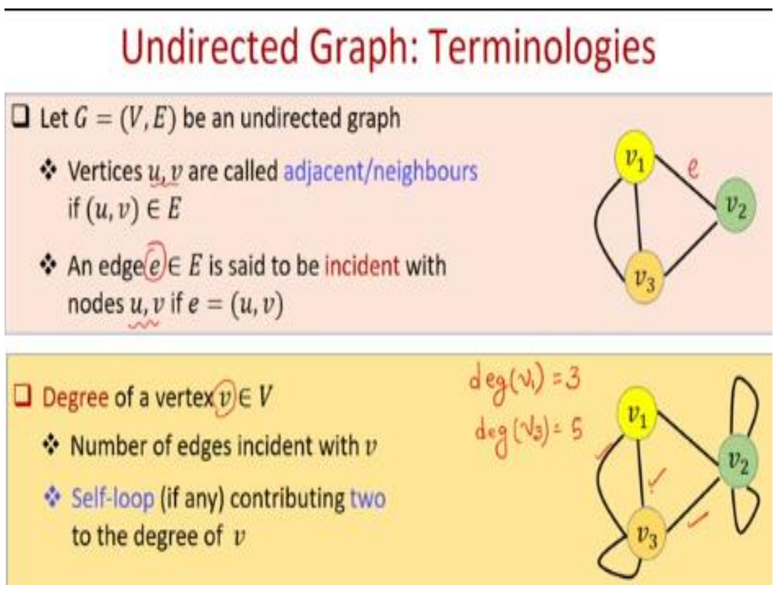

So, now let us introduce some terminologies related to undirected graphs. So, if you are given an undirected graph then a pair of vertices u, v are called adjacent or neighbors of each other. If the edge $(u, v) \in E$ b edge set, that means if u and v are the end points of an edge, then in an undirected graph then we call the vertices u and v to be adjacent or neighbors.

Here v could be u that is allowed. That means if I have a self-loop then the vertex v will be called adjacent to itself, right? Then if we are given an edge small e, then we will say that it is incident with the nodes u, v if u and v are the end points of the edge. So, for instance if I consider this edge

e, then v_1 and v_2 are incident with respect to this edge because they are the end points of this edge e and so on.

Now let us define next what we call as the degree of a vertex v. So, again I am giving the definition with respect to undirected graph, but the definition can be extended or generalized for directed graphs as well. So, what is the degree of a vertex? The degree of a vertex is the number of edges incident with v or in the simpler language, the number of edges which have v as one of its end point.

And the definition has a special case if we have a self-loop incident with the vertex v. If there is a self-loop incident with the vertex is v then that self-loop is counted as contributing 2 to the degree of the vertex of small v. So, for instance, if I take this undirected graph then the degree of the vertex v_1, let us find out what's the degree, how many edges are there incident with the vertex v_1. So, this is one of the edge, this is another edge, okay to the two edge and now we have this edge again incident with v_1 so, total three. Whereas if I consider the degree of the vertex v_3, then we have this edge, this edge and this edge and we have a self-loop. The self-loop will be counted twice, while counting the degree of the vertex v_3. So, total the degree of the vertex v_3 will be 5 and so on.

So, this degree of a vertex the definition is with respect to the undirected graph, the undirected graph need not be simple. In fact in this particular example between the vertex v_3 and v_1, we have two edges and both of them are counted while counting the finding the degree of the vertex v_3.

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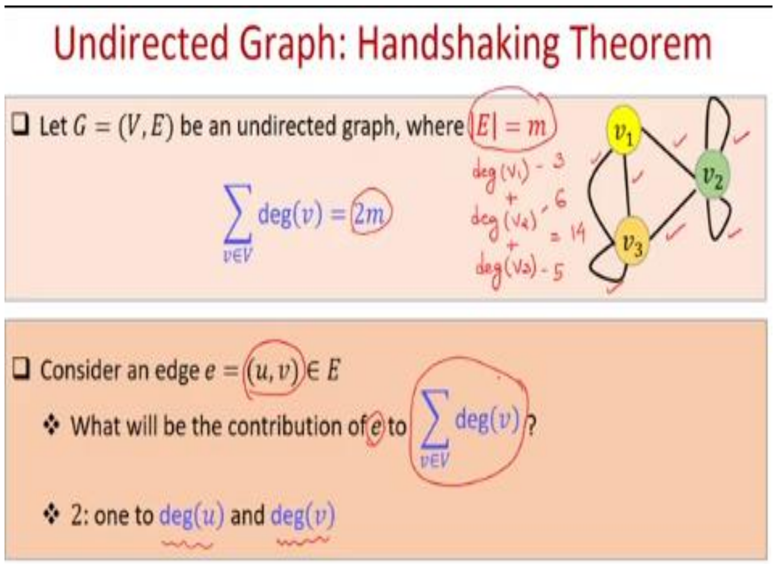

Next, we state a very fundamental fact about an undirected graph this is also called as the handshaking theorem. So, if you are given an undirected graph it may be simple, it need not be simple, it is just an undirected graph. And say the graph has m number of edges. Then what the theorem basically says is that if you sum the degrees of all the vertices in the graph then it will be twice the number of edges always.

So, you can verify this with respect to this example graph, so you take the summation of $\sum(v_1) + \deg\,(v_2) + \deg\,(v_3)$ It will be equal to twice the number of edges in the graph, so what is on? So, you have 1, 2, 3, 4, 5, 6, 7 and 8. So $2 \cdot 7 = 14$ and you can verify that. So, $\deg(v_1) = 3$, $\deg(v_2) = 6,\ \deg(v_3) = 5$.

So, let us prove this statement is true for any undirected graph. So, consider any arbitrary edge $e = (u, v) \in E$ in your undirected graph. It may be a self loop that means v=u or it may not be a self-loop. Let us try to find out that what will be the overall contribution of this edge when we are summing up the degrees of all the vertices in the graph? So, the claim is that the contribution of this edge will be 2 to the overall summation of degrees of all the vertices.

Because this edge will be contributing 1 when we will taking the degree of the vertex u and it will be again contributing 1 when we are taking the degree of vertex v and if u is same as v then as per the definition of the degree the self-loop will be counted twice, because v=u. So, that is why in the expression when we are summing up the degrees of all the vertices in the graph the contribution of the edge (u, v) will be 2. And hence it is easy to see that the summation of the degrees of all the vertices will be twice the number of edges in the graph.

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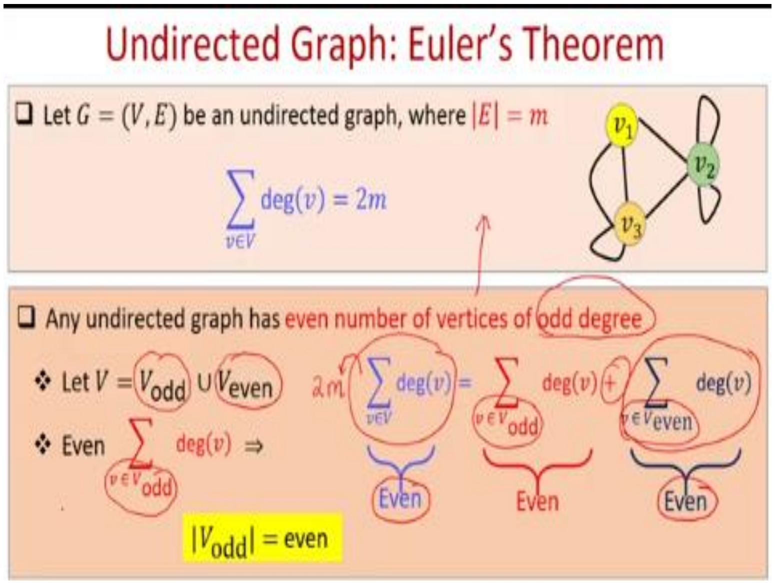

So, based on this fundamental fact we can derive another conclusion about any undirected graph and this conclusion is often called as the Euler's theorem. So, what it says is the following. So, what this theorem basically says? It says that you take any undirected graph then the number of vertices of odd degree will be always even. That means you will either have 0 number of vertices of odd degrees or 2 vertices of odd degrees or 4 vertices of odd degree or so on.

That means it would not be the case that the number of vertices of odd degree is odd and for that for deriving this conclusion we will use the previous fact namely the summation of the degrees of all the vertices in the graph is twice the number of edges and the proof is very simple. So, let V be the set of vertices in your undirected graph where $V = V_{odd} \cup V_{even}$ and $V_{odd} \cap V_{even} = \phi$. Because you cannot have a vertex in a graph which has both odd degree as well as even degree. A vertex will have either odd degree or even degree and hence it will belong either $v \in V_{odd}$ or $v \in V_{even}$ so these two sets constitute a partition of your set of vertices V. Now what I can say is that, if I take the summation of the degrees of all the vertices in the graph then that is equivalent to the summation of the following two quantities.

You take all the vertices in the set V_{odd} namely all the vertices which have odd degrees, and take the summation of the degrees of the respective vertices in the set V_{odd}. And you take all the vertices in the set V_{even} namely all the vertices which have even degrees and sum of their degrees and if you sum these two quantities that will give you the summation of the degrees of all the vertices in the graph.

Now I know that the left hand side of this equality is even, because as per the above theorem the left hand side is two times the number of edges. The number of edges could be odd or it could be even, but two times that number of edges will be an even quantity. So, I know that the left hand side is an even quantity and I also know that if I sum up the degrees of all the vertices which have even degrees, then that is also an even quantity.

The number of vertices of even degree it could be odd, it could be even it does not matter. But since I am adding several quantities each of which is an even quantity, the overall summation will be even. From that I can conclude the following that I can conclude that if I sum up the degrees of all the vertices in the set V_{odd} that also will be even because if that is not the case then you cannot have the difference of two even quantities.

Because that is simply coming from the fact that the summation of the degrees of all the vertices in the set V_{odd}, is the difference of two even quantities and the difference of two even quantities will be even, so what is the conclusion that we have drawn till now? We have concluded that if I take the summation of degrees of all the vertices in the set V_{odd} then that is an even quantity. That is possible only when the number of entities in the set V_{odd} is even, right?

So, because you are summing up several odd quantities and the summation of those odd quantities is even. That is given to you, that is possible only when the number of quantities that you have added is even. So, that is a fundamental fact about any undirected graph, irrespective of whether it is a simple graph or a non-simple or whether it is a regular it is not a simple graph.

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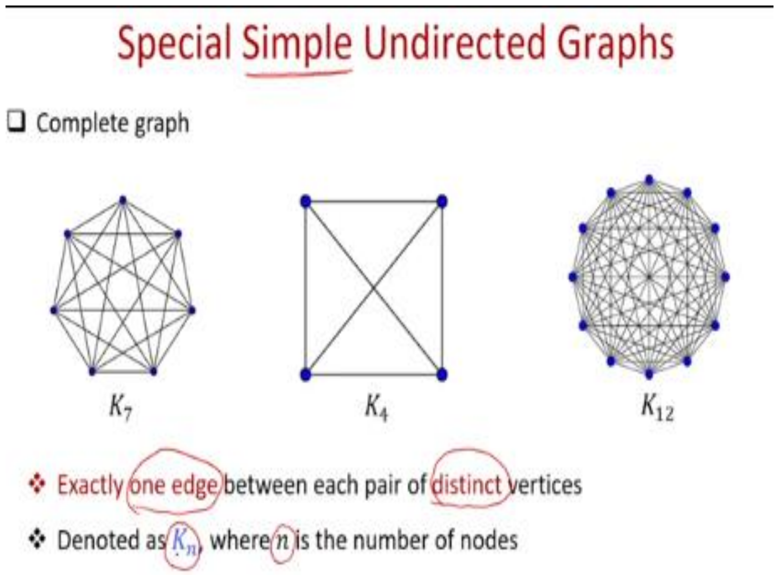

Now let us define some special types of undirected graphs. So, the first special type of undirected graph is a complete graph. And the property of this graph is that you have exactly one edge between each pair of distinct vertices. And since this is a simple graph and our property is that you cannot have more than one edge between every pair of distinct vertices. Automatically we have here the restriction that you cannot have a self-loop, because if you have a self-loop then that self-loop will be violating the definition here.

So, the requirement here is that you take the vertices, all the vertices between every pair of distinct vertices you will have exactly one edge. You do not have the option of 0 or 1, exactly 1 edge should be there between every pair of distinct vertices and if n is the number of nodes in a complete graph then we use this notation K_n to denote a complete graph with n nodes. So, these are the examples some of the examples of complete graphs with various values of n.

The complete graph with 7 nodes is this, complete graph with 4 nodes is this, complete graph with 12 nodes is this.

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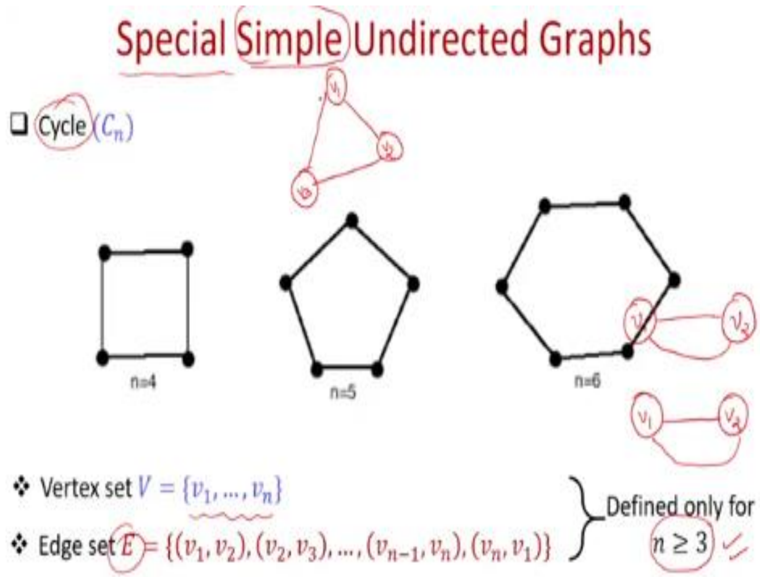

Then another special type of simple and directed graph is a cycle graph denoted as C_n. Here the vertex set will be consisting of n nodes and you will have $E = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}$. Now since the graph is simple, the cycle graph is defined only when the number of vertices is $n \geq 3$. Because if I try to define a cycle graph between involving just two nodes, then as per the definition the edge set will be the following. You have an edge between 1 and 2 and again you have an edge between 2 to 1 that will be the definition of the edge set as per this general definition.

But that will violate the property that it is a simple graph. So, that is why $n \geq 3$ and then only we can define a cycle graph. Well you cannot say that, why cannot I take this to be a cycle graph namely a graph where I have just an edge between 1 and 2? Because the interpretation of the cycle here is the following. The interpretation of the cycle is that once you go to the n^{th} vertex, you have an edge from the n^{th} vertex back to the first vertex.

So, without violating the requirement that your graph is a simple graph, this is possible only when n is greater than equal to 3. So, if I consider a cycle graph with three nodes then that will be something like this you, have v_1, v_2, v_3 and then you have an edge back from 3 to 1.

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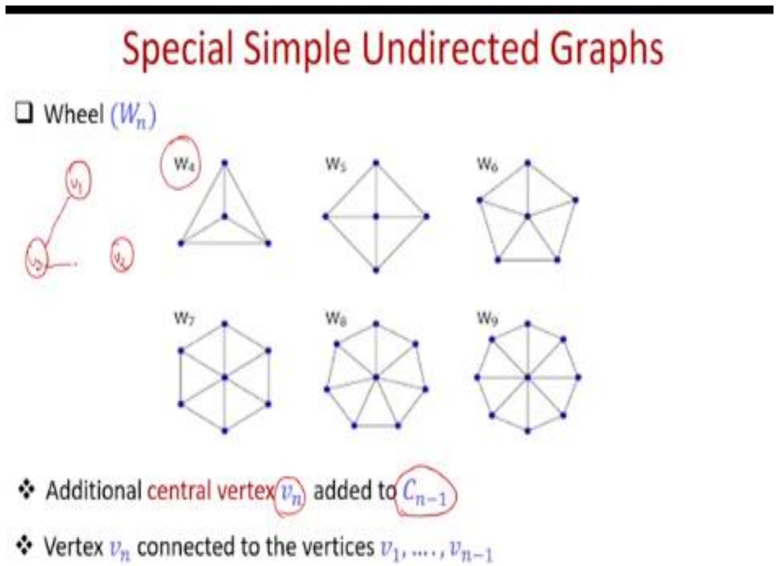

Then there is another special simple and directed graph called as the wheel graph. It is slightly different from the cycle graph, so what you do is you take a cycle graph involving n -1 nodes and then you add a central vertex which is the n^{th} vertex and the central vertex is now we add an edge involving this central vertex and all the vertices in your cycle graph C_{n-1}. So, for instance if I want to form W_4 then I take the cycle graph involving three nodes. Add the fourth vertex v_4 and add an edge from this fourth vertex to every other existing vertex in the cycle graph.

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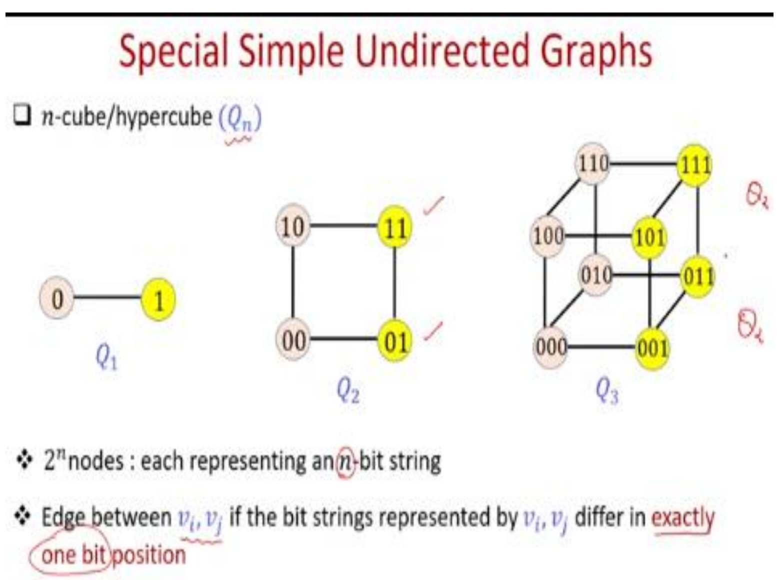

We also have another special simple undirected graph called as the n cube or the hyper cube denoted by this notation Q_n. So, this graph will have 2^n nodes, where each node represents a possible n bit string. So, remember that the number of bit strings of length n is 2^n, so each string is represented by a node in this hypercube graph and you have an edge between the i^{th} vertex and the j^{th} vertex if the bit strings represented by the i^{th} vertex and the j^{th} vertex differ in exactly one bit position. Otherwise the edge is not there between the i^{th} vertex and the j^{th} vertex. So, for instance if I take the graph Q_1 then I can have only two strings, two bit strings of length one, the bit string 0 and the bit string 1. They differ in exactly one bit position so the edge will be there. If I want to form the graph Q_2 the way I do it is as follows.

So, there will be, this will be the graph Q_2, because you will have four bit strings of length 2 and you can see that this edge is there because the binary string is 10 differs with the string 00 in exactly one bit position namely the first bit position. You do not have an edge between the bit string or the vertex one representing the bit string 10 and the vertex representing the bit string 01. Because these two bit strings differ in more than one position, so let me go back and try to understand that what exactly is the relationship between Q_2 and Q_1?

How exactly I can interpret Q_2 or relate Q_2 with the graph Q_1? So, the way I can construct the graph Q_2 is as follows. I take two copies of the graph Q_1, this is your first copy and this is the second copy. In the first copy I add the bit one at the beginning of all the nodes or all the bit strings represented by the nodes in the copy of Q_1. So, the copy of Q_1 that I would have taken is the following. I would have taken the vertex representing the bit string 0 the vertex representing the vertex bit string 1. What I am saying is you add 1 at the beginning of these bit strings.

You take the second copy of Q_1 and you add 0 at the beginning of all the strings represented by the notes in this copy of Q_1 and now you add the required edges depending upon the bit strings differ in exactly one bit position or not. In the same way if I want to define or get the graph Q_3 what I do the following, I take two copies of Q_2, one copy of Q_2, another copy of Q_2 and I extend the length of the bit strings of the nodes by adding one at the beginning of all the strings in the first copy of Q_2.

And I add 0 at the beginning of all the bit strings in the second copy of the Q_2 and then I add the required edges depending upon whether the two vertices differ in exactly one bit position. So, that is the definition of our Q_n graph.

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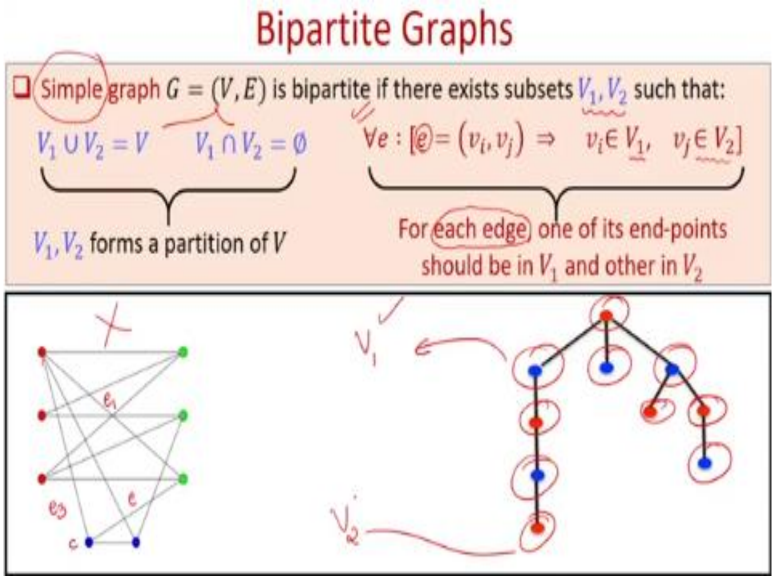

Now let us next define what we call as Bipartite graphs. So, if you are given a simple graph, then the graph is called bipartite, if we can find out two vertex sets V_1 and V_2 such that the following whole the vertex sets V_1 and V_2 should constitute a partition of your vertex set. That means $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$.

And the second property that I need from this partition V_1 and V_2 is the following. I need the following, you take any edge E in the graph the end points whose end points are v_i and v_j, then one of the end points should be in one of the subsets and the other end point should be in the other subset. $e = (v_i, v_j)$ then $v_i \in V_1$ and $v_j \in V_2$. So, what it means is that you take any edge of the graph, it cannot be the case that both its end points are in the same set. Set V_1 or in the set V_2, one of the end point should be in V_1 the other end point should be in V_2. So, for instance if I take this graph then, this graph is not a bipartite graph because I cannot find the required V_1 set and V_2 set. This is because if I focus on the specific portion of the graph, namely this portion is the triangle graph. So, if I call this as this edges that is e_1, e_2 and e_3 and if I try to find out a candidate V_1, V_2 satisfying this condition, I cannot do that.

Say, for instance I include this red colored vertex in the set V_1, so I let me call this vertices as a, b, c. So, suppose I include a in V_1 as per my requirement since the other end point of edge e_1 namely the vertex v should be the other subset or other partition. I should put the vertex b in the subset V_2, that is fine. Now what about the edge e_2? The other endpoint of the edge e_2 is c, so it cannot go in the same set V_2 because that will violate my requirement.

So, I can put c in the collection V_1, but as soon as I put c in the collection V_1, I see that I get a violation. Because I now have an edge namely e_3, where both the end points a and c are in the same collection V_1. So, that is why irrespective of how you try to come up with your candidate V_1 and V_2 you cannot find a partition V_1, V_2 for this graph satisfying this requirement. So, that is why this graph is not a bipartite graph.

But if I take this graph, then this graph is a bipartite graph and what will be the partition set for the vertex set V? What you can do is you take all the blue colored vertices and put them in the collection V_1 and you take all the red colored vertices and put them in the collection V_2. And now you can see that you take any edge e in this graph one of its end points will be in this V_1 and the other end point will be in V_2. It would not be the case that both its end points are either in V_1 or V_2. So, that is why this is an example of a bipartite graph.

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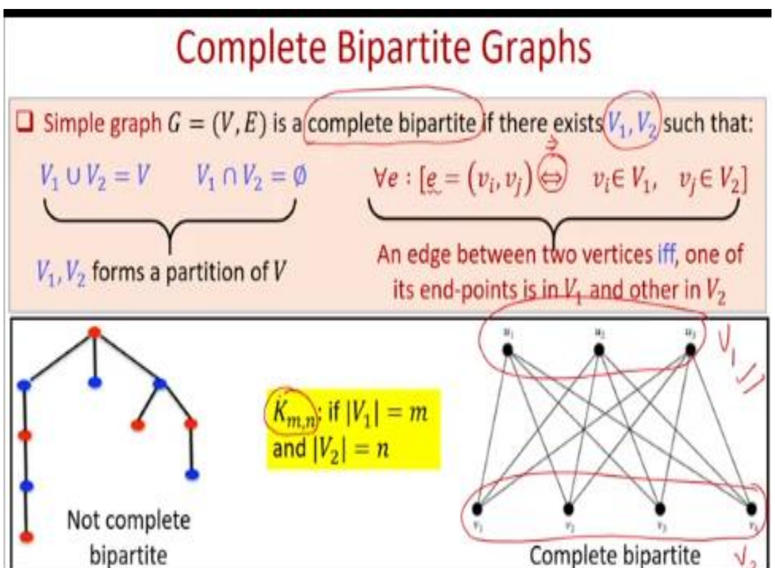

Now let us define next what we call as a complete bipartite graph? So, it is a special type of bipartite graph. So, first of all it is a bipartite graph namely it should be possible to come up with a partition of the vertex set, and that partition should have a special property. You take any edge in the graph one of its end point should be in V_1 other end point should be in V_2 that comes from the definition of the bipartite graph.

But this implication is now a bi-implication, that means it is the case that you have an edge in the graph between each and every vertex in the set V_1 and the set V_2. So, if you compare this bi-implication condition with the previous definition, in the previous definition it might be possible that you have some vertices in V_1 and some vertices in V_2 such that between those vertices you do not have an edge in the graph. So, for instance if I take this graph then this is a bipartite graph but this is not a complete bipartite graph. Why is not a complete bipartite graph? Because if I say for instance, you take this vertex it will be in one of the subsets given by your partition. And you take this blue colored node this will be in the other subset in your partition, but you do not have an edge between these two vertices in the graph.

So, that is why this implication was there only in one direction, but in a complete bipartite graph what I am saying is the implication should be a bi-implication. So, that is why this graph is not a bipartite graph but this graph is a bipartite graph because you can put the vertices u_1, u_2, u_3 in one subset and you can put the other vertices namely v_1, v_2, v_3, v_4 in the other subset namely V_2.

And you can check now that you take any edge in the graph one of its end point is in V_1 and the other end point is in V_2. So, that shows the implication is true in one direction and the implication is true in the other direction as well, you take any vertex in V_1 and any vertex in V_2. There exists an edge between those two vertices in your graph. So, that is why this is a complete bipartite graph. And we use this notation $K_{m,n}$ to denote a complete bipartite graph, where the cardinality of the two subsets in your partition are m and n respectively.

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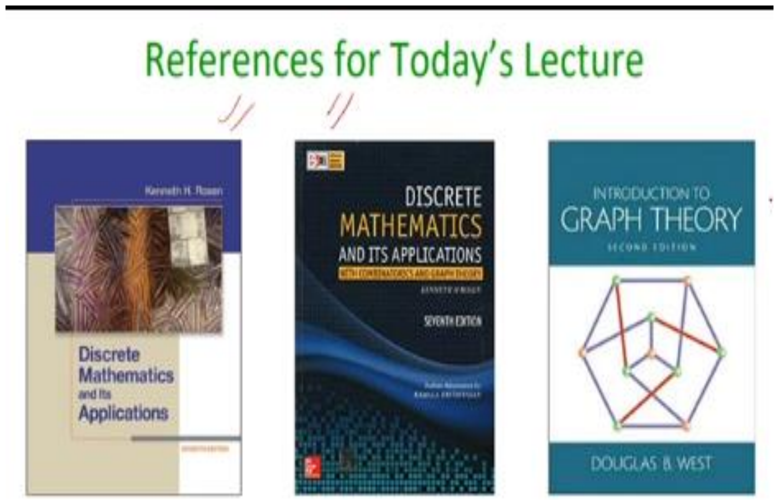

So, that brings me to the end of this lecture these are the references used for today's lecture. So, the basic concepts related to graph theory you can find in the Rosen book, but there is also this advanced or dedicated book for graph theory. So, this book on graph theory is very nice, it covers both the basic concepts as well as advanced concepts. And if you are interested to explore graph theory, I encourage you to get a copy of this book. Thank you!