Discrete Mathematics
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Lecture -04
Rules of Inference

Hello everyone, welcome to this lecture on rules of inferences.

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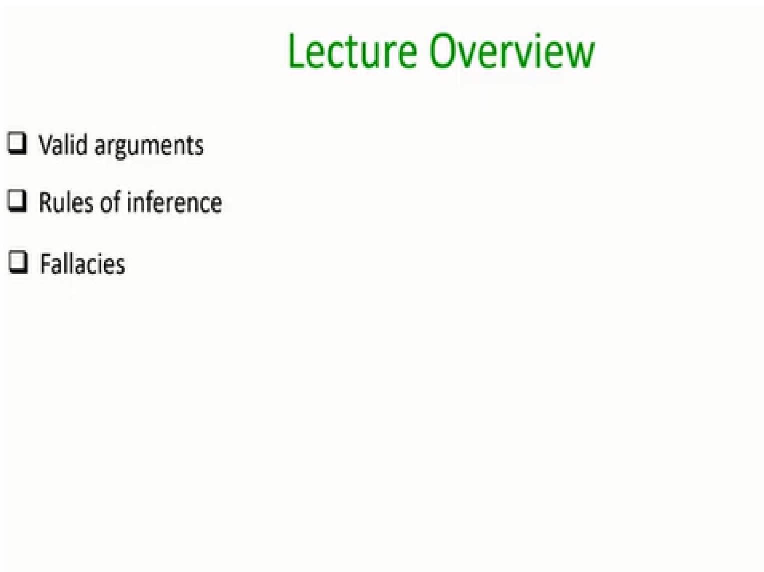

Just to recap in the last lecture, we introduced the SAT problem, we saw an application of the SAT problem namely the Sudoku puzzle solver, the plan for this lecture is as follows. We will introduce what we call as valid arguments, we will see rules of inferences and we will discuss some fallacies.

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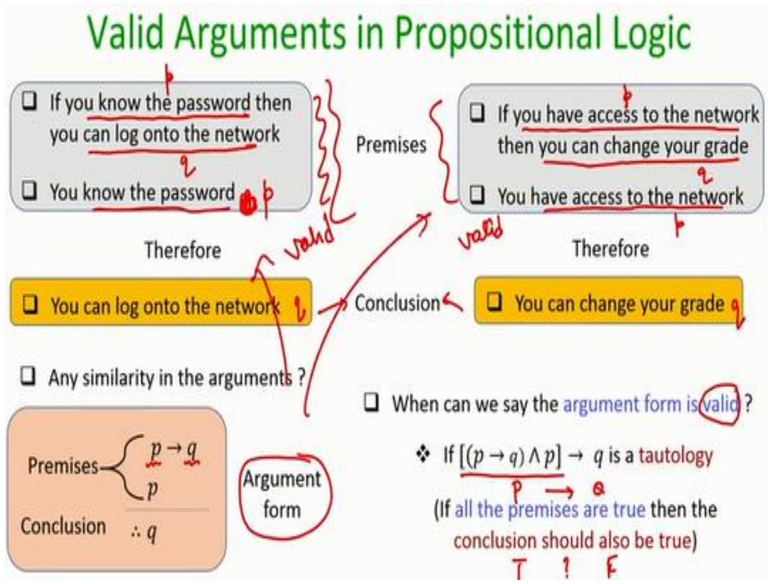

So, what we do mean by valid arguments in propositional logic? Suppose we are given a bunch of statements like this the statements are; if you know the password then you can login to the network and it is also given that you know the password therefore I am concluding that you can log on to the network. This is an argument which is given to you and we have to verify whether this argument is logically correct or not.

Then consider another bunch of statements which also looks like an argument, the statements that are given here are the following it is given that if you have access to the network then you can change your grade. When it is also given that you have access the network therefore I can conclude that you can change your grade. So, here you have a bunch of statements before the conclusion.

So those bunch of statements are called as premises. If you might be given one premises or two premises or multiple premises and based on those premises I am trying to derive a conclusion. So whatever is there before therefore is called a premise and whatever is there after therefore is called as conclusion. So before going into verifying whether these arguments are mathematical or logically correct or not, we want to argue here or we want to check here is that is there any similarity in the arguments.

Well, if I view these two arguments these two set of arguments in the English language sense then they are different because we are talking about different things. But if I try to extract out these two arguments, then there is a similarity that both these arguments have a common structure, they have a common template and what is the common template here the template is the following.

If you see the premise, the premise is of the form $p \rightarrow q$ and p and the conclusion is of the form q, why so? So you can see that, I can say that p represents the statement you know the password. I say that p represents the statement, you know the password and q represent a statement you can log on to the network. So that is why in the first argument the statement if you know the password then you can log on to the network can be represented by $p \rightarrow q$.

And it is also given that we know the passwords. So I have used the statement variable q to represent a statement, you know the passwords sorry, I have used the variable p to represent the statement you know the password therefore the second premise which is given to me is p. And what is the conclusion I am trying to do I am draw here, I am trying to draw the conclusion that you can log on to the network which represented by q here.

So I can say that this English language argument form can be represented by this template where in the premise I have I am given the statement $p \rightarrow q$ and p and therefore I am drawing the conclusion q. In the same way for the second set of arguments as in the second argument it is set of statements which are given to me they can be abstracted by this argument form why so? Again I can say that, let p represent the statement you have access to the network and let q represent the statement you can change your grade.

I have used p to represent you have access to the network and a conclusion, I am trying to draw here is you can change your grade, which is q. So again, the second argument the second set of English statements which are given in the argument can be also abstracted by this common form. So this common form where I do not worry about what is my p what is my q I have just a set of premise and I can corresponding conclusion, I call it as an argument form.

Now what I want to verify is whether this argument form is valid or not, whether it is correct or not by valid, I mean whether it is correct or not. What do I mean by argument form is valid my definition here is I will say my argument form is valid, if I can say that the conjunction of premises implies conclusion is a tautology, that means if I can prove that if all the premises are true then my conclusion is also true.

Because remember, anything of the form $p \rightarrow q$ is false only when p is true, but q is false. So that means I want to verify here that indeed q follows p or not, the capital Q follows from capital P that means all the conditions in my capital P are true whether I can come to the conclusion capital Q that is the case that this argument form will be a tautology. Because for all other three cases when p is capital P is false and when capital Q is false by default $P \rightarrow Q$ is true.

I have to only worry about a condition then my capital P is true and my capital Q is false I have to verify whether this is happening whether if it so happens that capital P is true, but capital Q is false and then I cannot say that capital P → capital Q is the tautology in which case I will say by argument form is invalid. So that is my definition of valid argument why I am interested in argument form here is that if I know how to verify whether a given abstract argument form is valid then it does not matter how do I instantiate my variable p, q and so on.

My p could be the statement if you know the password my q could be you can log on to the network, my p could be if you have access to the network my q could be you can change your grade and so on. If I know that this abstract form is valid then irrespective of the exact statements which I substitute for p, q my overall corresponding English language argument will be valid. So instead of individually verifying whether this argument the first set of English statement is valid or not, the second set of it is English statement is valid argument or not.

Instead of individually verifying them what I focusing here is I am focusing on the validity of the abstract argument form. If I can prove that abstract argument form is valid then it automatically implies that this is valid, this is valid, and so on.

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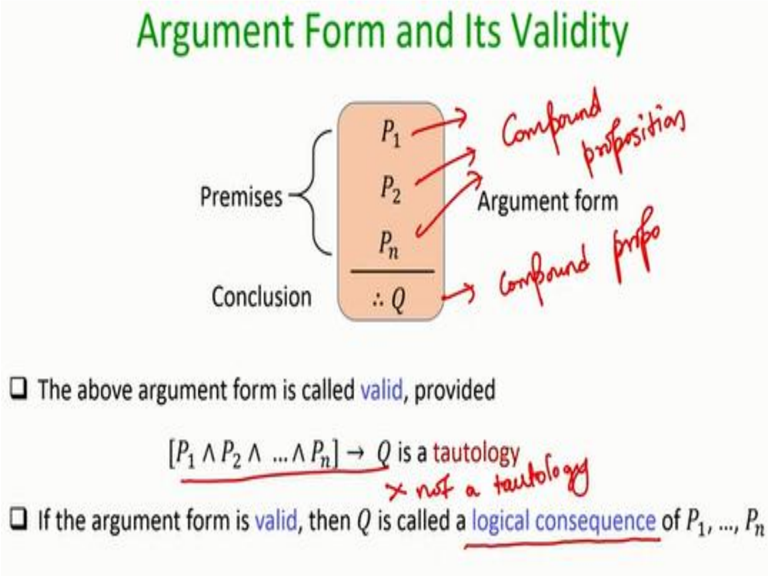

So in general, my abstract argument form is the following form. I will give a set of premises, namely n premises and each of them can be a compound proposition, all of them can be compound propositions and I will be given a conclusion which is also a compound proposition and my definition of a valid argument is the following. I will say that the above argument form is valid if the conjunction of premises implies the conclusion is a tautology.

If that is the case or if my argument form is valid and I will use the term that Q is a logical consequence of P_1, P_2, P_n, that means if you ensure all together that P_1, P_2, P_n, are simultaneously true. Then I can come to the conclusion Q, if my argument form is a valid argument form. If my argument form is not valid that means if this implication is not a tautology, then I cannot say that Q is a logical consequence of P_1 to P_n.

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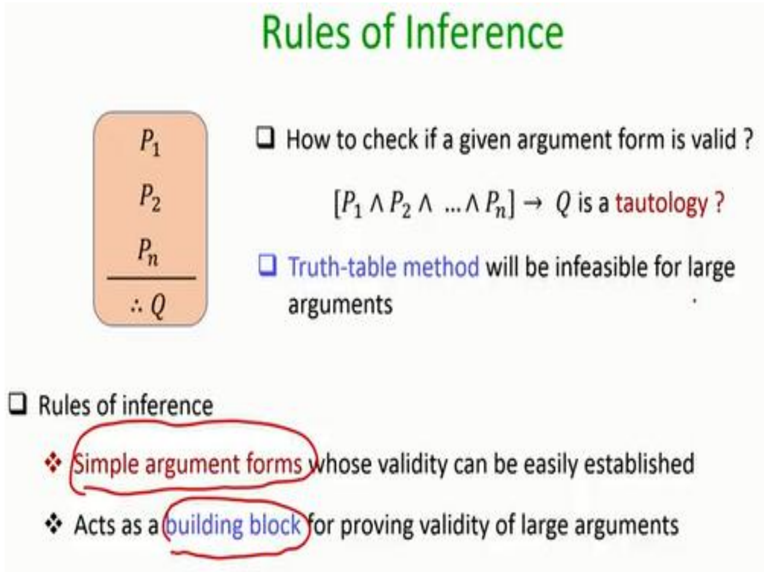

Now how do I check whether a given argument form is valid well? My definition says to check whether a given argument form is valid or not you check whether this implication is a tautology or not. The conjunction of premises, implication, the conclusion is a tautology or not and now you have the definition of tautology means that this expression. So I call this expression X implies, I represent this conjunction of P_1 to P_n by capital P, I have to check whether $P \rightarrow Q$ is a tautology or not. $P \rightarrow Q$ will be a tautology if this statement is always true.

Remember that $P \rightarrow Q$ is always true for the case when P is false, it does not matter whether Q is true or false in both the cases it will be true and if Q is true, it does not matter what is P. $P \rightarrow Q$ is always true. The only case I have to verify is when P is true, I have to check if it is so happens that P is true but Q is false. If that is the case, then this is not a tautology, okay? So that is the only case I am interested in.

So it turns out that I can verify whether the above implication is a tautology or not by using the truth table method that method I can always apply but truth table method will be infeasible for large arguments if I am given a large number of premises each involving many numbers of variables. So in general, I use rules of inference we can use rules of inferences which are very simple argument forms whose validity can be easily established using the truth table method.

And we give some names to this simple argument forms and then the idea is that we use this simple argument forms as a building block for proving the validity of large arguments. So when I will be proving the validity of the large arguments, I will not be using the truth table method, but I will be doing some kind of simplification where I will identify some parts of the argument form which I know is already true based on the validity of smaller argument forms, I will have some well known rules applied.

Now some well known names and if I can identify some portion in my complex argument form, I can easily say well that part is true so I do not we have to worry about that. That is a whole idea of rules of inferences. It is used for proving the validity of complex large argument form where I use simple argument forms whose validity has been already established using some well known methods say truth table method.

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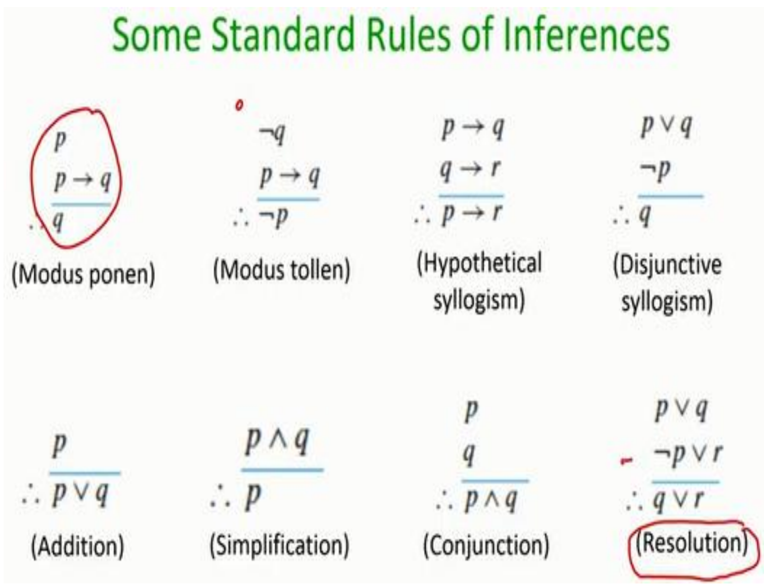

So, I will be listing down some standard rules of inferences each of them can be easily proved using the truth table method, the most popular here is what we call as Modus ponen. So what Modus ponen says is that if you are given the premises p and p → q, you can come to the conclusion q. To verify whether this is a valid argument form, you have to verify whether p conjunction p → q overall implies q is tautology or not.

And you can easily verify that this is a tautology using the truth table method, I am not going to do that it is an exercise for you the truth table will have 4 rows right p to be true, q, to be true p to be false, q to be false and so on and you can verify that for all possible cases or of all possible four combinations this overall expression is always true and hence I can always conclude the conclusion q from the premises p and $p \rightarrow q$.

Now why it is called Modus ponen and well there are some reasons for that I am not going to details. There is another well known rule of inferences, which is called as Modus tollen. It says the following that if you are given the premises $\neg q$ and $p \rightarrow q$ then you can come to the conclusion $\neg p$. So even though it is called by a different name it is given a different name and it might look a different argument form I can view it as a special form of Modus ponen.

So let us see how. You are given the premise $\neg q$ and remember the $p \rightarrow q$ is logically equivalent to $\neg q \rightarrow \neg p$. Because $\neg q \rightarrow \neg p$ is the contrapositive of $p \rightarrow q$ and contrapositive is always logically equivalent to the original implication and your conclusion here is $\neg p$. So if you see closely here the new thing that I have written here is of the same form as Modus ponen right, same form. Why?

Well you can say that negation of r is nothing but q some other variable and since negation of r is $\neg$ of q is r, so I can say $r \rightarrow s$ and I can say $\neg$ of p is denoted by s. And hence I am denoting the conclusion s and this is nothing but Modus ponen, which I know is a valid argument form. So even though Modus tollen is looking structurally different because you have the negation appearing here if I rewrite everything and do the substitution here I can bring this Modus tollen into the form of Modus ponen.

And, hence I do not have to separately prove by truth table that this Modus tollen is a valid argument form. So valid argument form I can use it in any simplification. In the same way I have this transitive law, which is also called as hypothetical syllogism which says the following. If you are given the premises $p \rightarrow q$ and $q \rightarrow r$, you can draw the conclusion $p \rightarrow r$ and again, I have to verify here whether the conjunction of $p \rightarrow q$ and $q \rightarrow r$ overall giving the conclusion $p \rightarrow r$ is a tautology or not.

And you can verify that it is indeed a tautology. I am not doing that, there will be 8 rows because you have 3 variables and you can quickly verify that in each of the rows the final result is true. We have disjunctive syllogism that says that you have if you are given disjunction of p and q and you will given the premise ¬ p then you can draw the conclusion q and so on. You have the addition law,, simplification law, conjunction.

And there is another law called as a resolution which again is a special form of Modus ponen, we will come back to this resolution later. Why this is an equivalent form of Modus ponen because of the following of sorry it is well, I can apply Modus ponen and I can apply the transitive law and show that by applying these two laws, I can get the resolution. I am not going into the details, I leave it as an exercise for you.

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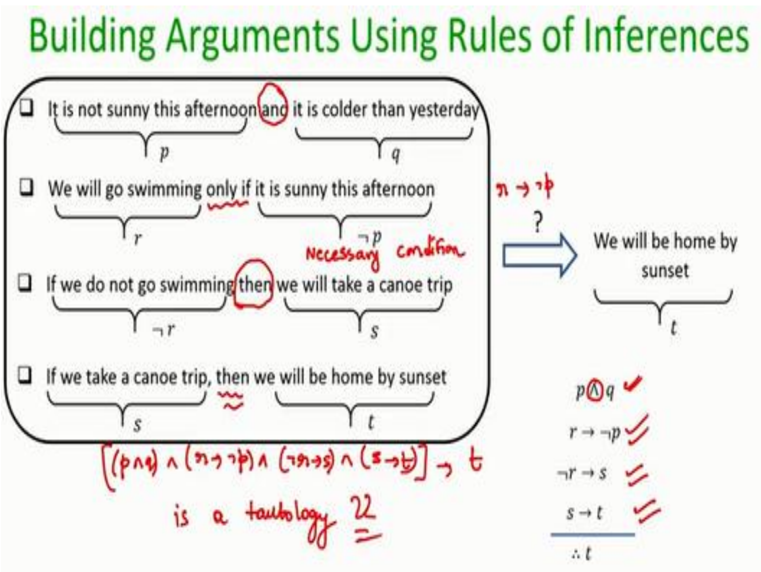

So now imagine that I am already given some well known rules of inferences. So whatever rules I have stated here, they are called as rules of inferences. I do not have to separately prove them they are now valid argument form and that is why I can use them at my discretion. So now I have to verify complex looking arguments using these rules of inferences. So imagine I am given this English argument, I am given a bunch of four statements, four premises and based on these four premises, I am trying to draw conclusion here.

So the first thing that I have to do here is I have to write down the abstract argument form here and for doing that I have to convert each of the English statements into some compound proposition. So what I do is I assign various propositional variables to represent different statements here. So suppose I use the variable p to represent a statement that it is not sunny this afternoon. So it is a declarative statement, it is a proposition.

I am assigning the propositional variable p for that. In the same way say I assign the propositional variable q to represent a statement it is colder than yesterday. My second premise is we will go to swimming. So this statement has not occurred till now. So I am assigning a new variable r here for that and it is given here that we will go to swimming only if it is sunny this afternoon.

Well, I have already assigned a variable p to denote the statement it is not sunny this afternoon, so it is sunny this afternoon will be represented by ¬ p. Now my third premise is if we do not go swimming then we will take a canoe trip. So I have already used the variable r to represent we will go swimming. So ¬ r will represent we will not go swimming and this statement we will take canoe trip is coming for the first time.

So, I will present I use introduce a variable s for that and it is easy to see that the fourth premise, I do not have to introduce anything for this statement if we take a canoe trip but I have to introduce a variable t for representing the statement we will go home by sunset. What is the conclusion I am trying to draw? We will be home by sunset I am using the variable t for that. So the first step here is I have assigned truth variables for various statements involved in this argument form.

Now, I have to write down the abstract argument form. What is the first statement? It is not sunny this afternoon and it is colder than yesterday. So we have an occurrence of ‘and’ here right? So that is why it is p conjunction q. What is the second statement? We will go swimming only if it is sunny this afternoon. So remember whenever we have an occurrence of only if, then whatever is occurring after only if is the necessity condition, this is the necessary condition not a sufficient condition.

So you go back to the lecture where we have introduced the implications operator, right? So this statement will be represented by $r \rightarrow \neg p$ that means $\neg p$ is a necessary condition for r that means if it is not sunny this afternoon definitely I will not go to the swimming that is what is the interpretation of this statement. So that is why it will be represented by $r \rightarrow \neg p$.

What is the third premise here? The third premise is a statement of the form if something then something. That is why it is negation $r \rightarrow s$ and my fourth statement is again a statement of the form if something then something, I have used a variable s and t that is why it is of the form $s \rightarrow t$ and what is the conclusion? I am trying to draw the conclusion t. So now I have to verify whether this argument form is valid or not as per the definition, I have to verify whether the conjunction of p and q and $r \rightarrow \neg p$ and $\neg r \rightarrow s$.

And $s \rightarrow t$ overall implying t is the tautology or not, that is what I have to verify. If it is the tautology, then I will say that this is a valid argument otherwise, I will say it is an invalid argument. Well, I can use the truth table method here to verify whether this is a tautology or not, but how many variables I have? I have p, q, r, s, t, 5 variables. So 32 rows, well it still manageable but might be time consuming.

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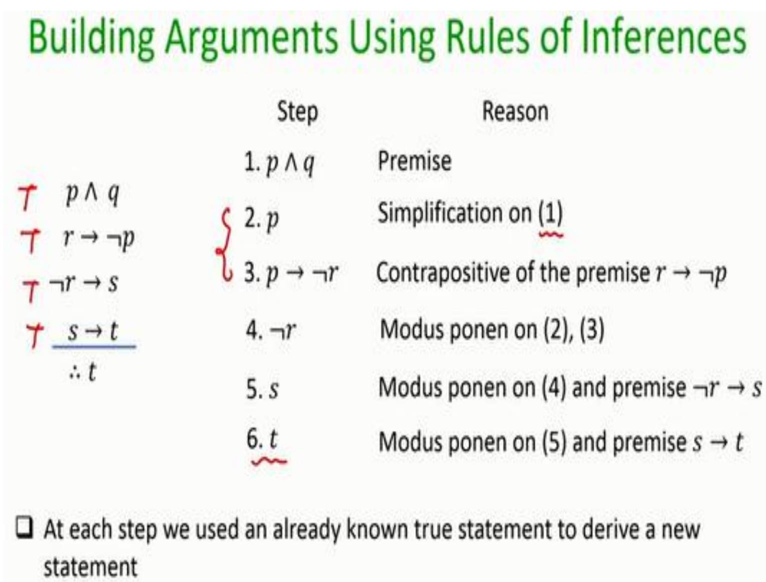

So what I will do instead is I wont touch the truth table here, but still I will be able to show you that how we can verify whether this argument form is valid or not using well known standard rules of inferences. So here is the proof, so I start with some premise here. So remember I have to show that if premises are true, I have to check whether if premises are true then conclusion is true.

I have to verify that, then only this is a tautology that is a definition of valid argument. So I am assuming that all my premises are true that is, this is true, this is true, this is true, this is true, and then I have to check whether my conclusion is also true or not. So since p and q is true, I am starting with that, I apply the simplification law and why I am writing it as one, one because the first statement is I am starting with the premise p and I am giving a number to that the number that I have given to that statement that premises the conjunction of p and q.

So I can say that since, my premise p and q is true. I can apply the simplification law on that and come to the conclusion p. Now, I can say that since my premise $r \rightarrow \neg p$ is true, I can apply the contrapositive law on that and come to the conclusion $p \rightarrow \neg r$; that means if $r \rightarrow \neg p$ is true, of course $p \rightarrow \neg r$ will be true; that is what is the interpretation of this statement.

And I am giving this number 3 to this new statement, which I am deriving from already known statements. Now what I can say is, I can apply the Modus ponen on statement number two and statement number three. So I have derived p to be a true statement, I have derived $p \rightarrow \neg r$ to be a true statement and now I can say that if both statement two and statement three are true then by applying the Modus ponen and on that I come to the conclusion negation r, that is a new true statement, which I can apply to my bunch of true statements.

And now I can say that if negation of r is the true statement and anyhow I am given the premise $\neg r \rightarrow s$ which I am assuming to be true based on these two statements by applying the Modus ponen, I can come to the conclusion s and again, I can say that if s is a true statement and if my premise $s \rightarrow t$ is the true statement based on these two things I can come to the conclusion t. So what I have done here.

At each step, I have used already known true statement which might be either given already as part of premises or which I might have derived in some previous step to derive new statements which will be true and by doing this process I ended up coming to the conclusion t and hence I can say that this argument form is a valid argument form that means if I assume all the premises to be true, based on those things by applying rules of inferences properly at each step, I keep on deriving new conclusions.

And, I end up coming to the final conclusion which is there in my argument form and hence my argument form here is a valid argument form and now you can check here that I have not at all touched truth table here, I never said okay since p and q is true then both of them will be true and then go into the following propositions and so on. I never did an argument of that form, I just used whatever premises are given to me and I kept on deriving new conclusions and ended up showing you the final conclusion.

So that is how we use rules of inferences to prove whether complex argument forms are valid or not.

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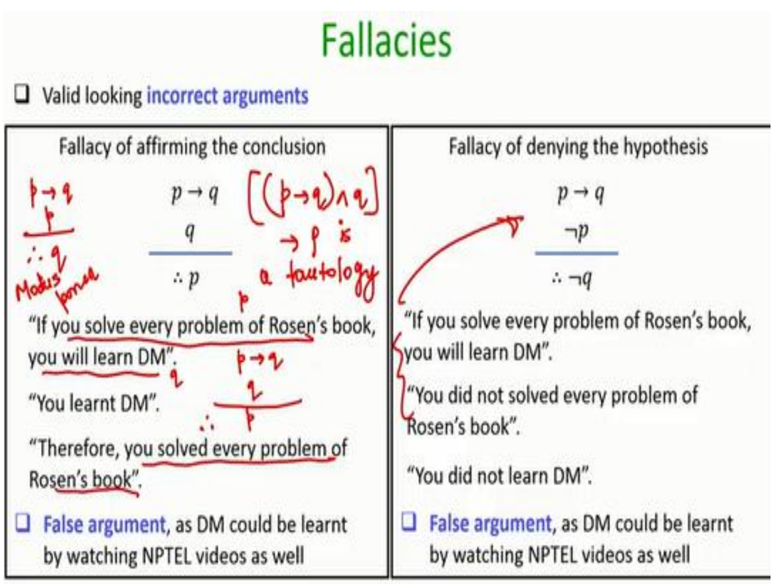

Now there are some well known fallacies which are incorrect arguments but on a very high level it might look a valid argument but they are very subtle incorrect arguments. So there I will be showing you two common incorrect arguments or two common fallacies. The first fallacy is that of affirming the conclusion. So consider this argument form: your premises are $p \rightarrow q$ and q and you are drawing the conclusion p.

So to verify this you have to verify whether $p \rightarrow q$ and q, implies p is a tautology or not. Well, it is not a tautology, the problem here is the following consider this following argument, I make the premise, I give you the premise that if you solve every problem of Rosen's book, you will learn discrete maths and it is already give also given that you have learnt discrete maths, okay?

Now based on these two premises, I am trying to draw the conclusion that you have indeed solved all the problems of Rosen's books, is this a valid argument? Is this a valid reasoning? Well, this is an invalid reasoning because it might be possible that even without solving any of the problems of Rosen's books you have learn discrete maths by some other mechanism say for instance by watching the NPTEL videos, this discrete maths course and without even touching any problem of Rosen's books.

So this argument, by this English argument forms in this argument form. So let p represent a statement at you solve every problem of Rosen's books and q represent a statement at you will learn discrete maths. So that is why this is $p \rightarrow q$. Another premise that is given is you learn discrete maths that means it is given q to the true therefore the conclusion that I am trying to draw here is that you solved every problem of Rosen's books, which is p.

And as I am giving here, this is an invalid argument here, because it might be possible that you have learned discrete maths even without touching any problem of the Rosen's books. So this is called as fallacy of affirming the conclusion this is different from your Modus ponen and remember your Modus ponen was $p \rightarrow q$ and p, therefore q, this is different, right?

Modus ponen is a valid argument form, but this fallacy of affirming the conclusion is incorrect. It may or may not be true. So let us see another fallacy here, this is the fallacy of denying the hypothesis the argument form here is $p \rightarrow q$, negation p and these are the two premises and the conclusion you are trying to draw is negation q. So an instantiation of this abstract argument form is the following say again, my premises are if you solve every problem of Rosen's books, you will learn discrete maths.

You do not solve every problem of Rosen's books. So these two premises come under this abstract form $p \rightarrow q$ and $\neg p$. And therefore I am drawing the conclusion you would not learn discrete maths. Again, as you can see, this is a false argument because you might have learned discrete maths by just watching NPTEL videos if they are very good without even solving any of the problem of Rosen's books.

So these two are very common fallacy might look very similar to Modus ponen and but they are not valid argument form. So that brings me to the end of this lecture, just to summarize in this lecture we have introduced argument forms we have defined valid argument forms when do we say an argument form is valid. We have seen various rules of inferences and how we use rules of inferences to verify complex argument forms are valid or not. Thank you.