Discrete Mathematics
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Module No # 08
Lecture No # 38
Tutorial 6: Part II

Hello everyone, welcome to the second part of tutorial 6.

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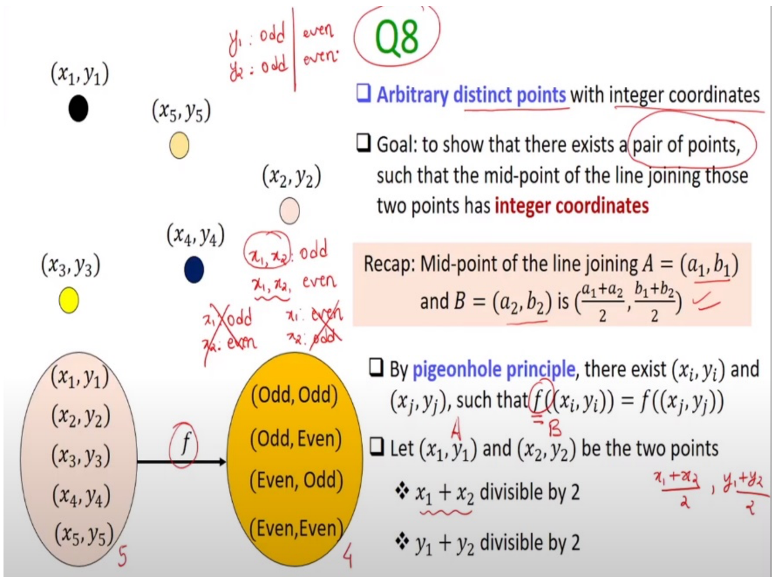

Let us start with question number 8. You are given here arbitrary distinct points in 2 dimensional planes. Each point will have an x-coordinate, y-coordinate and the points are having integer coordinates. So they are arbitrary points except that they are distinct. So, I am denoting the points, their respective coordinates as $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4$ and x_5, y_5. And our goal is to show that irrespective of the way these 5 points are chosen arbitrarily they are always exist a pair of points such that if you consider the midpoint of the line joining those 2 points it has integer coordinates.

So just to recap if you have 2 points, a point with coordinates (a_1, b_1) and another point with coordinates (a_2, b_2) then the midpoint of the line joining these 2 points is given by the formula $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$. And we want to apply here pigeonhole principle. So remember for pigeonhole principle we have to identify the set of pigeons and the set of holes here and then the mapping which relates the pigeon and the holes.

So let us do that. So consider the set of 5 arbitrary points which are all distinct and have integer coordinates. We are trying to map this point depending upon what is the nature of their x-coordinate and y-coordinate. So depending upon whether the x-coordinate is even, or x-coordinate are odd, or whether the y-coordinate is odd, or the y coordinate is even, I have 4 possible combinations.

And my function f maps these 5 points to the corresponding pair; say if x_1 is odd and y_1 is even then I will say that $f(x_1, y_1)$ is (odd, even) and so on. That is the mapping here. So now, it follows from pigeonhole principle that we have now 5 items here in the set A and 4 items in the set B then there always exist a pair of points among these 5 points say (x_i, y_i) and (x_j, y_j) such that both of them are mapped to the same ordered pair.

So it could be any 2 out of those 5 points; it could be the first 2 points, it could be the last 2 points, it could be the third point or the fourth point and so on; we do not know. It depends upon the exact 5 points that we chose. So, without loss of generality assume that out of those 2 points which are guaranteed to be mapped to the same ordered pair are the first 2 points.

So say (x_1, y_1) and (x_2, y_2) be the 2 points such that the corresponding f output of the f function for these points are the same. Now we want to inspect what happens to the midpoint of the line joining these 2 points A and B. So as per the formula the midpoint of the lines joining these 2 points A and B will be $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

And since both the points A and B are mapped to the same ordered pair; so for instance it could be the case that both x_1 as well as x_2 are odd or it could be the case that both x_1 as well as x_2 are even. So irrespective of the case $x_1 + x_2$ will always be divisible by 2. If both of them are even definitely sum of 2 even quantities is divisible by 2. Whereas if both of them are odd then also the sum of 2 odd quantities is divisible by 2. And as per our assumption it is not the case that x_1 is odd and x_2 is even that is not the case because we are considering the case when the output of the f function on these 2 points A and B are the same.

In the same way we cannot have the case where x_1 is even and x_2 is odd because that is not the property of the point A and B. Due to the exactly the same reason, the type of y_1 and y_2 coordinates are the same. Either they are both odd or both of them are even right. And again in this case it is easy to see that $y_1 + y_2$ will be divisible by 2. And that shows that this statement is a correct statement.

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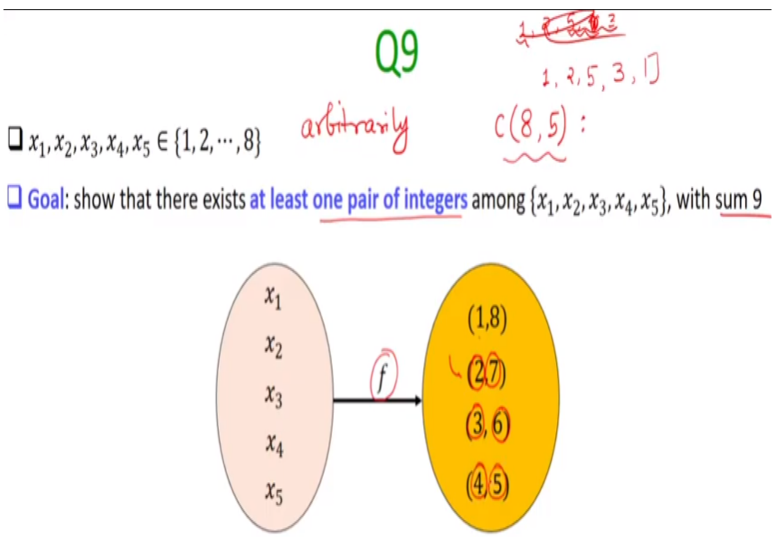

So let us go to question number 9. Here you are given the following. You are choosing 5 integers from the set 1 to 8 arbitrarily. Our goal is to show irrespective of the way you choose those 5 points there always exists at least one pair of integers among those chosen 5 integers whose sum is 9. So say you pick 1, 2 and 5 and then if you pick 3 then you still do not have any pair of integers whose sum is 9. But as soon as you pick the fourth point, so if you pick 4 that is the fifth number then you have 5 and 4 which is summing up to 9.

If you pick 6 as the fifth number, then you have 3 and 6 summing up to 9. If you have if you pick 7 as the fifth number, then you have 7 and 2 summing up to 9. If you pick 8 as the fifth number, then you have 1 and 8 summing up to 9 and so on. So you can verify this by an example but we want to prove it irrespective of the 5 numbers that we are going to pick.

So one way of proving this is that you take all possible 8 choose 5 ways of picking 5 numbers and for each of those combinations you show that the statement is true but that will be an overkill because this is a relatively large value. Instead we will apply the pigeonhole principle and again for applying the pigeonhole principle we have to identify the pigeons and the holes and the mapping. So my pigeons here are the 5 integers among the numbers 1 to 8 that I am picking arbitrarily and my holes are the ordered pairs of distinct integers in the set 1 to 8 whose sum will give you 9.

So you have either the ordered pair (1, 8) or the ordered pair (2, 7) or ordered pair (3, 6) or the ordered pair (4, 5). And you do not have any other ordered pair from the set 1 to 8 summing up to 9. And my function f basically maps these x_i values to the corresponding ordered pair depending upon whether x_1 is 1 or 8 I will say that $f(x_1)$ is either $(1,8)$. Or if x_1 takes either the value of 2 or the value of 7 then I will say $f(x_1)$ is (2, 7).

Or if my x_1 is either 3 or 6 then I will say that $f(x_1)$ is (3, 6) or if my x_1 is either 4 or x_1 is either 5 then I will say that $f(x_1)$ is (4, 5). That is the interpretation for my mapping f.

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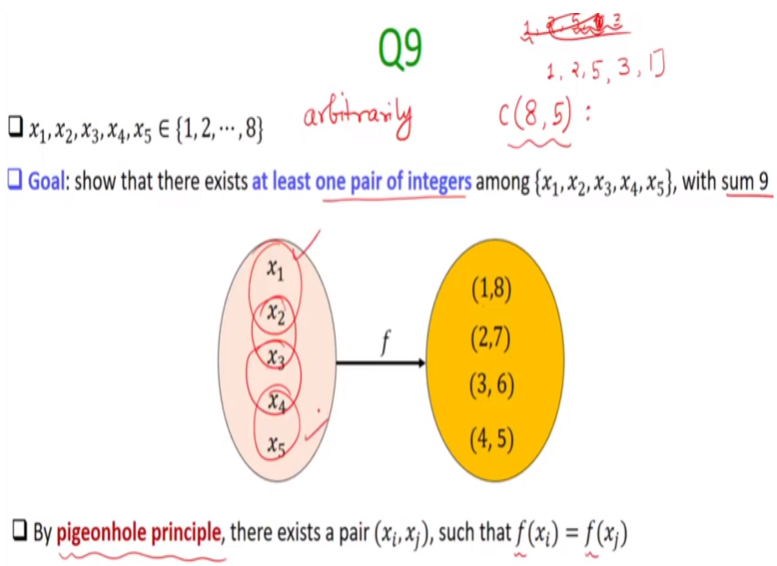

Now it follows simply from pigeon-hole principle that there always exists a pair or two values out of the 5 numbers say (x_i, x_j) such that $f(x_i)$ and $f(x_j)$ are the same. It could be say the first 2 values, the last 2 values, the second or the third value, the third or the fourth value, or the first value or the fifth value; it could be any 2 values out of those 5 numbers.

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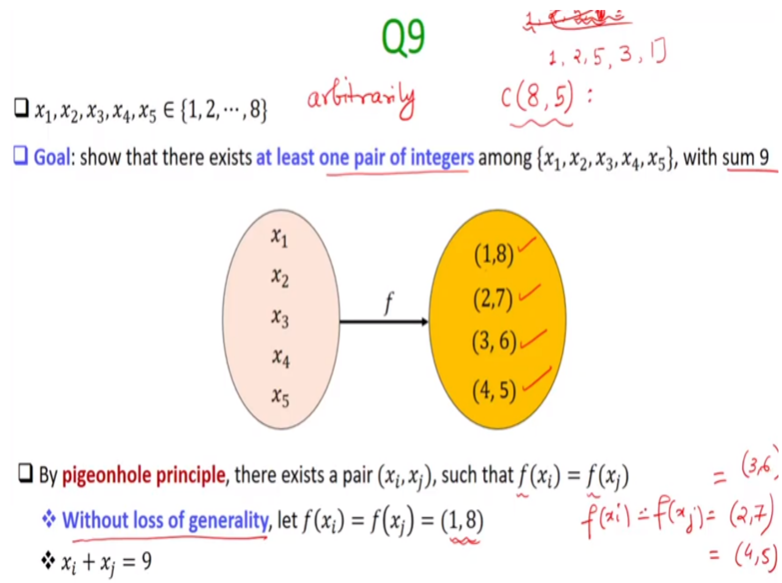

We do not know which one. So again without loss of generality, suppose both of them got mapped to (1, 8); we do not know what is the identity of x_i or x_j and we do not know the corresponding mapping as well. It could be either (1, 8), (2, 7), (3, 6) or (4, 5). So, without loss of generality; that means whatever reasoning we are giving here for the case where $f(x_i) = f(x_j) = (1,8)$ hold, the same argument will hold even if $f(x_i)$ is same as $f(x_j)$ is equal to say (2, 7); the same reasoning will hold symmetrically for that case as well.

Symmetrically for the case when it is (4, 5), symmetrically for the case when it is (3, 6) and so on. So that is why we do not consider the remaining 3 cases. We just consider the case when $f(x_i)$ and $f(x_j)$ is (1, 8).

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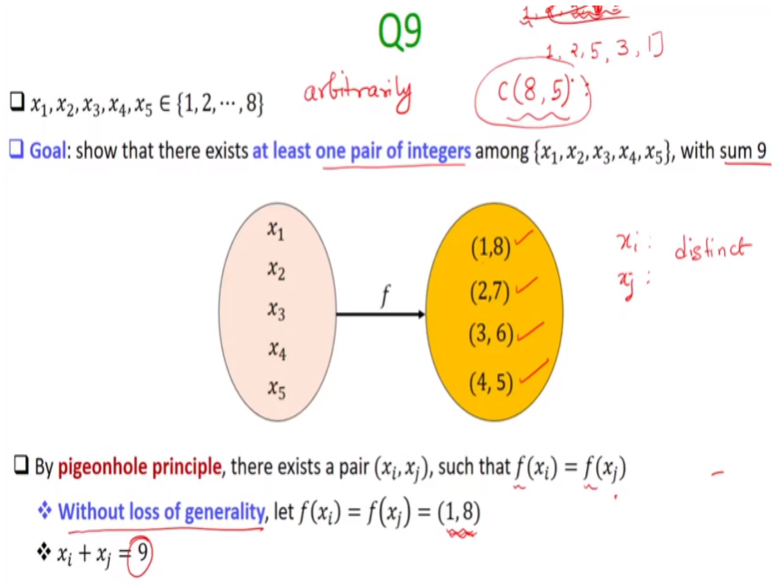

If that is the case then since your x_i and x_j are distinct and they got mapped to (1, 8) that means either x_i is 1 and x_j is 8 or x_i is 8 and x_j is 1. Irrespective of the case, the sum of x_i and x_j is 9. So now you can see that even without enumerating all possible $C(8,5)$ arrangements or combinations of picking 5 numbers out of these 8 numbers we ended up arguing in a very simple fashion that our statement is true using pigeonhole principle. It shows the power of this proof strategy or counting mechanism basically.

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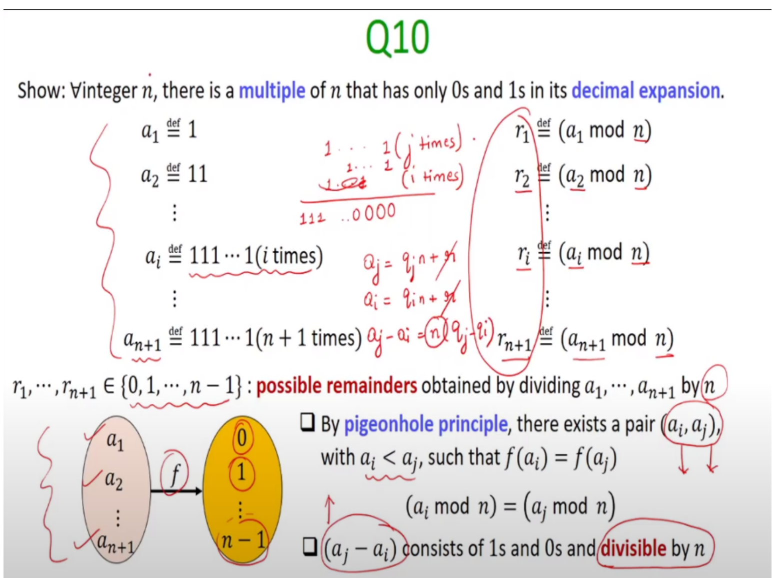

So question 10 we want to prove a universally quantified statement. Namely, we want to prove that you take any integer n, there is always a multiple of n which has only the digits 0's and 1's in its decimal expansion. So before going into the proof if you want to take few examples say $n = 1$ then I always have the number 1 which is a multiple of 1 and which has only the digit 1 in its decimal expansion.

Remember it is not mandatory that you have both 0's as well as 1 in the decimal expansion. The only restriction is we have to show that in the decimal expansion you only have either the digits 0's or 1's. If you take $n = 2$ then I can take the number 10 which is a multiple of 2 and which has only 1's and 0's and in its decimal expansion. If I take $n = 3$ then I can take the number 111 which has only the digit 1 in its decimal expansion and which is divisible by 3.

So at least by taking few examples we found that the statement is true. But this is a universally quantified statement and we cannot prove a universally quantified statement just showing examples for a few cases. So we have to give the proof for arbitrary n. Again, we are going to apply here pigeonhole principle. So let me define a few decimal numbers here.

I define the first decimal number to be 1. I define second decimal number as 11, the i-th decimal number as a decimal number consisting of i number of 1's and the $n + 1$ decimal number which has the digit 1, $n + 1$ number of times. Let me define another set of values. So my value r_1 is the remainder which I obtain by dividing a_1 by n. Similarly, I define r_2 to be the remainder obtained by dividing a_2 by n. I define r_i to be the remainder obtained by dividing a_i by n.

And in the same way I define r_{n+1} as the remainder obtained by dividing a_{n+1} by n. Now what can I say about this remainders? It is easy to see that these remainders belong to the set 0 to $n - 1$ because of the simple fact that you divide any number by n the only possible remainders could be 0 if it is completely divisible by n or the remainders could be $1, 2 \dots n - 1$. Now you have to apply the pigeonhole principle.

So my pigeons are the numbers a_1 to a_{n+1} that I have constructed here. And my holes are basically the remainders which I can obtain by dividing these $n + 1$ numbers by n. And I have n possible remainders and my function f map the numbers to the corresponding remainder which I have obtained by dividing that number by n. So you have more number of numbers and less number of remainders.

So it follows from the pigeonhole principle that you always have a pair of numbers a_i and a_j out of this $n + 1$ numbers which gives you the same remainder if you divide a_i and a_j by n. I do not know the remainder it could be either 0, or the remainder could be either 1, or the remainder could be $n - 1$.

I do not know what are the individual remainders that a_i and a_j are going to give on dividing by n. But what I know is that they are leaving the same remainder. And again without loss of generality assume that a_i is occurring before a_j in my sequence here. Now what can I say about this number $a_j - a_i$. So a_j will be a number which has j number of 1's and a_i is another number which has i number of 1's. Both of them gives me the same reminder on dividing by n.

So if I take $a_j - a_i$then this will be a decimal number which will have trailing 0's and then at the leading positions you will have the 1's. That means it is a decimal number which has only the characters 1s and 0's. But what can you say about its divisibility by n. This number will be divisible completely by n because a_j gives you the same remainder, say r, so I can say a_j is some $q_j * n + r$ and a_i also gives me the same remainder r, so I can write a_i as some $q_i * n + r$.

Then if I take $a_j - a_i$ the effect of r cancels out and I get that its completely now a multiple of n. So, I showed you constructively here that irrespective of what is your n, I can always give you a number which is divisible by n and which has only 1's and 0's in its decimal expansion right.

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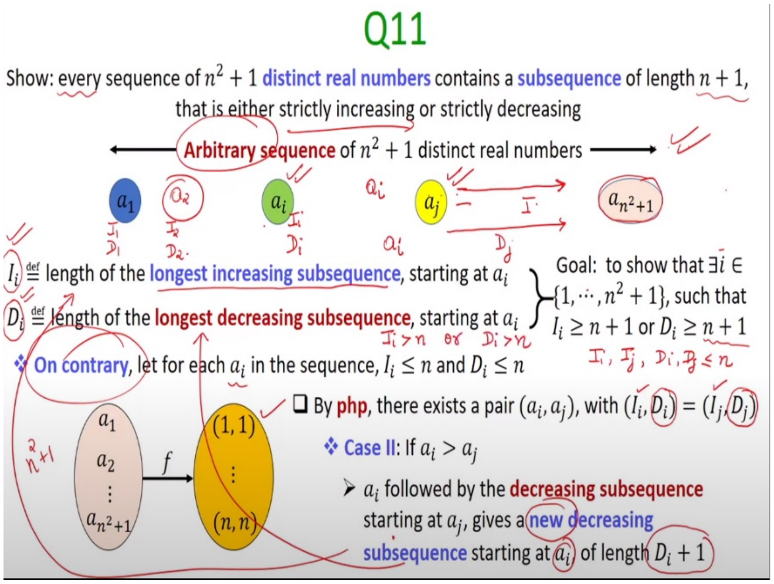

So now let us go to question number 11 which is really a very interesting question. Here we want to show the following that you take any sequence of $n^2 + 1$ distinct real numbers. They are arbitrary real numbers; may be positive, negative in any order you take them. The only condition is that they have to be distinct. Then the claim is that irrespective of the $n^2 + 1$ real numbers that you have in your sequence you always have a subsequence of length $n + 1$ which is either strictly increasing or strictly decreasing.

First of all what what is a strictly increasing sequence? A sequence of the form $(a_1, a_2, \ldots)$ where $a_1 < a_2 < a_3 < \cdots < a_{i-1} < a_i \ \ldots$. Whereas if I have a sequence of the form $(a_1, a_2, a_3, \ldots)$ where $a_1 > a_2 > a_3 \ldots > a_{i-1} > a_i > \cdots$ then it is a strictly decreasing sequence.

Now what does a subsequence means? A subsequence mean here that the values may not be consecutive. That means I am allowed to miss few numbers. In the sense, say I take a sequence 1, 3, 0, -5, 2, 8 and so on. Then I can choose to pick 1 and then exclude 3 and 0 and -5. This is a subsequence. In the same way I can pick a subsequence saying 3, 2 and 8 that means I skip 0, I skip -5.

So what this question basically says is that irrespective of the way your $n^2 + 1$ distinct real numbers are chosen you always have a subsequence. By that I mean that you have a set of $n + 1$ values going from left to right but need not be in consecutive locations; some of the locations might be skipped. But the number of values are $n + 1$ such that if you view those $n + 1$ values they are either strictly increasing or strictly decreasing. That is what we have to prove.

Again, if you want to convince yourself whether this is indeed a true statement or not you can take some concrete values of n, try to draw any possible sequence of $n^2 + 1$ for that value of n and you can verify that this statement is true. But now we want to prove it for any arbitrary sequence. How do we do that? So let the arbitrary sequence of $n^2 + 1$ distinct real numbers be denoted by a_1 to a_{n^2+1}.

Why I am taking arbitrary here? Because I want to prove this statement for every sequence. So this is a universally quantified statement and to prove a universally quantified statement I can use the universal generalization principle by proving that a statement is true for some arbitrary element of the domain. My domain here is the set of all possible sequences of $n^2 + 1$ distinct real numbers. I am just taking one candidate element from that domain arbitrarily.

I do not know the exact values of a_1 ... a_j ... a_{n^2+1}. What I will do is to prove this statement, I will use pigeonhole principle along with proof by contradiction. So let me first define two values. I define I_i as the length of the longest increasing subsequence starting at a_i. So a_i will have some value depending upon what is the arbitrary sequence and it will have some various possible increasing subsequences starting at a_i.

One might be of length 1, a trivial increasing subsequence starting at a_i is the valuea_i itself. That is a subsequence of length 1. But I might be having a subsequence of say length 2 which is strictly increasing and starting at a_i. I might have a subsequence of length 3 starting at a_i and so on. So whatever is the length of the longest increasing subsequences starting at a_i, I am denoting by I_i.

In the same way, I define D_i as the length of the longest decreasing subsequence starting at a_i. I might have several strictly decreasing sequences starting at a_i. In fact the sequence a_i itself is a subsequence of length 1 which is strictly decreasing. But I might be having a subsequence of higher length which is strictly decreasing and starting at a_i. So the length of the longest decreasing subsequence starting at a_i I am denoting it as D_i. So that means with a_1, I have associated the values I_1 and D_1. With a_2, I would have associated the valueI_2 and D_2.

And similarly with a_i, I would have associated the value I_i and D_i, with a_j I would have associated the value I_j and D_j; and with a_{n^2+1} I would have associated the value I_{n^2+1} and D_{n^2+1}. It is easy to see that I_{n^2+1} will be 1, D_{n^2+1} is 1. Because I have only one sub sequence starting at a_{n^2+1} namely the value a_{n^2+1} itself.

It is both an increasing subsequence starting at a_{n^2+1} as well as it is a decreasing subsequence starting at a_{n^2+1} because there is nothing after the number a_{n^2+1}. Now, what is my goal? The question basically asks me to show that there always exist some i or some value a_i such that there either exists an increasing subsequence of length $n + 1$ that means I_i is greater than equal to $n + 1$ or there is a decreasing subsequence of length $n + 1$. That means D_i is $n + 1$.

I have to show the existence of one such number a_i in this subsequence. I prove that by assuming a contradiction. So assume that the statement is false and that means for each a_i in the sequence, the value $I_i$is at most n. That means you take any number in the sequence the maximum length increasing subsequence of length n and the maximum length decreasing subsequence is also of length n.

What does that mean? That means if I try to pair all I_i and D_i pairs then they can take the values in the range $(1,1)$ to (n, n) namely n^2 possible pairs. These are the possible values of (I_i, D_i) pairs. But how many numbers I have in the sequence? I have $n^2 + 1$ values in the sequence that I have chosen. That means I have more pigeons and less holes. What does that mean? So by PHP here I mean pigeonhole principle.

So pigeonhole principle guarantees me that you definitely have a pair of values here say a_i and a_j. Such that your I_i and I_j are same. That means the length of the longest increasing subsequence starting at a_i is the same as the length of the longest increasing subsequence starting at a_j. And in the same way the length of the longest decreasing subsequence starting at D_i is same as the length of the longest decreasing subsequence starting at D_j.

And as per my assumption I_i, I_j, D_i, D_j are all upper bounded by n because I assume the contradiction. Now how do I arrive at a contradiction here? So there could be two possible cases with respect to the magnitude of a_i and a_j.

The first case $a_i < a_j$. If that is the case, then what I can say is the following. I can say that you take the increasing subsequence starting at a_j; what is its length? Its length is I_j and if I put the value of a_i at the beginning of that subsequence then that gives me now a new increasing subsequence starting at a_i and of length I_{i+1}. But that goes against the assumption that the length of the longest subsequence starting at a_i wasI_i. So that is how I arrive at a contradiction.

On the other hand if I take the case when $a_i > a_j$ then I have to just give a symmetric argument. What I can say is the following. I know that there is a decreasing subsequence starting at a_j and its length is D_j. My claim is if you take that subsequence and put an a_i at the beginning then that now give me a new decreasing subsequence starting at a_i and the length of this new decreasing subsequence is $D\_\{i + 1\}$.

Which now goes against the assumption that the length of the longest decreasing subsequence starting at a_i was D_i. That means in both the cases I arrived at a contradiction and that shows that whatever I assumed here that means the value of each I_i and each D_i was upper bounded by n is incorrect. That means there is at least one $a_i$where either I_i is greater than n or D_i is greater than n. I do not know what exactly is that a_i. So I gave you a non-constructive proof here. But I argued that the existence of such a_i is guaranteed.

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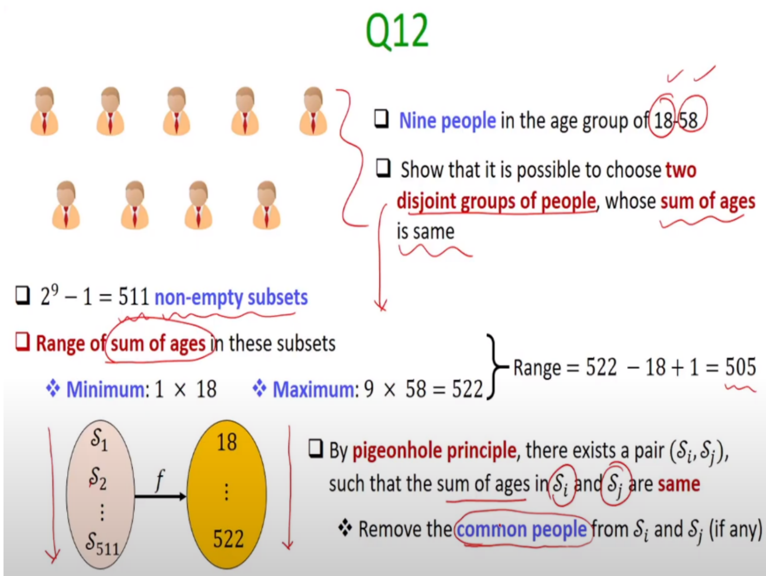

Now let us go to question 12. In this question we want to show the following you are arbitrarily picking 9 people in the age group of 18 to 58. That means the minimum age it is allowed is 18 the maximum age is allowed 58. Now we want to prove that irrespective of what exactly are their ages, as long as they are in the range 18 to 58 it is always possible to choose 2 disjoint groups of people out of this 9 people whose sum of ages is the same.

Again, we will do this by pigeonhole principle. So the first thing is since we want to argue about a non-empty set of people because when I want to consider the age of the people there have to be people in the group. So I have to focus on non-empty subset. So if I have 9 people then the number of non-empty groups that I can form out of those 9 people need not be disjoint is 511. And now what I can say about the range of the sum of ages in these 511 subsets.

If I consider the minimum sum of ages possible in a group it could be 18. This is possible only when I have a group of just consisting of one person and that person has age 18. That is a minimum possible sum. Whereas the maximum possible sum can occur when in my group I have all the 9 people person 1, person 2 and up to person 9 and each of them has age 58.

That is a maximum possible value of sum of the ages in a group we picked from 9 people. That means the range of possible sums here is 505. So now let us apply the pigeonhole principle. My pigeons are the various possible non-empty set of people that I can form out of this 9 group of 9 people. So I have 511 possible subsets and my holes are the range of sum of ages. That means what can be the sum of ages if I consider the various possible subsets given that the ages could be in the range 18 to 58.

So I have more pigeons than holes so by pigeonhole principle I can say that they always exists a pair of group S_i and S_j such that the sum of ages of the people in S_i and S_j are the same. But my question wants me to show that the group should be disjoint. So how do I argue that? I can always form disjoint groups of people out of this S_i and S_j. Well if they are already disjoint then I have showed the existence of 2 groups having the same sum of ages.

But if the sets S_i and S_j are not same; if they have some common people just remove the common people from both the set S_i and as well as S_j. The common people in the set S_i and S_j were contributing the same amount to the sum of ages in the set S_i as well as in the set S_j. So if I remove those common people the same amount will be removed from the sum of ages in S_i and S_j. And now where I will get 2 disjoint groups of people having the same sum of ages.

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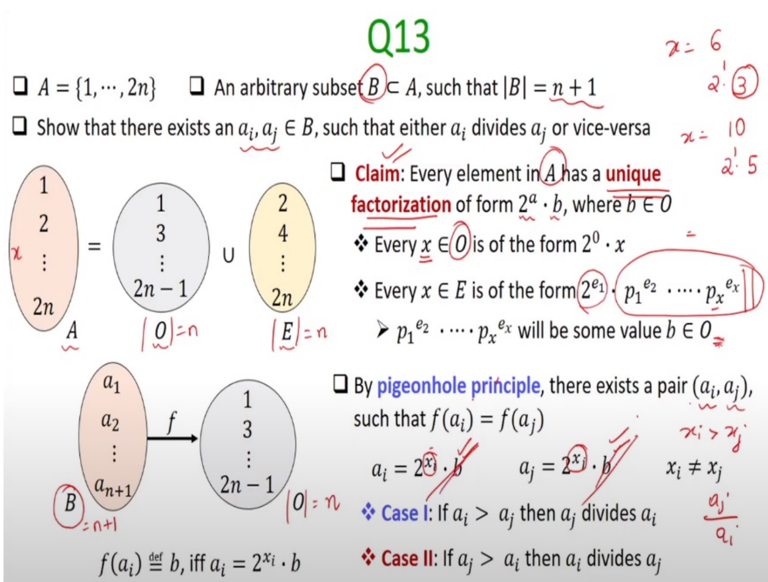

In question 13 you are given the set of A consisting of the numbers 1 to $2n$ and we want to show that if I pick an arbitrary subset B consisting of $n + 1$ elements from the set A and irrespective of the subset there always exist a pair of values such that one divides the other. And this is again a very interesting question. So for applying the pigeonhole principle what I do is I divide this set A

into 2 disjoint subsets namely the subset consisting of the odd values and the subset consisting of the even values.

Both of them will have the cardinality n. Now my claim is the following: you take any value in the set A, it has a unique factorization of the form that you have some power of 2 multiplied by the remaining value where the remaining value will be a number, specifically an odd number in the subset 1 to $2n - 1$. For example, if your number x that you are taking in the set A is already an odd number then I can write it in the form $2^0 * x$.

So in this case my a will be 0 and my x will be x itself. So my statement is true. Whereas if your x would have been say 6 then 6 can be written as 2 into 3. So you have 2^1 times an odd value. If your x is say 10 then you can write it as $2 \quad 1 * 5$. If your x is say 20, then you can write it as $2^2 * 5$. So you can see that irrespective of the case, whether your x is odd or even, this claim is always true.

So for x being odd this statement is always true. But the statement is true even for a general x which is even because for such x where x is either 2, 4, or $2n$; I can express it in the form 2 power sum positive exponent e_1 followed by the remaining values. And this is because of the fundamental theorem of arithmetic that every integer has a unique prime factorization. The claim is that if I consider the remaining prime factorization here then that will be an odd value.

And that odd value will be in the set O here and it is easy to verify that. So that means this claim is true. Now based on this claim I have to apply the pigeonhole principle. For applying the pigeonhole principle I do the following. Let B be the arbitrary set of $n + 1$ values that I have chosen. And I mapped those arbitrary chosen values to the leftover value in its unique factorization that this claimed guarantees.

So a_1 will be written in the form of some $2^{x_1} * b$. So a_1 will be mapped to this b_1; a_2 will be written in the form of some 2^{x_2} into leftover thing. That left over thing is an odd number in the set O. So a_2 will be mapped to b_2 and so on. That is a mapping f here. Now what is the cardinality of set O? That is n; that means my number of holes is n. But the number of pigeons is $n + 1$. That means by pigeonhole principle it is guaranteed that there exists a pair of values a_i and a_j out of this n+1 values.

Where a_i is sum 2^{x_i} into some left over thing which is an odd value. And a_j is some 2^{x_j} multiplied by the same leftover value. I do not know the exact value of that left over odd value b. But that left over odd value b will be the same; that is a guarantee. And exponents $x_i \neq x_j$ because I am considering the distinct vales a_i and a_j. But what is guaranteed is that the leftover odd value here that is there as per this unique factorization claim will be the same. Now I have 2 possible cases if a_i is greater than a_j then clearly a_j divides a_i because if I divide a_j by a_i then the effect of b goes out and the exponent x_i is greater than exponent x_j. So, whatever is leftover that will be the quotient and the remainder will be 0.

Whereas if a_j is greater than a_i then again the effect of b vanishes and $2^{x_j}/2^{x_i}$ that will give you 0 remember. So irrespective of the case my statement is correct.

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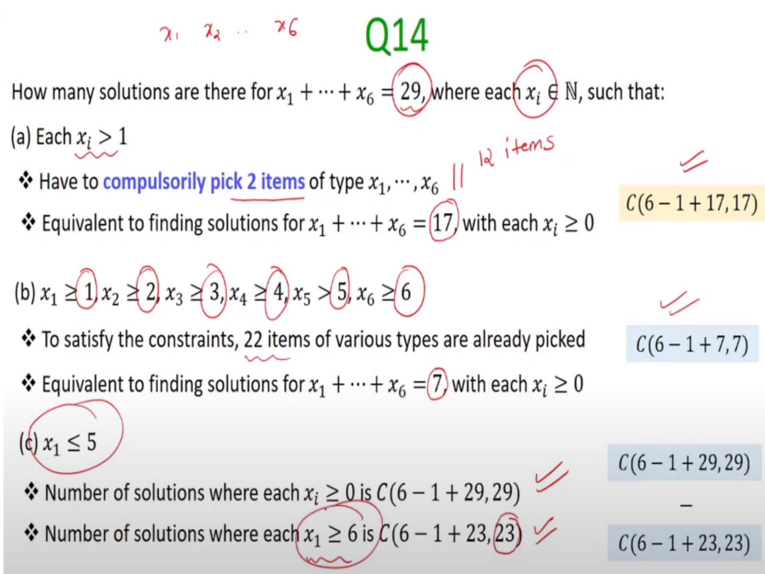

Now let us go the last question. Here we want to find out how many solutions are there for the equation $x_1 + \cdots x_6 = 29$ where there are various possible restrictions on x_i. So in part a, we have the restriction that each x_i has to be greater than 1. So you can imagine that you are given here bills of type x_1, bills of type x_2 and bills of type x_6. We have to pick total 29 bills with the restriction that you have to definitely pick more than one bill of each type.

That is the interpretation of this first restriction. That means I have to compulsorily pick 2 items of type $x_1, x_2 \dots x_6$. That means I had already picked 12 items compulsorily. That means now I am left over with the problem of picking 17 bills in total, out of this 6 different bill types where there are no restrictions. And remember as per the formula for the number of combinations with repetitions the answer is $C(6-1+17,17) = C(22,17)$.

In part b, the restriction is $x_i \geq i$. Again, if I interpret this restriction that means I have to definitely include one bill of type x_1, 2 bills of type x_2, 3 bills of type x_3, 4 pills of type x_4, 5 bills of type x_5 and 6 bills of x_6. That means I have already picked 22 bills of various types. That means now my goal was to pick 29 bills; 22 definitely I have already picked. So, I am left over with the problem of picking 7 bills where those 7 bills can be of type x_1, x_2 to x_6 in any possible order, no restrictions. So again, from the formula for number of r-combinations with repetition the answer will be $C(6-1+7,7) = C(12,7)$.

In part c, the restriction is that $x\_1 \backslash le$ 5. So, what we do here is the following. We first find out the number of solutions where there is no restriction. That means x_1 maybe 0 as well; those solutions are also included in this quantity. And now I try to find out those solutions where this condition namely x_1 less than equal to 5 is violated. That means find the number of solutions where x_1 is greater than equal to 6.

That means definitely I have to pick 6 bills of type x_1 which further implies that now I am interested to pick the remaining 23 bills without putting any restriction that how many bills of different types I have to choose. The number of solutions for this case will be this. But this is not what we want. We want to find out the number of solutions which do not have this condition. So what I do? I subtract this value from the set of or from the number of solutions that I have without any restrictions and that will give me the answer.

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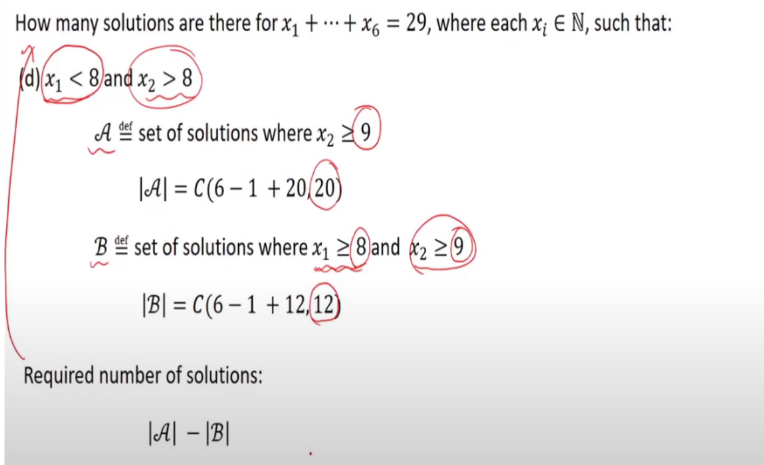
Q14



The last part here, my restrictions are $x_1 < 8$ and $x_2 > 8$. Let us first try to find out the number of solutions where $x_2 \ge 9$. That means just try to satisfy the second restriction here. The number of solutions will be this because if x_2 is greater than equal to 9 that means 9 bills of type x_2 definitely have to be chosen. That means now I am left with the problem of picking 20 bills from bills of 6 types without any restrictions.

And now let us try to find out the number of solutions where this first restriction is violated, namely x_1 is greater than equal to 8 and x_2 is greater than equal to 9. So, what basically I am trying to do is the set A that I have defined here it has all those solutions where x_1 is less than 8 as well as x_1 is greater than 8. So, I am trying to take out those solutions where x_1 is greater than equal to 8 from this set A. I am denoting that set as B and the cardinality of the set B is this because if I am supposed to satisfy x_1 greater than equal to 8 and x_2 greater than equal to 9 that means I have already picked 17 bills. My goal will be now to pick 12 more bills from bills of 6 types without any restrictions. This will be the number of ways the number of solutions. And as I have said from the interpretation of the set A and B the required number of solutions is the difference of these 2 cardinalities which we can easily find out. So with that we finish our tutorial number 6. Thank you.