Discrete Mathematics
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Module No # 07
Lecture No # 35
Solving Linear Homogenous Recurrence Equations – Part I

Hello everyone, welcome to this lecture on solving linear homogenous recurrence equations part 1.

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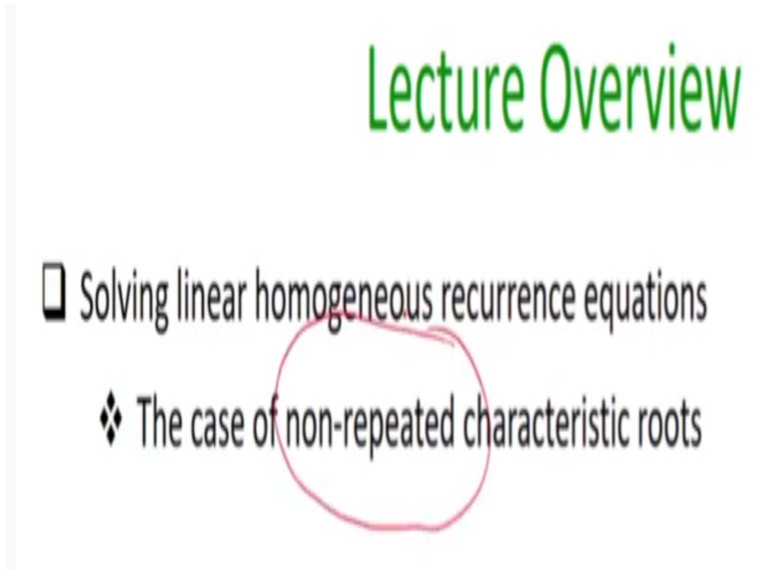

So just to quickly recap, in the last lecture we discussed how we can solve counting problems by formulating recurrence equations and we also started discussing about how to solve the recurrence equations. Because when you want to count certain number of things using recurrence equations then there are two parts. First thing is formulating the recurrence equation and the second thing will be finding the closed-form formula or the solution for the recurrence equation.

Because, until and unless you do not have a closed-form formula you may not be able to come up with a solution. You have to solve the recurrence equation. So we already discussed the iterative method in the last lecture. In this lecture, we will continue our discussion on solving linear homogenous recurrence equations. And we will discuss one category, namely when we have non-repeated characteristics roots.

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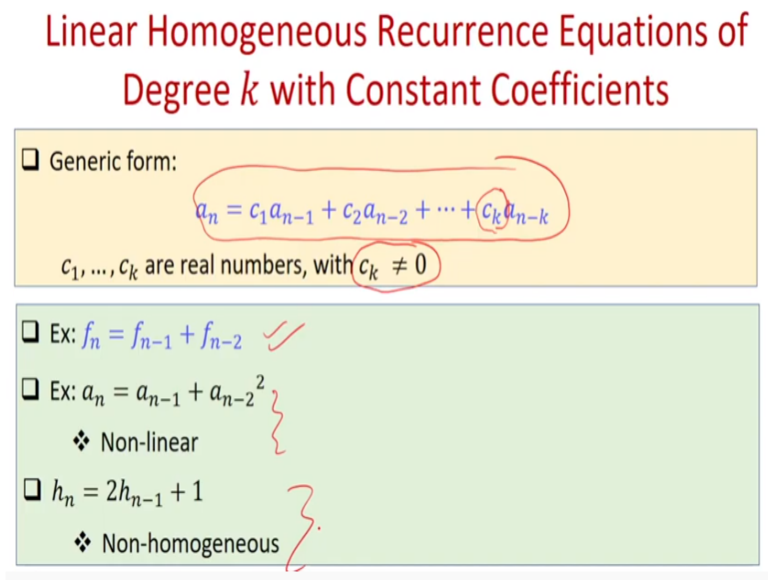

So just to quickly recap, what exactly are linear homogenous reference equations of degree k? The general form is $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$. You have an infinite sequence where the n-th term of the sequence depends upon the previous k terms i.e., a_n is always dependent on a_{n-k}, or in other words $c_k \neq 0$. The recurrence equation for the Fibonacci sequence is an example of linear homogenous equation.

$a_n = a_{n-1} + a_{n-2}^2$ is a non-linear equation and $h_n = 2h_{n-1} + 1$ is a non-homogenous equation. So we are interested to come up with the general method for solving recurrence equations of this type.

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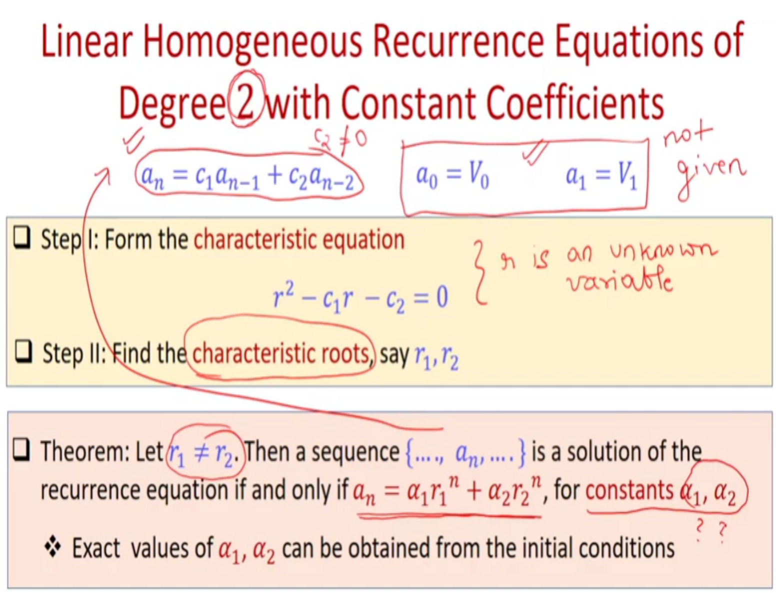

So, I will first demonstrate the process assuming that we have a linear homogenous recurrence equation of degree 2. When I say degree 2 that means the n-th term of that infinite sequence which I am interested to find out depends upon the previous two terms, namely, it depends on a_{n-1} and a_{n-2} where $c_2 \neq 0$. c_1 can be 0 but c_2 definitely cannot be 0.

And you may or not may not be given initial conditions. So again, recall in the last lecture we discuss that if you are not given initial conditions then there could be multiple number of sequences or solutions satisfying the recurrence condition. Because since the initial conditions are not given you are free to put any value as the initial condition; any term as the initial condition. And if once you freeze that initial conditions, that determine what will be the remaining terms of the sequence?

So in this case I am assuming that you are given the initial conditions. Say the initial conditions are $a_0 = V_0$ and $a_1 = V_1$. So the first step here will be to construct what we call as characteristic equation and the characteristic equation will be an equation in an unknown r.

So r is an unknown variable here whose value is not known. This characteristic equation will be a quadratic equation in r. Why quadratic? Because right now we are considering degree 2 recurrence equations. And the form of the characteristic equation will be $r^2 - c_1 r - c_2 = 0$. So that is why it is important that your recurrence condition should be of this form.

Now since this is a quadratic equation, we will have 2 roots for this equations. So I call those roots as r_1 and r_2 and those roots are called as *characteristic roots* because they are the roots of this characteristic equation. Now there could be 2 possibilities: the roots r_1 r_2 and are distinct or $r_1 = r_2$ and they could be the same. When I say non-repeated I mean the former case where the roots r_1 and r_2 are different.

So once you have solved the characteristic equation you will have the value of the characteristic roots and you can check whether you are in this case or not. Now if you are in this case then we can prove that any sequence which is the solution of the recurrence equation that is given to you will be of the form $\alpha_1 r_1^n + \alpha_2 r_2^n$.

So for the moment imagine that you are not given the initial conditions. You are just interested to find out one possible sequence satisfying the given recurrence condition. Then what this theorem says is, any infinite sequence whose n-th term satisfies this recurrence condition will have its n-th term of the form $\alpha_1 * r_1^n + \alpha_2 * r_2^n$ for some arbitrary constants α_1 and α_2.

That is what the theorem says and now if you are given this initial conditions that means you are interested to find out the sequence whose initial terms are V_0 and V_1 as well. Then the exact values of this constants α_1 and α_2 can be obtained by utilizing the initial conditions.

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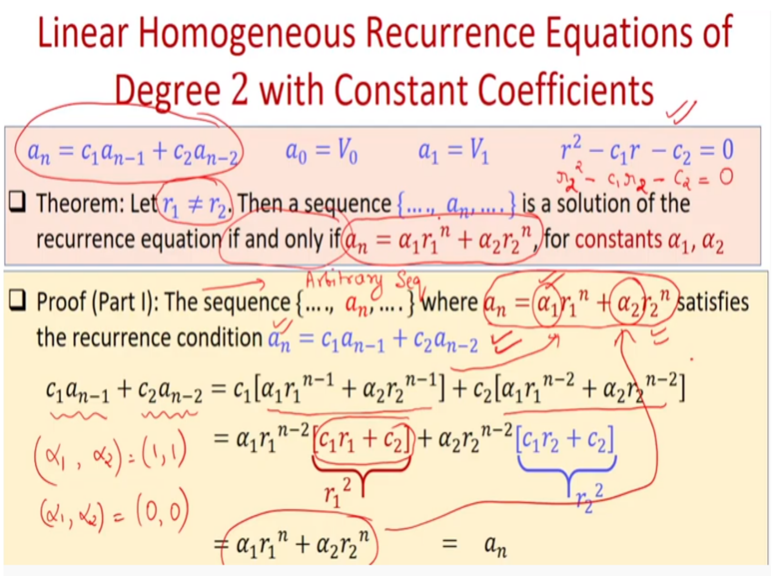

So, let us first prove this theorem statement here. So what are the things which are given to you? Your goal is to find out an arbitrary sequence whose n-th term satisfies this recurrence condition. You have found the characteristic equation, you solved the characteristic equation and the roots are distinct. Our goal is to prove that you any arbitrary sequence whose n-th term is of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ where α_1 and α_2 are constants, satisfies the given recurrence condition.

So let us prove that. And this is true irrespective of the initial conditions i.e., irrespective of the initial conditions the n-th term of that sequence will be of this form. So let us prove that and the proof is very simple.

So, what is the n-th term of the arbitrary sequence that we are considering? Or to put it another way, our goal is to show that if the n-th term is of this form then this recurrence condition is satisfied. So let us prove that whether indeed it satisfies this recurrence condition or not. So what is the recurrence condition? The recurrence condition says that $c_1a_{n-1} + c_2a_{n-2}$ (where a_{n-1} and a_{n-2} are the $(n-1)$-th and $(n-2)$-th terms of the arbitrary sequence respectively) should be equal to the n-th term of this arbitrary sequence. That is what we have to prove.

By substituting $n - 1$ in this formula we obtain that the (n - 1)-th term will be $\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}$. And its $(n-2)$-th term will be $\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}$; again, just obtained by substituting $n = n - 2$ in the formula for the n-th term of the sequence.

Now we will solve it. So we will rearrange the terms and after rearranging the terms we get $\alpha_1 r_1^{n-2}[c_1r_1 + c_2] + \alpha_2 r_2^{n-2}[c_1r_2 + c_2]$. And now what we are going to do is, we are going to utilize the fact that r_1 and r_2 are the roots for this characteristic equation. That means both r_1 as well as r_2 satisfies the condition $r^2 - c_1r - c_2 = 0$. That means $r_1^2 - c_1r_1 - c_2 = 0$ or in other words $r_1^2 = c_1r_1 + c_2$. So that is why I can substitute this part by r_1^2 and similarly your r_2 also satisfies the characteristic equation.

So we also have $r_2^2 - c_1r_2 - c_2 = 0$ which in other words implies $c_1r_2 + c_2$ is r_2^2. So by substituting this we get $\alpha_1 r_1^n + \alpha_2 r_2^n$. And what is this? This is nothing but the n-th term of the arbitrary sequence. Thus we have proved part 1. So we have shown that you take any arbitrary sequence, if its n-th term is of this form then definitely that satisfies the recurrence condition.

Now, any value of the constants α_1 and α_2 will give you a sequence which satisfies the given recurrence conditions. So I can have $(\alpha_1, \alpha_2) = (1,1)$ and that will give me one arbitrary sequence satisfying the given recurrence condition. I can put $(\alpha_1, \alpha_2) = (0,0)$ and that also will be satisfying the recurrence condition and so on.

In fact the arbitrary sequence where all the terms are 0, trivially satisfies the recurrence condition. But we are not interested in such trivial solutions. So this proves the theorem in one direction. That means you know how to find out at least one sequence satisfying the recurrence condition. But the theorem statement is an if and only if statement. It basically says that if at all there is a solution, then it has to be of this form where the n-th term is some constant times r_1^n plus another constant times r_2^n.

We had shown right now that you give me a sequence whose n-th term is of this form, namely $\alpha_1 r_1^n + \alpha_2 r_2^n$, it will satisfy the recurrence condition. But now I want to prove the other way around, that if at all there would have been a solution, the structure of the n-th term of that solution will be constant times r_1^n plus another constant times r_2^n.

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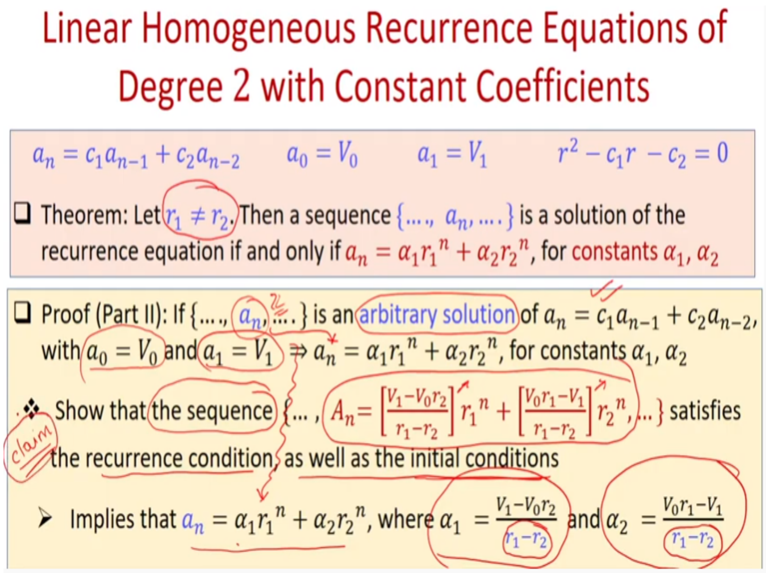

So that will be the part 2 of the proof. So now, here we are assuming that suppose there is some solution satisfying this recurrence condition and the initial conditions that are given to you then we want to prove that this n-th term of the sequence is of this form for some constants α_1 and α_2. And the proof strategy here will be as follows.

We will first prove that you take another sequence different from the sequence that we are considering right now or the solution that you are considering right now. So I am taking some another sequence whose n-th term is of the form $A_n = \left[\frac{V_1 - V_0 r_2}{r_1 - r_2}\right] r_1^n += \left[\frac{V_0 r_1 - V_1}{r_1 - r_2}\right] r_2^n$. Then I will show that this satisfies not only the recurrence condition but also the initial conditions. The above claim automatically implies that the n-th term of the arbitrary solution we are considering is of the form $\alpha_1 r_1^n + \alpha_2 r_2^n$ where the constants $\alpha_1 = \frac{V_1 - V_0 r_2}{r_1 - r_2}$ and $\alpha_2 = \frac{V_0 r_1 - V_1}{r_1 - r_2}$. And why so?

This is because I cannot have two different arbitrary sequences satisfying the same recurrence condition and having the same initial conditions. As per my claim, I have one sequence satisfying the recurrence condition as well as the initial conditions. And I am also given another sequence whose structure I do not know; whose a_n I do not know, which also satisfies the recurrence condition as well as the same two initial conditions.

Then as we discussed in the last lecture, if I want to satisfy simultaneously the initial conditions as well as the recurrence conditions then there can be only one possible sequence, you cannot have multiple possible sequences whose terms are different but they are satisfying the initial conditions as well as the recurrence conditions. That cannot happen. Because if the initial conditions of the two sequences are same then that automatically implies that all the following terms of the sequences are also going to be the same, because both of them satisfies the recurrence condition.

So assuming my claim is true, I end up showing that the arbitrary solution whose n-th term I do not know is of this form: constant times r_1^n plus constant times r_2^n. Because as per my claim there is another sequence whose n-th term is $\left[\frac{V_1 - V_0 r_2}{r_1 - r_2}\right] r_1^n += \left[\frac{V_0 r_1 - V_1}{r_1 - r_2}\right] r_2^n$.

By the way the reason I am highlighting this $(r_1 - r_2)$ here in the denominator is that these constants α_1 and α_2 are well defined even though in the denominator I have $(r_1 - r_2)$. This is because I am considering the case where r_1 and r_2 are distinct and if r_1 and r_2 are distinct their difference will not be 0. That is why constant α_1 and α_2 that we are considering in this proof are well defined.

So what is left now? We have to now show this claim. We have to prove that this claim is true.

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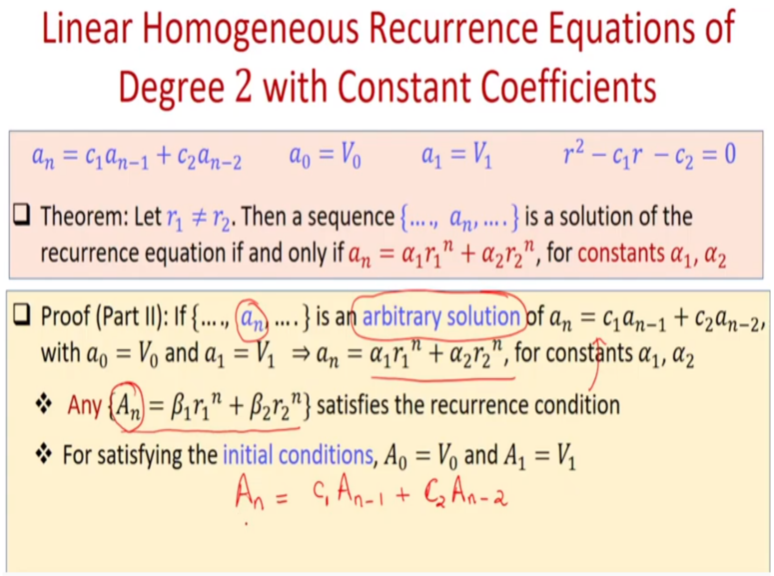

So let us prove this claim. So here we are given some arbitrary solution for the recurrence condition as well as satisfying the initial condition. We have to prove that if that is the case then the n-th term of that arbitrary solution is of this form. For that, we first observe from the proof of the part 1 of this theorem that any sequence, irrespective of the initial conditions, whose n-th term is of the form $A_n = \beta_1 r_1^n + \beta_2 r_2^n$ where β_1 and β_2 are constants always satisfies the recurrence condition.

This is what we proved in the proof of part 1. We use A to differentiate from the sequence whose n-th term is a_n i.e., the sequence whose n-th term is A_n is different from the arbitrary solution whose n-th term is represented as a_n.

We know that this satisfies the recurrence condition. That means we know that $A_n = c_1 A_{n-1} + c_2 A_{n-2}$. The proof is similar to that of part where we utilized that r_1 is a characteristic root, r_2 is a characteristic root and substituted r_1^2 with $c_1 * r_1 + c_2$ and so on.

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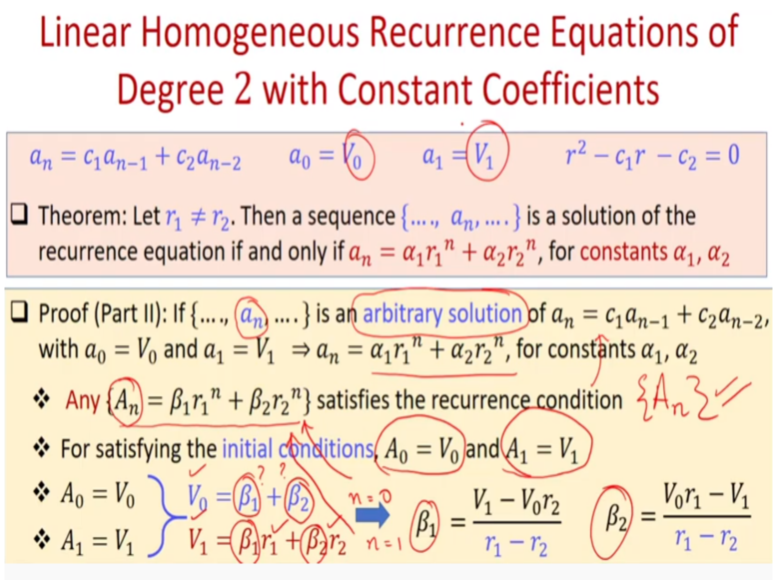

Now, if I want to find out the initial terms of that A series we would like that $A_0 = V_0$ and $A_1 = V_1$. If that is the case, then I get 2 equations in β_1 and β_2. How? If I substitute $n = 0$ here, I get 1 equation, and if I substitute $n = 1$ here, I get another equation. So now what are the things known to me? V_0 and V_1 are already given to me because they are the initial conditions.

β_1 and β_2 are the unknowns. And r_1 andr_2 are known to you. So you have now two equations in 2 unknowns and you can solve them and get the value of β_1 and β_2. That means I have now formed a concrete sequence, namely the A sequence, whose n-th term is $A_n = \beta_1 r_1^n + \beta_2 r_2^n$. I know it satisfies the recurrence condition.

And I also know the values of β_1 and β_2 for which this A series will satisfy the given initial conditions, namely I know the values of β_1 and β_2 for which A_0 would have given me V_0 and A_1 could have given me V_1.

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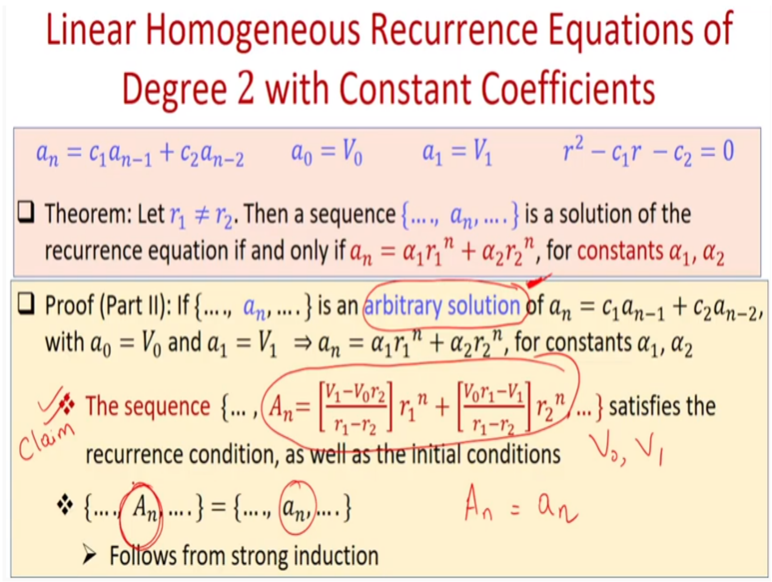

So our goal was to prove this claim; and we proved it. We proved that indeed any A sequence, a sequence whose n-th term is this, satisfies the recurrence condition as well as the initial conditions V_0 and V_1. What this means? This means that now I have 2 different sequences, A sequence satisfying the recurrence condition as well as the initial condition. And arbitrary solution which I assumed satisfying the given recurrence condition as well as the initial conditions.

And now both these 2 sequences are same because as I said earlier I *cannot* have 2 different sequences with the *same initial condition* but different n-th term simultaneously satisfying the recurrence condition. It is possible to have two different sequences A and B if we do not put the restriction that their initial conditions are the same. But I cannot have 2 difference sequences A sequence and B sequence whose n-th terms are different while satisfying the same recurrence conditions with identical initial terms.

We have two sequences satisfying the recurrence condition and both of them are satisfying the initial conditions. That is possible only when $A_n = a_n$. That means the arbitrary solution that you considered here, it is of the form some constant times r_1^n plus some constant times r_2^n. Because that is the term of this A sequence. So that completes the proof for the part 2 here.

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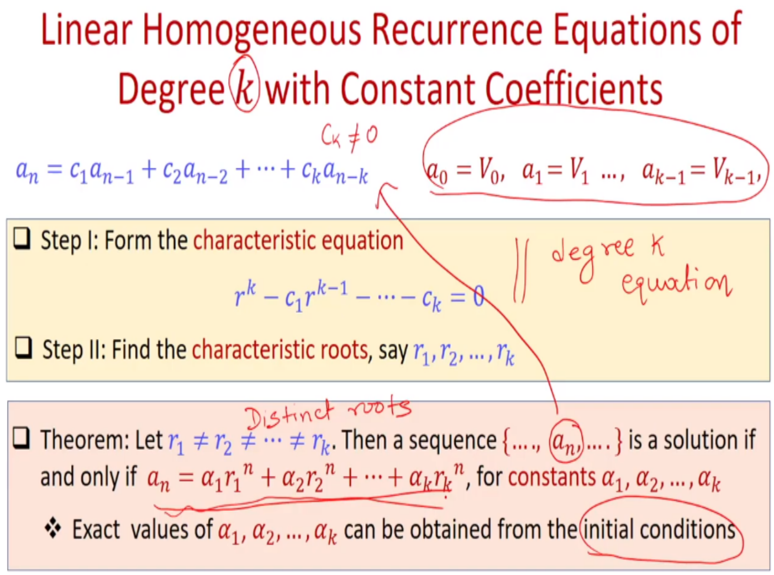

Till now, we focused on the case of degree 2. Now we will try to extend or generalize this theory for the case of degree k linear homogenous recurrence equations. So remember the degree k equation, the general formula is this where c_k is not allowed to be 0. And you may or may not be given the initial conditions. If you are not given the initial conditions then you will stop with showing the closed-form formula for the n-th term in terms of some arbitrary constant.

Those constants you can put as any constant. But if you are given the initial conditions as well then you can solve and find out those concrete constants. So, what will be the process in the general case? We will first form a characteristic equation; this will be a degree k equation. Next we will solve it and find out characteristic roots. Let us denote the characteristic roots by $r_1, r_2 \ldots r_k$.

Now there could be multiple cases. The case that we considered when k was 2 and that we are going to consider in this case, is the case of distinct roots. Namely, when all your k characteristic roots are different. Then extending the theorem that we proved for the case of $k = 2$, we can show that any sequence which satisfies this recurrence condition will have its n-th term of the following form.

Some constant times r_1^n plus another constant times r_2^n and continuing like that some constant times r_k^n. This will be the general form of the solution satisfying the recurrence condition. The exact values of these constants $\alpha_1, \alpha_2 \ldots \alpha_k$ can be obtained from the initial conditions.

So if you are given the initial conditions then by substituting $n = 0, n = 1\ \ldots n = k - 1$; we will get k equations. In this, k unknowns α_1 to α_k and then we can get the concrete values of the constants α_1 to α_k. But if you are not given the initial condition then the only thing that we can do is we can just find out the general form of the solution. It is up to us what constants $\alpha_1, \alpha_2 \ldots \alpha_k$ we substitute. That will determine the sequence satisfying the given reference condition. But if you want a unique condition then you also need to have the initial conditions available.

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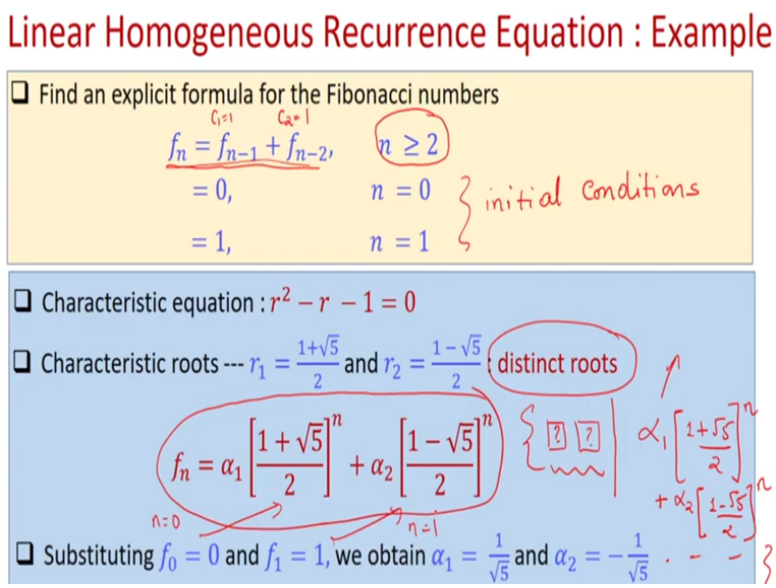

So now let us see an example where we will apply the method that we had discussed. We now want to find out an explicit formula for the n-th term of the Fibonacci sequence. So just to recall, the n-th term of the Fibonacci sequence is the following. Then n-th term depends on the previous 2 terms and the initial conditions are this. So the first step will be to find the characteristic equation so here $c_1 = 1$ and $c_2 = 1$.

So that is why our characteristic equation will be $r^2 - r - 1 = 0$. Now if I solve the characteristic equation and find the characteristic roots then I see that I obtain 2 distinct roots. That means I can apply the theorem that we have discussed in this lecture. What I will say is that the n-th term of any sequence whose n-th term is of the form $f_n = \alpha_1 \left[\frac{1+\sqrt{5}}{2}\right]^n + \alpha_2 \left[\frac{1-\sqrt{5}}{2}\right]^n$ will satisfy the recurrence condition.

Now whatever value for α_1 and α_2 I substitute that will determine a different Fibonacci sequence. If someone just gives me the recurrence condition and not the initial conditions, and asks me to find out a sequence satisfying this recurrence condition, I can say that any sequence whose n-th term is $f_n = \alpha_1 \left[\frac{1+\sqrt{5}}{2}\right]^n + \alpha_2 \left[\frac{1-\sqrt{5}}{2}\right]^n$ for any value of the constants α_1 and α_2 will satisfy the recurrence condition without worrying about what are the first 2 terms. But in this case, I am given the initial conditions. So if I am given the initial conditions, I will utilize them to find out exact value of α_1 and α_2 which is consistent with the initial conditions of actual Fibonacci sequence.

So I am interested to find out the sequence whose zeroth term is 0 and the next term is 1. That means I have to substitute $n = 0$ and $n = 1$ in this general formula. And then I will get 2 equations in α_1 andα_2 and by solving them I can get the exact values of my constants α_1 and α_2. They will be $\frac{1}{\sqrt{5}}$ and $\frac{-1}{\sqrt{5}}$. And then I will say that here is the exact solution of the actual Fibonacci sequence which satisfies not only the initial conditions but also the recurrence condition.

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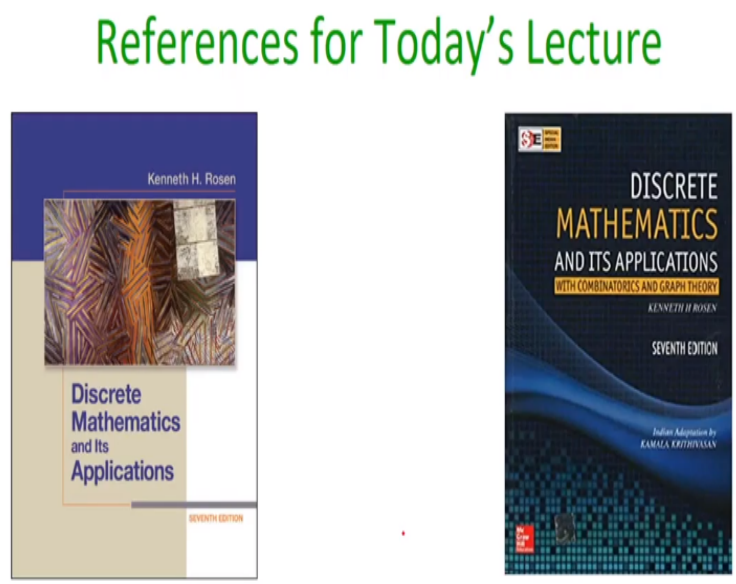

So that brings me to the end of this lecture. These are the references for today's lecture. Just to summarize in this lecture we started discussing about how to solve linear homogeneous recurrence equations of degree k. And we saw one of the cases, namely we saw the case where the characteristics roots are distinct. In this case we saw how to find the characteristic root and if all the characteristic roots are different then we know what will be the general solution and depending upon whether the initial conditions are available or not we can find out the exact solution. Thank you.