Discrete Mathematics

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Module No # 07

Lecture No # 33

Permutation and Combination

Hello everyone, welcome to this lecture on permutations and combination.

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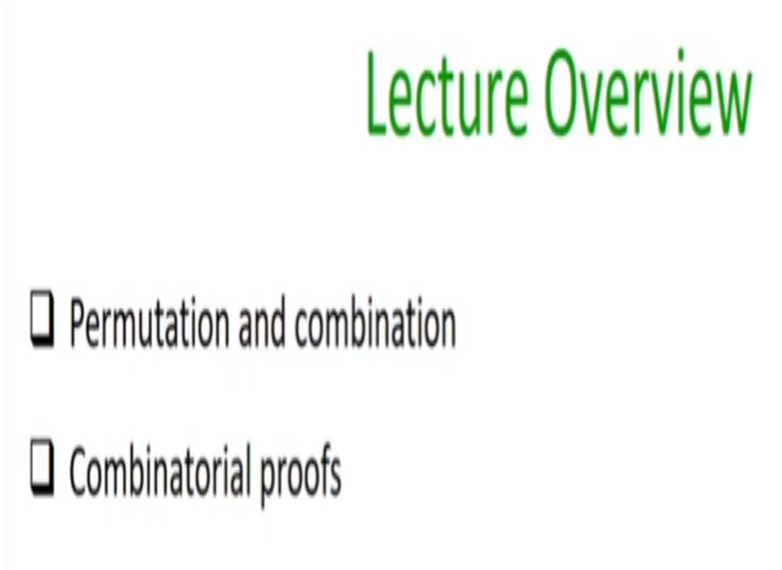

Just to quickly recap, in the last lecture we started our discussion on combinatorics and we discussed the basic counting rules like the sum rule and the product rule. So in this lecture, we will recall the concepts related to your permutation and combination that you might have studied during your high school. And we will also discuss about the combinatorial proofs.

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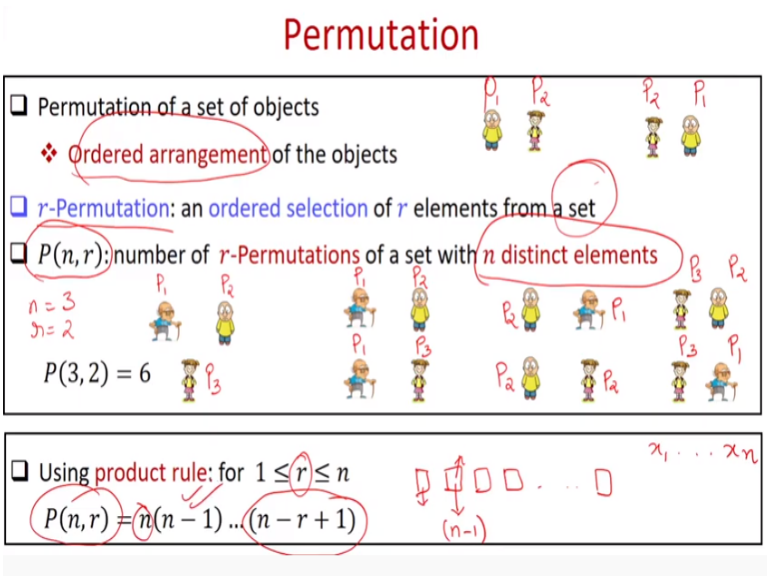

So to being with, what is a permutation of a set of objects? As the name suggests, it is an ordered arrangements of objects and when I say the ordered arrangement of the objects, that means the ordering of the objects matter here. So for instance if I consider 2 persons, person number 1 and person 2 then these two orders are different. If I consider person number 1 followed by person number 2 then this order is different than the ordering where the person number 2 is appearing before the person number 1.

So, we define what we call as r-permutation and r-permutation is nothing but an ordered selection of r elements from a set. So you are given a set which has certain number of elements, of course it should have r or more number of elements. If you select r elements in an ordered fashion then that is called an r-permutation and the number of such r-permutations from a set consisting of n distinct elements is denoted by this quantity or this permutation function $P(n, r)$.

So you are given a set with n distinct objects and we want to find out how many r-permutations I can have from this set. So for instance if I consider $n = 3$ and $r = 2$ then $P(n, 2) = 6$. Why? Say you have 3 persons; person 1, person 2 and person 3. So you have a set of 3 objects or 3 persons here and I want to find out how many 2 permutations I can have; how many ordered selection of 2 elements I can have from this collection.

So pictorially these are the 6 possible ordering. I can choose person number 1 followed by person number 2, that's one ordering. I can choose person number 1 followed by person number 3, that's another ordering. I can choose person number 2 followed by person number 1 as one of the orderings.

I can have person number 2 followed by person number 2 as another ordering. In the same way I can have person number 3 followed by person number 2 as one of the orderings and I can have person number 3 followed by person number 1 as another order. So these are the different possible 2 permutations that you can have from this collection of 3 people. So now it's easy to see that if I apply the product rule then I can derive the formula $P(n, r) = n * (n-1) * (n-2) \ldots (n-r+1)$.

Of course, for this formula to make sense you require your $r \in \{1, \ldots, n\}$; otherwise you get into the issues of negative quantities. So how exactly we get from product rule the output of $P(n, r)$ function to be this? So, you can imagine that I have r slots to be occupied. And I have n objects to choose from. I have object number x_1 to x_n. Now when it comes to the first slot, here I can put either object number x_1 or object number x_2 or object number x_n.

So that is why I have n choice for the first object or first slot here. Now once I have decided which of the n objects to put in the first slot corresponding to that I have now $n-1$ options or $n-1$ objects to choose from to put in the second slot and so on.

By the way here repetitions are not allowed because right now I am considering the case of selecting or forming r-permutations where in the permutations repetitions are not allowed. So that is why when I am considering the second slot here. I can't consider the object which I have already assigned to the first slot. So that is why I have only $n-1$ options instead of n options to choose from when it comes to the second slot. The formula becomes different if in the permutation that I am forming repetitions are allowed. So now it is easy that by applying the product rule I get the output of this $P(n, r)$ function to be this value.

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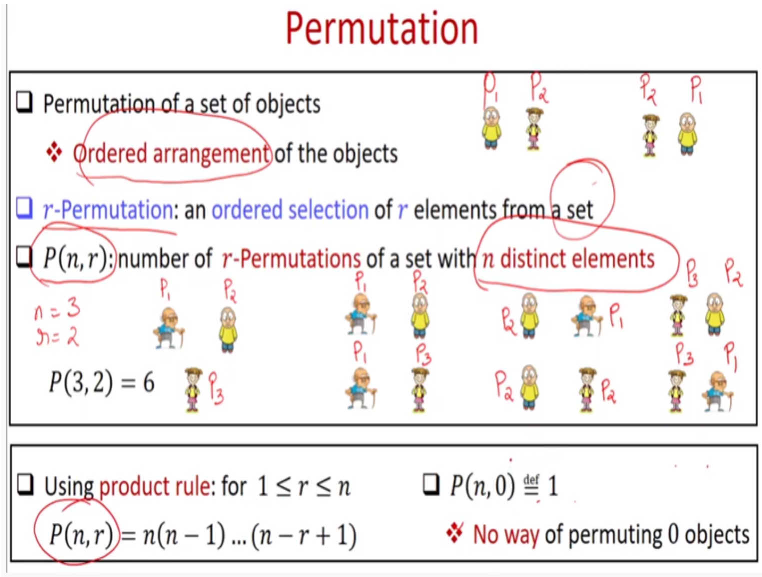

Now, we can define $P(n, 0) = 1$, namely, no way of permuting 0 objects. So if you are given n objects and you don't want to select any objects or you don't want to permute any object then that can be considered as 1 way of doing that. Because there is no way; so no way is considered as the only way of permuting 0 objects. So that is why we define $P(n, 0) = 1$. That is defined; it is not coming from the product rule, that is coming as part of our definition.

So now we have the value of $P(n, r)$ where r is non-zero and in the range 1 to n and we have the value of $P(n, r)$ when $r = 0$.

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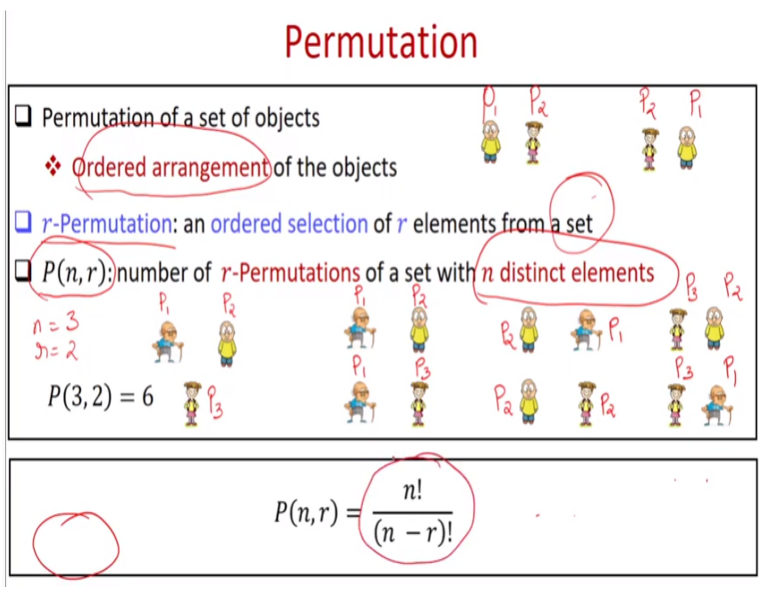

So if I unify these 2 values I get that $P(n, r) = n!/(n - r)!$. You can easily verify that.

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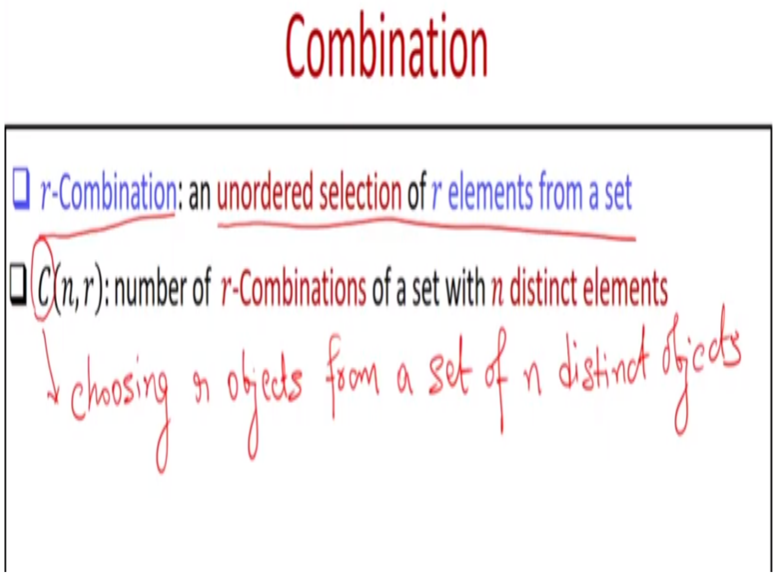

Now you consider the case when we are selecting objects but while selecting the objects the order does not matter. That means we are now interested in unordered selection of r elements from a set and each such unordered selection is called as an r-combination. So that means if I am now selecting 2 objects out of 3 objects then it does not matter whether I pick x_1 before x_2 or whether I pick x_2 before x_1. So the arrangement (x_2, x_1) will be considered the same irrespective of whether x_1 comes before x_2 or whether x_2 comes before x_1.

So we use this notation $C(n, r)$. This is often treated as choosing r objects from a set of indistinct objects. And this function basically denotes a number of r combinations that we can have for a set which has n distinct elements.

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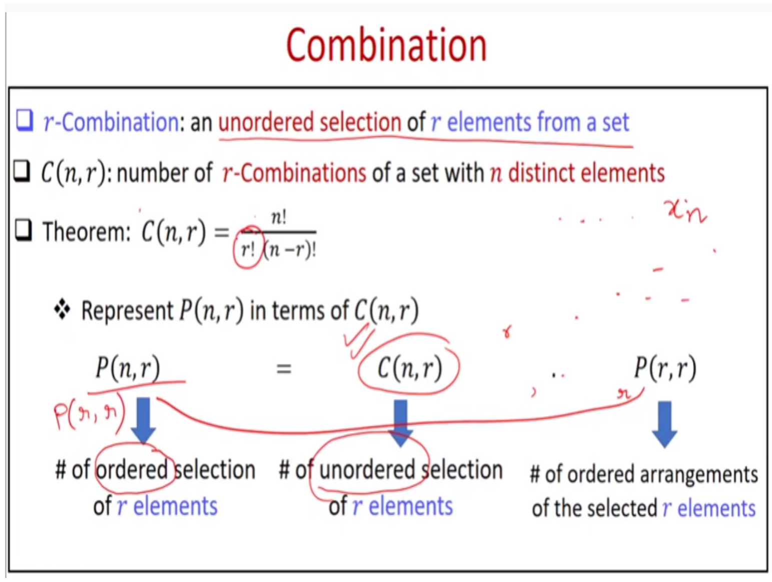

So again you must have studied it during your high school that the output of this function or the value of this function is nothing but $\frac{n!}{r!(n-r)!}$. So there are several ways of deriving that. The simplest way could be to find out a relationship between the permutation function and the combination function. So my claim is that $P(n,r) = C(n,r) \cdot P(r,r)$. Why so?

Because if you see, the left-hand side it is nothing but the number of ordered selection of r elements. So you are given a set with n objects and you are interested to find out how many ordered r permutations you can have over a set consisting of n elements that's nothing but your function $P(n,r)$. My claim is that the number of ordered selection of r elements is nothing but the following.

You first find out the number of unordered selection of r objects or equivalently find out how many ways you can first select those r elements which you want to order in your permutation. And that you can do in $C(n,r)$ ways. Now once you have decided which r elements you are going to put in your permutation; right now you are considering the unordered selection of those r elements, if you have decided the r objects then the number of ordered arrangements of those selected r elements, if you multiply that with the number of unordered selection of r elements that will give you the total number of ordered selection of r elements.

So it's like saying the following. You have $\{x_1, \ldots, x_n\}$. You first decide the r objects for permutation. Where the order does not matter as of now. This can be done in $C(n, r)$ ways. Now once you have decided that I have selected object number $x_{i_1}, x_{i_2} \ldots x_{i_r}$.

We have selected this r objects. Every arrangement, every possible ordered arrangement of these r elements will give you 1 possible permutation. And how many ordered arrangement of this r objects you can have? You can have $P(r, r)$ such ordered arrangements. So that will give you the total number of r permutations that you can form from this subset of r elements $x_{i_1}, x_{i_2} \ldots x_{i_r}$.

So that's a relationship between the P function and C function. So now your goal is to find out the value of the $C(n, r)$ function so you just take this $P(r, r)$ in the denominator. And $P(r, r)$ is nothing but $r!$. That's how we get the value of the C function.

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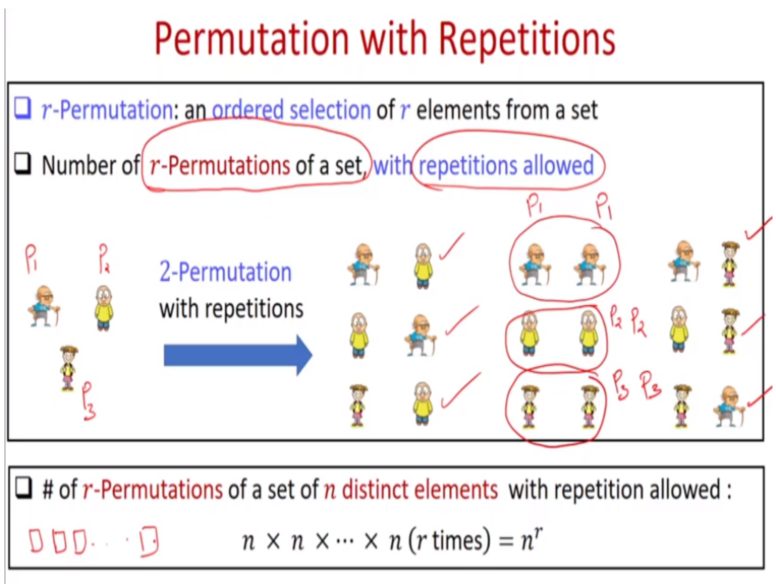

Fine, so till now we considered the case of permutations and combinations where repetitions were not allowed. Now we will consider the case where even in the selection the repetitions are allowed as well. So we are now interested to first find out the number of r permutations of a set of objects where I am allowed to have repetitions. So for instance if I consider a set with 3 persons; person 1, person 2, and person 3.

Now if I ask you how many 2-permutations I can have over this set where, I can repeat the person when I am forming the permutation. It now turns out that instead of 6 possible permutations I will now 9 possible permutations. The 6 possible permutations which we had earlier where repetitions were not allowed they will be still present. So those permutations are still present here.

So these were the 6 permutations which were there earlier when the repetitions were not allowed but now since I am allowing you repetition I can have a permutation where I have P_1 followed by P_1. I can have a permutation where I have P_2 followed by P_2 and I can now have a new permutation where I have P_3 followed by P_3. These are all allowed now because repetitions are allowed in this case. So now again if I want to find out the number of r-permutations of a set of n distinct elements where repetitions are allowed then it turns out to be the product of n, r number of times. Because I have to fill r slots and the first slot can be occupied in n ways.

And for each of those n ways in which I can occupy the first slot I can fill the second slot also in n ways. Because repetitions are allowed. And now corresponding to each of the ways in which I would have filled the first 2 slots, I have n ways to fill the third slot and so on. So that is why the total number of r-permutations of n distinct elements that I can form is n^r.

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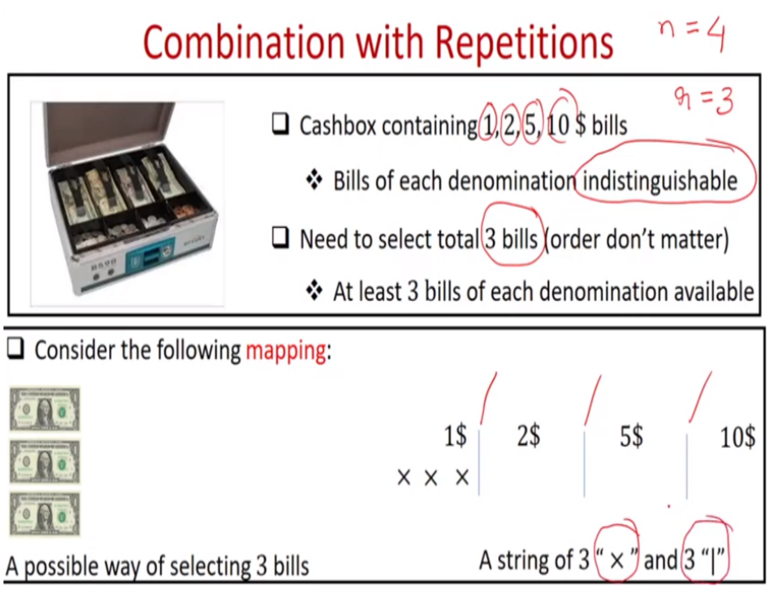

Now let's try to find out number of r combinations where repetitions are allowed and this is slightly tricky. So before going into the derivation of this formula let me give you a motivating example and then we will try to relate this example with the problem of coming up with the number of r-combinations where repetitions are allowed. So consider the following, you must have seen cash box when you go to super market and do the billing.

So in cash box you have various slots available each slot is occupied with the currency of some specific denomination. So imagine you have cash box which has 4 slots; the first slot has bills of 1 dollar right, many bills of 1 dollar, the second slot has many bills of 2 dollars, the third slot is for 5 dollar bills and fourth slot is for 10 dollar bills. By the way in this problem when we are finding the number of r-combinations with repetition, so I consider that the objects in the set from which you want to find out the r-combinations each object has many copies available and each of those copies are indistinguishable.

So for instance in this example I assuming that in the 1 dollar bill slot you might have several 1 dollar bills and all those 1 dollar bills are indistinguishable that means you can't say that if I choose the 1 dollar bill which is on the topmost position then that will consider different from the 1 dollar bill which is present at the bottom right.

So I won't consider those things here because I will be making the assumption here that the bills of each denomination are indistinguishable. That is important while deriving the formula. Now suppose my problem is the following, I want to select total 3 bills from the bills which are available in the cash box and here order does not matter because I am interested to find out the number of r-combinations where $r = 3$ here and $n = 4$ here.

You have 4 types of objects namely object of type 1 dollar bill, bills of type 2 dollars, bills of type 5 dollars, and bills of type 10 dollars. And you have many copies of them. At least 3 copies of bills of each denomination is available and your goal is to find out how many ways you can pick 3 bills in total where the order does not matter. So for instance one way of picking 3 bills is you pick all 1 dollar bill that means you pick 3, 1 dollar bills or you may decide to pick 1 bill of 1 dollar and 1 bill of 2 dollar.

Or you may decide to pick one 5 dollar bill, one 2 dollar bill and one 1 dollar bill. Or you may decide to pick all the 3 bills of 10 dollar type and so on. So these are the various ways. Our goal is to find out how many such selections are possible? So for finding that consider the following mapping. So what I do is I think in my mind that in your cash box the bills of various denominations are separated by a boundary.

So you can imagine that boundary is nothing but a vertical line so you have a vertical line or a boundary between the 1 dollar bills and the 2 dollar bills in the cash box. Similarly you have a boundary here, you have a boundary here. Now suppose I pick bills of 1 dollar in 3 numbers. So remember my goal is to pick 3 bills in total. So one way of doing that is I pick three 1 dollar bills. So I represent this selection by saying that I have picked three 1 dollar bills.

So I put 3 cross under the heading 1 and I don't put any cross under 2 dollar denomination, 5 dollar denomination, and 10 dollar denomination. So I can say that there is mapping here between a possible way of selecting 3 bills. So here is a way of selecting 3 bills where I have picked three 1 dollar bills and corresponding to that I am giving you a string consisting of 3 crosses or 3 cross and 3 vertical lines.

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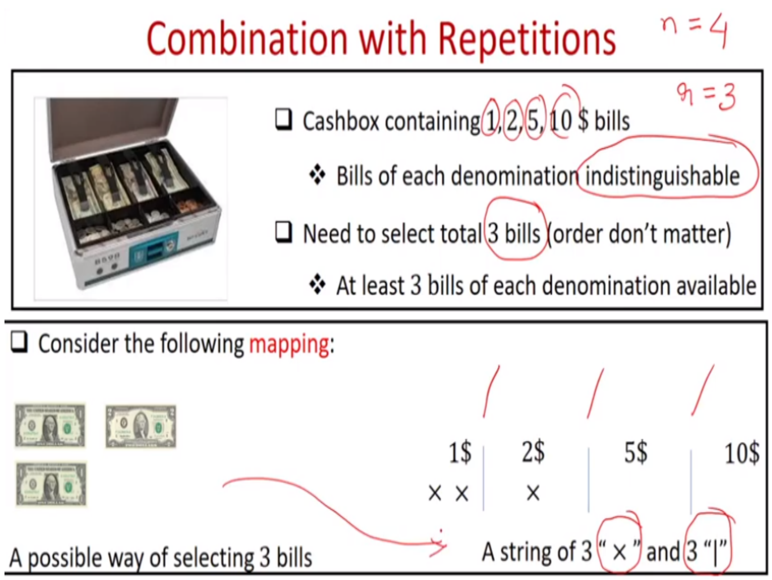

Now consider this way of selecting 3 bills; I pick two 1 dollar bills and I pick one 2 dollar bill. So my claim is corresponding to that I can associate this string where I put 2 cross under 1 dollar because I have chosen 2 bills of type 1 dollar and I put 1 cross under the 2 dollar denomination because I have chosen 1 bill of 2 dollar. Under, 5 dollar and 10 dollar I do not put any cross. So I can say that corresponding to this way of selecting 3 bills I can associate this string of 3 cross and 3 vertical lines.

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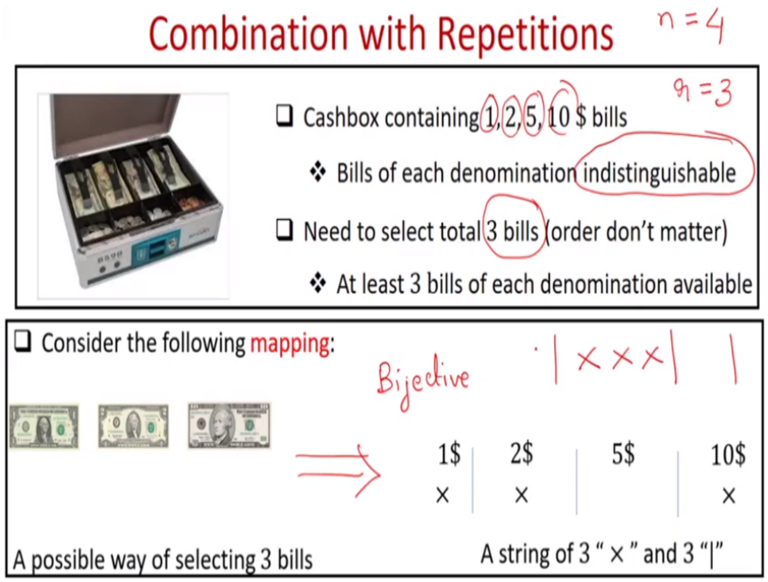

In the same way if my way of picking 3 bills is the following namely picking one 1 dollar bill, one 2 dollar bill, and one 10 dollar bill then the corresponding string will be, you put 1 cross under 1 dollar 1 cross under 2 dollar and 1 cross under 10 dollar.

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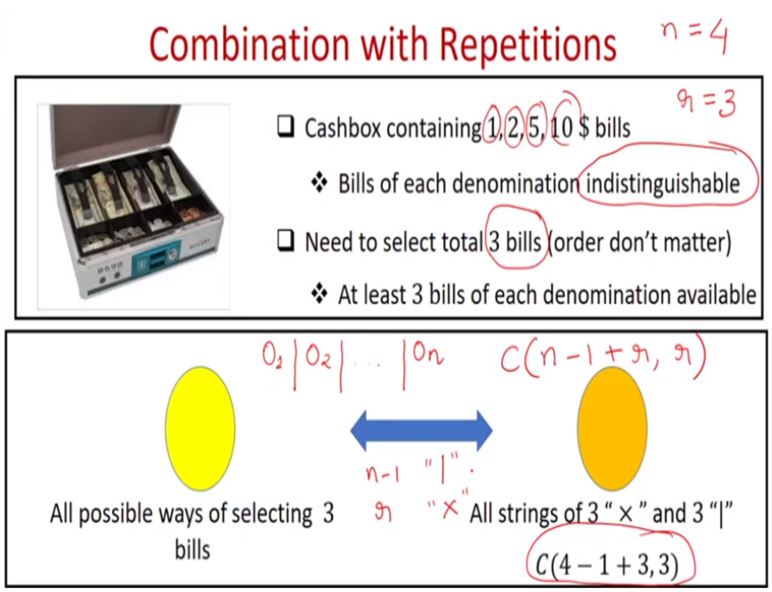

And now I can say that I have a bijective mapping, namely one-one and onto mapping. So whatever mapping I have discussed till now my claim is that, that is a bijective mapping; namely a one-to-one onto mapping between the set of all possible ways of selecting 3 bills out of bills of 4 denominations in an unordered fashion. And a set of all strings of length 6 where there are 3 crosses and 3 vertical lines. My claim is that the mapping that I have discussed is a bijection.

Of course it is an injective mapping; from this direction to this direction. Why so? Because, you take 2 different ways of selecting 3 bills the corresponding string of 3 cross and 3 vertical lines will be different. And my claim is that this mapping is surjective as well. You give me any string of 3 cross and 3 vertical lines; I can tell you a corresponding way of picking 3 bills. So for instance, if you give me a string like say, "|xxx|", then what is the corresponding way of picking 3 bills here?

So here no bill of type 1 dollar will be picked and all the 3 bills of 2 dollar type will be picked and no bill of 5 dollar will be picked and no bill for dollar for 10 dollar will be picked. So this mapping is surjective as well. So since I have 2 sets here and I have defined the mapping one-to-one namely injective and surjective as well, this mapping is bijective function. And that means the total number of way of selecting 3 bills is equivalent to finding the total number of string of length 6 which has 3 cross and 3 vertical lines.

And it's easy to see the number of such strings is this. Why? Because your string is of length 6 and you have to choose 3 positions where the cross has to appear. Because once you decide the 3 positions where the cross is going to appear, automatically at the remaining 3 positions from the problem definition the vertical line will be present. So you don't have to worry about the positions or the 3 positions where the vertical lines are going to appear because once you have chosen the 3 positions where the cross are going to appear the remaining things are automatically frozen for vertical line to appear.

And why I am writing it in this form "4 - 1 + 3" because in general, the general formula for general n and r will be $C(n-1+r, r)$. Because how many vertical lines will be there if the number of objects of various types is n? So you will have n slots because you have objects of n types.

So you will have object of type 1 and then there will be a vertical line and then you will have object of type 2, you will have 1 vertical line, and like that you have an object of type n. So how many vertical lines will be there $n-1$ and how many cross positions you have to fill? You have to fill r number of cross positions. So that is why the length of the string will be $n-1+r$ because it will have $n-1$ number of vertical lines and r number of crossings.

So that is why the general formula for r-combinations where repetitions are allowed is $C(n-r+1, r)$.

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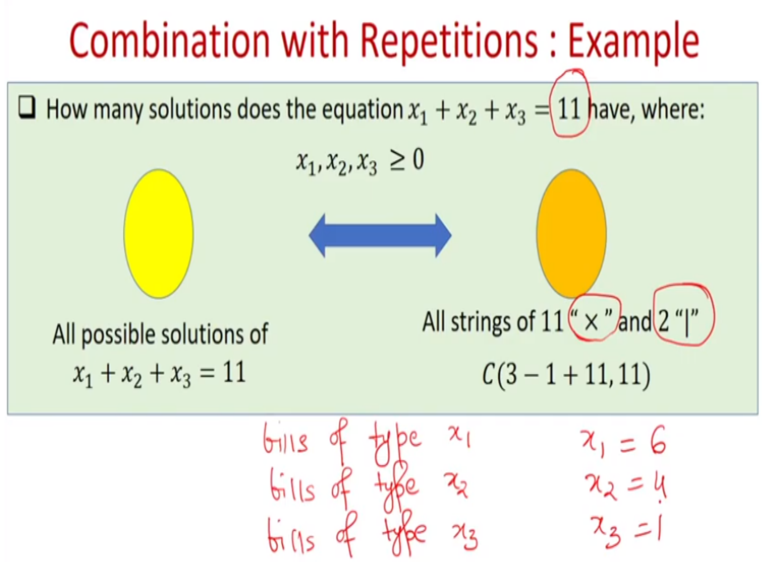

So now let's see some examples of combination with repetitions. This is a very interesting formula which is used in lots of counting problems. So suppose I want to find out the number of integer solutions for the equation $x_1 + x_2 + x_3 = 11$ where$x_1, x_2, x_3$are allowed to be greater than equal to 0. So my claim is that this, the number of solutions is nothing but the number of all strings of 11 crosses and 2 vertical lines.

This is because you can interpret this formula; you can interpret this problem as the following. You have bills of type x_1, bills of type x_2 and bills of type x_3. Your goal to pick total 11 number of bills. You can pick all the 11 bills of type x_1. That is one possible solution which corresponds to $x_1 = 11$ and $x_2 = 0$ and $x_3 = 0$. Or you can pick 10 bills of type x_1 and 1 bill of type x_2 which corresponds to x_1 in 10 and x_2 being 1.

Or you could have $x_1 = 6$, $x_2 = 4$ and $x_3 = 1$ which corresponds to picking 6 bills of type x_1, 4 bills of type x_2 and 1 bill of types x_3. And here order does not matter. So that's why your problem now reduces to picking total 11 number of bills from bills of 3 possible denominations where order does not matter, and repetitions are allowed and we have derived just now that the formula for that is nothing but this quantity.

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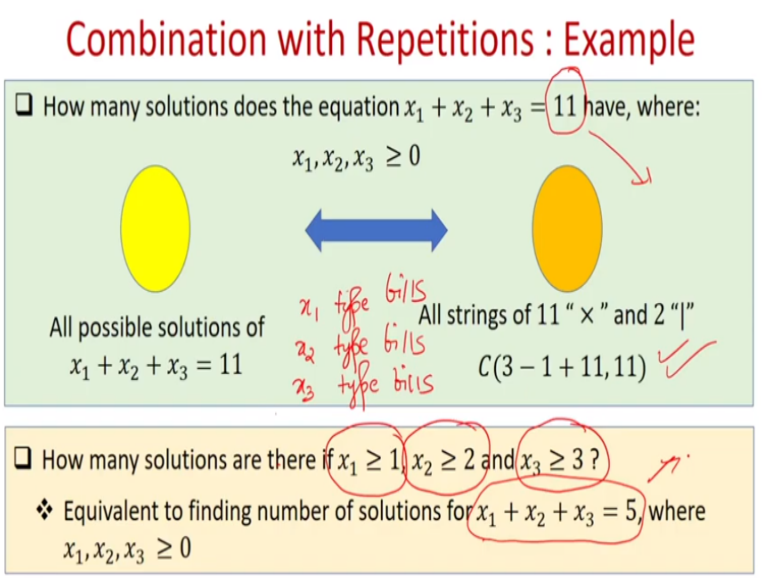

Now suppose I change the problem slightly. I am now interested to find out the number of solutions where $x_1 \geq 1$, $x_2 \geq 2$ and $x_3 \geq 3$. Earlier it was allowed that $x_1 = 0$ but now x_1 is not allowed to be 0. x_1 has to be at least 1, x_2 has to be at 2, x_3 has to be at least 3. So my claim is that this is equivalent to finding the number of solutions for a new equation, $x_1 + x_2 + x_3 = 5$ where there is no restriction on x_1, x_2, x_3.

That means any of them can be 0. Why so? Because again if I consider the bill analogy you have bills of denomination x_1, x_1 type bills, you have x_2 type bills and you have x_3 type bills. Now $x_1 \geq 1$ means definitely I have to choose 1 bill of type x_1. My goal is to pick 11 bills right out of those 11 bills 1 bill has to be of type x_1. 2 bills have to be of type x_2 and 3 bills have to be of type x_3.

That means the problem already states that I have already chosen 6 bills definitely I have chosen 1 bill of type x_1 and 2 bills of typex_2. So total 3 bills I have already chosen. And 3 bills of type x_3 that means 3 more bills are already chosen. So $3 + 3 = 6$ so 6 bills are already chosen. My goal was to pick 11 bills. So now I am left with the problem of choosing 5 bills and now for choosing these 5 remaining bills I have no restriction.

I can pick all of them of type x_1 or all of them of type x_2 or all of them of type x_3. Or 1 bill of type x_1 and 4 bills of type x_2 or 1 bill of type x_1, 2 bills of type x_2 and 2 bills of type x_3 and so on.

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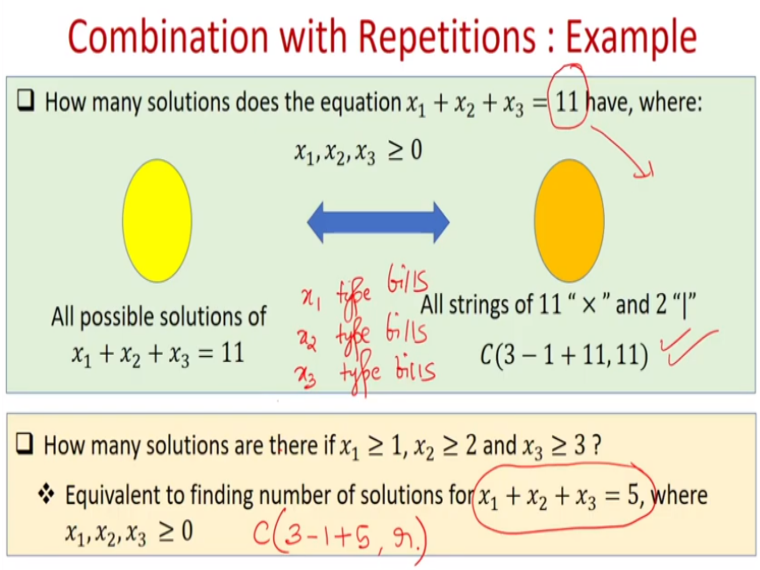

And now we know how many ways I can satisfy this equation where there are no restriction on x_1, x_2, x_3. That will be nothing but $C(3 - 1 + 5, r)$.

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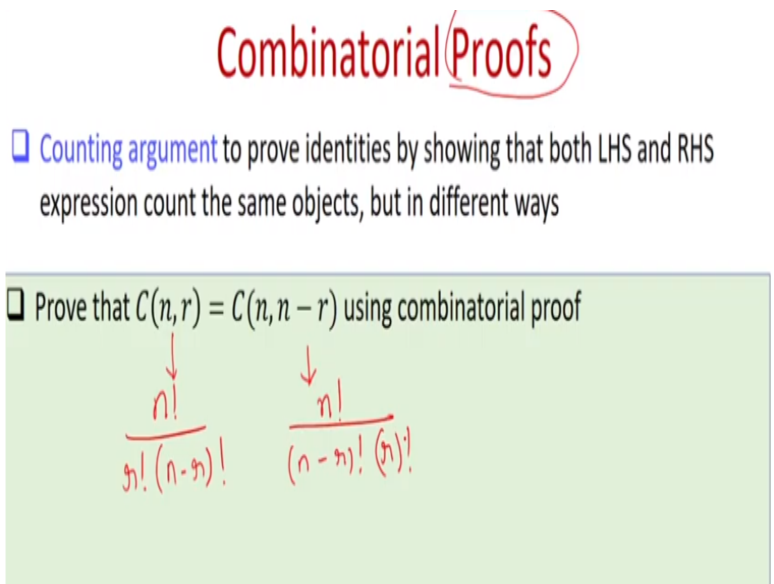

Now let us go to the last topic for today's lecture namely combinatorial proofs and again I am sure that you have studied it during your high school. So what exactly are combinatorial proofs. This are some common proof strategy which we often use in combinatorics. Namely it's a counting argument to prove identities where you have something on the left-hand side and something on your right-hand side and you want to prove mathematically that your expression in the left-hand side and the right-hand side are same.

But to do that we use a counting argument and we prove that the expression in the left-hand side and the expression on the right-hand side count the same number of objects but in different ways. But nowhere in the proof we actually expand our expressions on the left-hand side or right-hand side and show by simplification that left-hand side is same as right-hand side. We do not do that.

That is not the goal of a combinatorial proof. So let us see a very simple combinatorial proof which you must have definitely studied. We will want to prove that the value of $C(n,r)$ is the same as the value of $C(n, n-r)$. Of course one way of doing that is I expand $C(n,r)$ and rewrite it as $\frac{n!}{r!(n-r)!}$. And I expand my right-hand side. And in this case actually both the expressions are same.

So I could have simply said that they are same. But that is not the goal of combinatorial proof. In general when we are giving a combinatorial proof for proving LHS and RHS are same we do not expand or simplify the expressions in the left-hand side and right-hand side. If you do that, that's not a combinatorial proof. You will get 0 marks if you are asked to prove something by combinatorial proof and you end up simplifying expressions.

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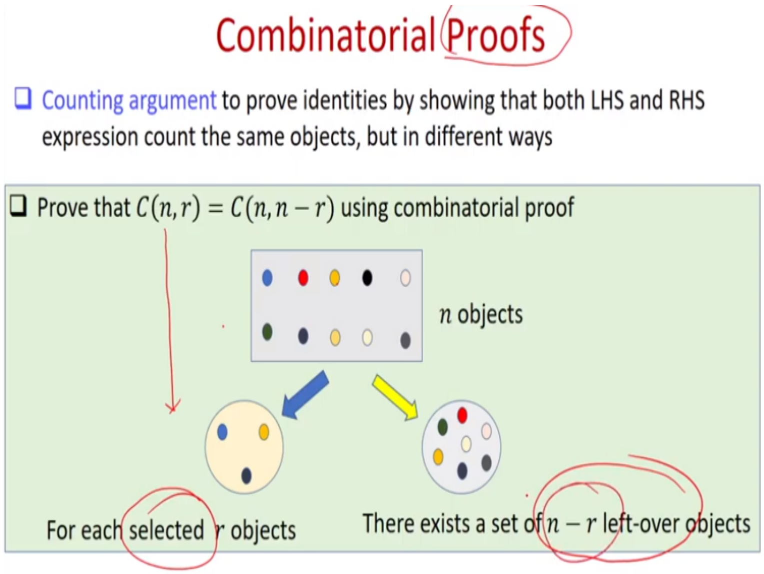

The way we are going to prove this equality using combinatorial proof, is the following. Suppose you are given n objects then your left-hand side is nothing but the number of ways in which you can pick r objects out of those n objects. That's the interpretation of $C(n,r)$ function. Now it turns out for each of the ways in which you can select r objects there is a way of excluding $n-r$ objects.

So I can reinterpret my problem and say that instead of worrying about how many ways I can pick r objects out of n objects, I instead count the number of ways I will decide or the number of ways I will choose the $n-r$ objects which I want to leave. Because once I decide that these are the $n-r$ objects which I am going to leave that automatically gives me the r objects which I will be taking or considering.

So that's why it is easy to see that the LHS expression and RHS expression are same. And we are counting 2 different things here. The left-hand side expression basically counts the number of ways you would have selected the objects. Whereas the right-hand side expression counts the number of ways in which you would have left objects. And there is a mapping. Whatever you left corresponding to that you are left with object which you are picking.

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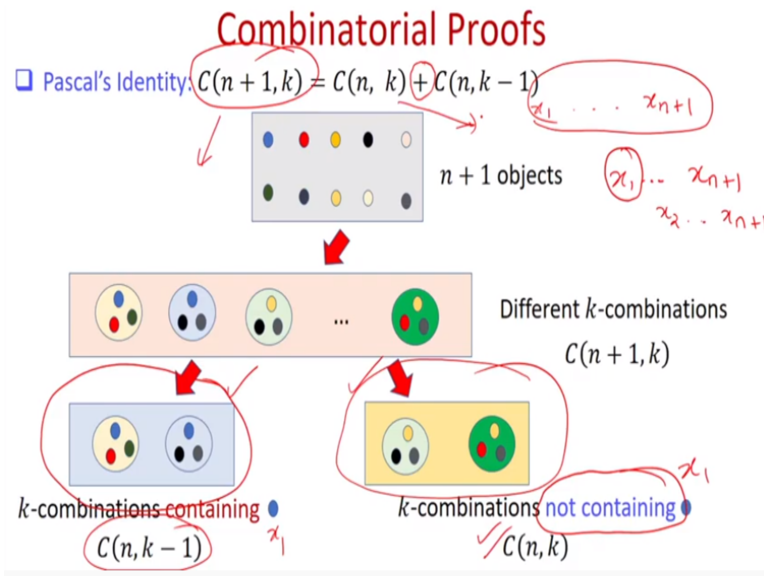


Now let's prove an interesting combinatorial identity using combinatorial proof. This is often called as Pascal's identity. So to prove this consider a collection of $n+1$ objects. So I am calling those $n+1$ objects as say x_1 to x_{n+1} they are distinct. Now what is my left-hand side expression? That denotes the total number of k-combinations I can have out of those $n+1$ objects. I have to pick k objects. I can do that in $C(n+1,k)$ ways. That is the left-hand side.

Now I have to show that the same thing can be counted by adding these 2 quantities that are there in the right-hand side. How do I do that? So my claim is that different k-combinations that I can have out of x_1 to x_{n+1} can be divided into 2 groups. That will take care of the addition that we have on your right-hand side expression. I have to somehow show that the total number of different k-combinations that I can have can be divided into 2 categories, 2 disjoint categories to be more specific and those 2 disjoint categories are the following.

You consider all k-combinations that you can form out of those $n+1$ objects where a specific object is always present. Say the object x_1 is always present. And the number of such k-combinations is nothing but $C(n,k-1)$. Because if the object x_1 is always going to be included in the k objects which you are finally choosing, then you have to worry about how many ways you can pick the remaining $k-1$ objects out of x_1 to x_n.

So you had x_1 to x_{n+1} so you are always going to choose x_1 that is the category we are right now considering. So now you are left with n objects namely x_2 to x_{n+1} and you have to choose $k-1$ objects out of this remaining n objects which you can do in these many ways. In each such k-combination you include the object x_1 that will give you category 1 of k-combinations.

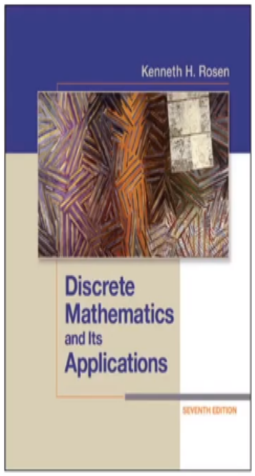
Whereas category 2 k-combinations are the one where the object x_1 is never included. And it is easy to see that the number of different k-combinations of this category is $C(n, k)$. Because if you are not going to includex_1 then your problem is still to choose k objects and now you are left with only n objects to choose for those k objects. You can choose your k objects only from the collection x_2 to x_{n+1}.

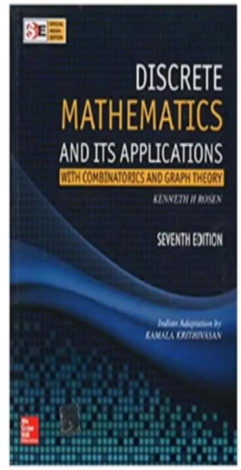
So you are left with only with n possibilities and the number of k-combinations that you can now have in the second category is this. And now if I focus on the total or the different k-combinations that I can have out of this x_1 to x_{n+1}, I can have either a k-combination of category 1 or a k-combination of category 2. Namely in the k combination either, x_1 is there or x_1 is not there. I cannot have any third possible category.

And these 2 categories are disjoint. There is no k-combination where x_1 is present as well as x_1 is absent. So if I sum the total number of k-combinations that I have in category 1 and the number of k-combinations that I have in category 2 that will give me the total number of k-combinations that I can have for a set consisting of $n+1$ objects. And that's precisely your right-hand side. And this is a combinatorial proof because now I have not expanded my left-hand side expression, I have not expanded my right-hand side expression and simplified them. I am just giving a counting argument and proving that LHS and RHS are counting the same things.

(Refer Slide Time: 38:09)

References for Today's Lecture





So that brings me to the end of today's lecture. These are the references used for the today's lecture. Just to summarize, in this lecture we introduced permutations, combinations, we saw the formula for permutations and combinations both with repetitions and without repetition. And we also discussed about combinatorial proofs.