**Discrete Mathematics**
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**Lecture -26**
**Tutorial 4: Part II**

Hello everyone welcome to the second part of tutorial 4.

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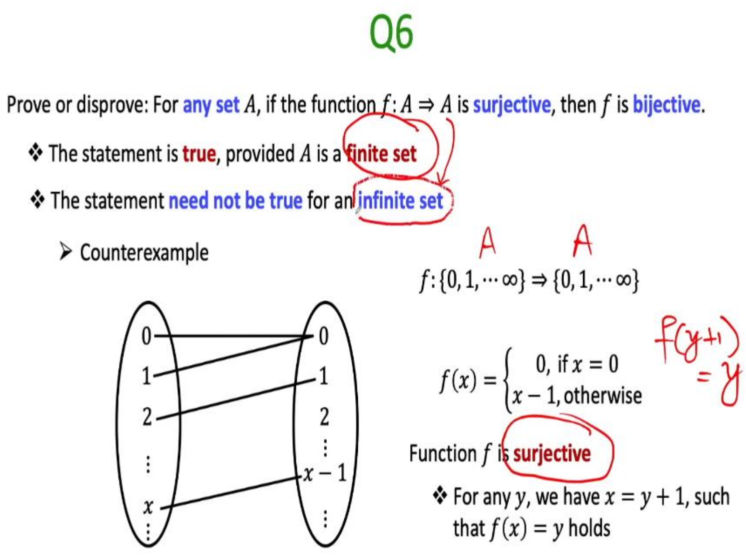


So, we begin with question number 6. In question number 6 you are asked to either prove or disprove the following. You are given a function $f: A \rightarrow$A and you are given that the function is a surjective. Then the question is, is it necessary that the function is a bijective as well. It turns out that the statement is true provided your set A is a finite set. Because indeed if the set is a finite set and the function is from the same set to itself and surjective.

Then we can show it is a bijective function as well. So, we will touch upon this fact sometime later in this course. But the statement need not be true if the set A is an infinite set. So, here is a counter example. So, imagine the function f given from the set of set 0 to infinity to the set 0 to infinity, . So, that is my set A and the function is defined as follows. The mapping $0 \rightarrow 0$ and the mapping of $1 \rightarrow 0$.

So, clearly the function is not injective, not 1 to 1 and what about the mapping of the remaining elements. The mapping of element 2 is 1 the mapping of element $x \rightarrow (x - 1)$ and so on. So, it is easy to verify that the function is indeed a surjective function because you pick any element y from the set 0 to infinity, the pre-image for that element y will be y + 1. Because $f(y + 1) = y$ as per the function f we have defined here.

So, clearly my function is a surjective function. But it is not one to one and that is why it is not the bijective function. The problem due to which we cannot say it is a bijection is because the set over which the function is defined can be an infinite set.

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## Q7

Let $R$ be an **equivalence relation** on a set $A$, where $|A| = 30$ and let $R$ partition $A$ into equivalence classes $A_1, A_2, A_3$. If $|A_1| = |A_2| = |A_3|$, then what is $|R|$?

- $A_1, A_2, A_3$ constitute a **partition** of $A$
  - Since $|A_1| = |A_2| = |A_3|$ and $|A| = 30$, we get that $|A_1| = |A_2| = |A_3| = 10$

$$R = \{(i, j) : i, j \in A_k, \text{for } k = 1, \ldots, 3\}$$

- Elements from $A_1$ contribute to $10^2 = 100$ tuples $(i, j)$ to $R$
- Elements from $A_2$ contribute to $10^2 = 100$ tuples $(i, j)$ to $R$
- Elements from $A_3$ contribute to $10^2 = 100$ tuples $(i, j)$ to $R$

$$|R| = 100 + 100 + 100 = 300$$

In question number 7, you are given an equivalence relation over a set A, where the set A has 30 elements. And since it is an equivalence relation, the relation partitions the set A into three subsets each of equal size. So, the question asks you how many ordered pairs are there in that equivalence relation? So, since the subsets $A_1, A_2, A_3$ constitute a partition of the set A and it is also given that the size of each subset is same and since the number of elements in the set A is 30, we get that the size of each subset in the partition is 10.

Recall, when we showed that for every equivalence relation there is a partition and for every partition there is an equivalence relation, we showed that if you are given a partition how you get the corresponding equivalence relation whose equivalence class will be giving you that partition.

So, in that construction our equivalence relation was consisting of all ordered pairs of the form (i, j), where for every subset $A_k$ in your given partition if the elements (i, j) are present in that subset you add the ordered pair (i, j). So, based upon this fact we get here that the elements of the subset $A_1$ within the partition will contribute to ten square ordered pairs of the form (i, j) and they will be added to the relation R.

Similarly, you have 10 elements within the subset $A_2$ and they will contribute to 10 square ordered pairs as per our construction in the relation R and in the same way you have 10 elements in the subset $A_3$ and they will contribute to 10 square number of (i, j) ordered pairs or order tuples in the relation so, as a result the number of ordered pairs in our equivalence relation with 300.

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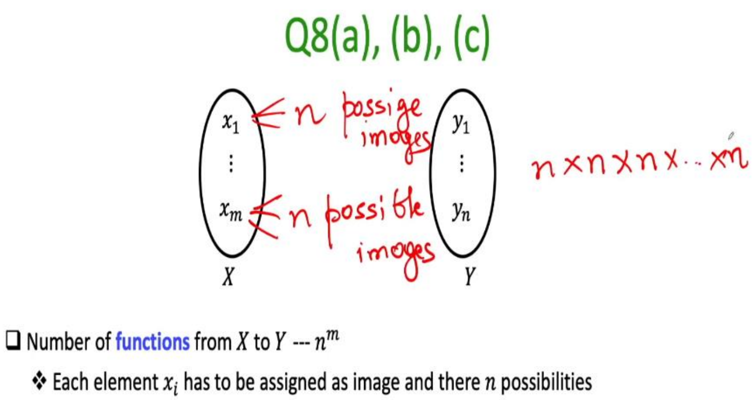


In question 8 part (a), (b), (c) we are supposed to count certain things. So, you are given two sets X and Y consisting of m and n number of elements. So, I am calling the elements of $X = \{x_1, \dots, x_m\}$ and the elements of the $Y = \{y_1, \dots, y_n\}$. We are supposed to find out the number of functions from the $X \to Y$. It turns out that the number of functions will be $n^m$, why so?

Because when we want to build a function from the set $X \to Y$, each element $x_i$ from the set X has to be assigned an image that is the definition of a function. Now how many ways I can assign an image for the element $x_i$? Well I can assign $y_1$ as the possible image for $x_i$, I can pick $y_2$ as the possible image for $x_i$ and in the same way i can pick $y_n$ as a possible image for the element $x_i$. So, there are n possibilities when it comes to assigning image for an element $x_i$.

And the image for $x_i$ and the image for $x_j$ they are independently picked, there is no dependency between the images of $x_i$ and images of $x_j$ that is important here because we are just interested in counting the number of functions. That means it might be possible that the image of $x_i$ is same as the image of $x_j$ and so on. So, there is absolutely no restriction on the way we can pick the images for $x_i$ and we can pick the images for $x_j$.

So, based on all these observations we can say that I have n number of possibilities when it comes to assigning image for $x_1$. And like that for each of the elements from the X set I have n possible images which I can choose. And that is why the number of functions are, nothing but the number of ways I can pick the images for each of the element from the X set that is $n \cdot n \cdot \ldots n$ , $m$ times which is nothing but $n^m$.

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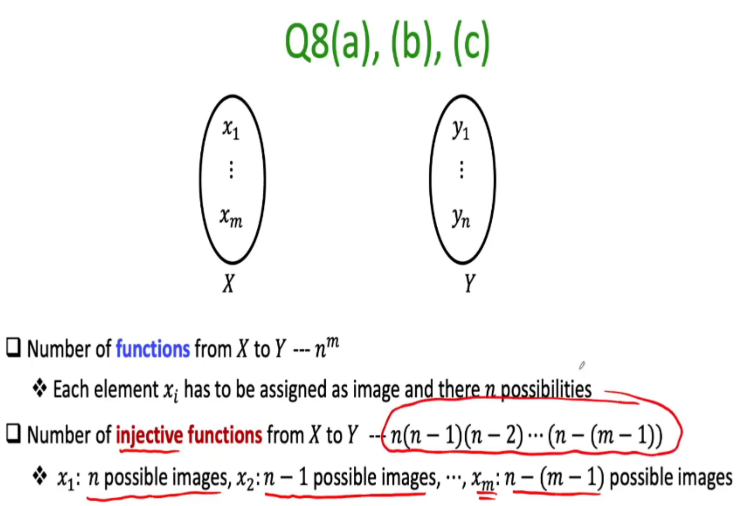


Part b asked you to find out the number of injective functions from the X set to Y set. Well we will be using more or less similar argument that we used for part A except that now we cannot say that the images for every element $x_i$ is chosen independently. Because now we are counting or we are interested in the injective functions and in injective functions for every distinct element $x_i$ from the set X you have to assign a unique image.

You cannot have both $x_1$ and $x_2$ getting mapped to the same element in the Y set. So, that is why when it comes to selecting image for $x_1$, I have n possibilities. But once I have decided the image for the element $x_1$, I cannot assign that image to be a possible image for element $x_2$, that is why for $x_2$ I have n - 1 possible images and like that when once I have fixed the images for $x_1$, $x_2$ and $x_{m-1}$.

When I am assigning the image for the mth element from the X set that image has to be different from all the images which I have selected for the previous elements of the X sets. That means I have only these many number of possible images, namely n - m - 1 possible images to assign for the element $x_m$. So, that is why the total number of injective functions will be now $n \cdot (n-1) \cdot (n-2) \cdot \ldots (n-m-1)$.

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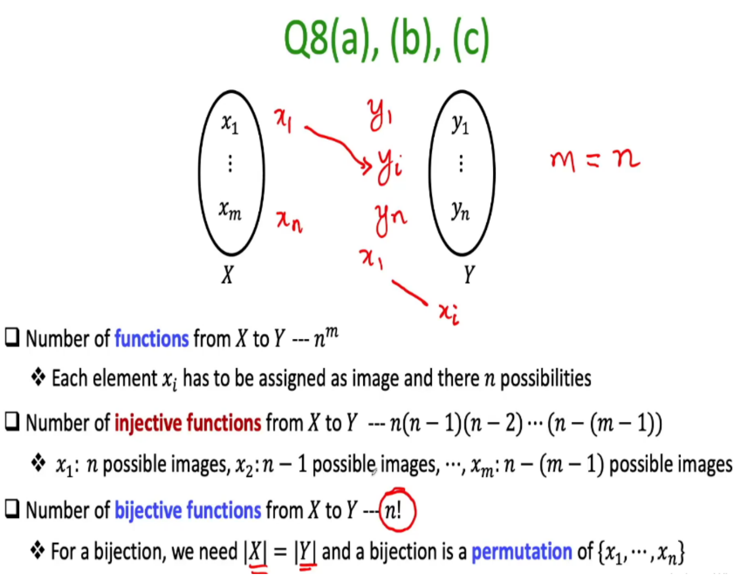


Part c asks, you to find out the number of bijective functions from X to Y. So, the first thing to observe here is that for a bijection from X to Y we need $|X| = |Y|$. It is very easy to verify that if their cardinalities are different, then we cannot have a one to one and onto mapping from the X set to the Y set. Now if the cardinality of the X and the Y set are same.

That means I am talking about the case where m =n then any bijection from the X set to Y set can be considered as a permutation of the elements $x_1$ to $x_n$. Because I can imagine that I have n number of elements here and I have also n number of elements here and each $x_i$ has to be assigned a unique image. So, that can be interpreted if $x_1$ is assigned as the image $y_i$ as per your bijection, then I can imagine that $x_1$ is getting shifted to the ith position, that way I can think of bijection between the X set to the Y set.

Even though $x_1 \to y_1$, I can interpret in my mind $y_i$ to be same as $x_i$ and as a result if I do this logical mapping, I can interpret a bijection from the X set to the Y set as nothing but a permutation of $x_1$ to $x_n$. And how many permutations I can have for n elements, for us I can have $n!$ number of permutations. So, that will be the number of bijective functions from the set X to the set Y.

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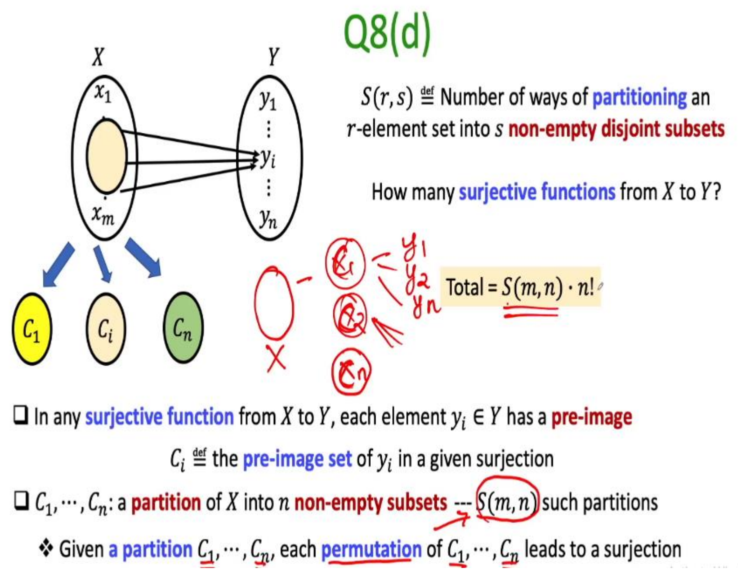


In part d of question 8 we are introducing a function, the S function this function is also called as Stirling function of type 2. And this is a very important function when it comes to combinatorics we will encounter it later again. So, what exactly is this function this is a two input function it takes an input r and an input s and it denotes basically the number of ways of partitioning an r element set into s non-empty disjoint subsets. Of course, $s \leq r$.

You have a bigger set call it A, $|A| = r$ and basically we want to find out how many ways I can split this bigger set into a pairwise disjoint subsets, basically s number of pairwise disjoint subsets such that their union gives back you the original set. So, there might be several ways of dividing this bigger set A into s number of pairwise non-empty disjoint subsets. The number of divisions is nothing but the value of the stirling function of type 2.

Now using this stirling function we have to count the number of surjective functions possible from the set X to the set Y. So, since we are interested to find out the number of surjective function, remember in a surjective function each element from the codomain set should have at least one pre-image. Well it can have more than one pre-image as well. So, let me define the set $C_i$ to be the pre-image set of any element $y_i$ from the co domain set namely whichever elements could be the possible pre-images for the element $y_i$ the collection of those pre-images I am calling it to be the $C_i$ set. Now it is easy to see that if I take any surjective function with respect to that surjective function, if I focus on this pre-image set of element $y_1$, the pre-image set of the element $y_2$ and like that the pre-image set of the element $y_n$. Then each of the collection of those subsets the collection of those pre-image sets will constitute a partition of your set X namely the domain set, why so?

So, it is easy to see first of all that the intersection of these pre-image subsets will be empty set. You cannot have an x present in both $C_1$ and say $C_i$. That means you have two possible images for the element x which is a violation of the definition of any function. And it is also easy to see that if I take the union of these pre-image sets I will get back the domain X, Y.

Because as per the definition of a function each element x from the set X will have an image. So, that is the way I can interpret any surjective function you give me any surjective function and if I focus on the collection of pre-image sets of various elements from the co-domain that will constitute a partition of the domain set. So, now what we will do is how many such partitions can we have for the set X into n non empty disjoint subsets?

I can have $S(m, n)$ number of such partitions. So, now we are interested to find out how many surjective functions we can have. So, what we can say is, if you want to construct a surjection you first divide your set X into n pairwise non-empty disjoint subsets, call them as $C_1$ to $C_n$. How many such partitions you can have? $S(m, n)$ number of such partitions. Now once you have divided your set X into n pairwise non empty disjoint subsets, each permutation of those subsets leads to a surjection.

So, what I am trying to say is you have divided your set X into $C_1, C_2, \dots, C\ n$. And now $C_1$ could be the pre-image set of either $y_1$ or $y_2$ or it could be assigned as a pre-image set of $y_n$. Now once we have decided that $C_1$ is going to be the pre-image set of which element from the co domain.

We next have to assign the subset$C_2$ to be the possible pre-image set of any element from the Y set except for the element which has been assigned to this subset $C_1$. And like that I can continue for the remaining subsets in my partition. So, that is why I am saying here once you have decided the subsets within your partition you take any permutation of that corresponds to a surjection. So, that is why the total number of surjective function will be $S(m, n) \cdot n!$.

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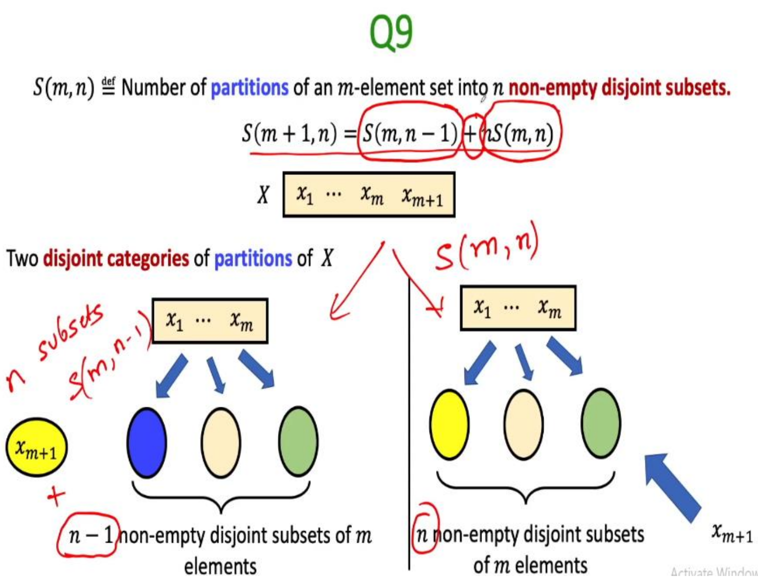


In question 9 we are continuing with the notion of our stirling numbers and you are supposed to prove that the stirling function satisfies this recurrence condition. So, to prove this statement consider a set X which has m + 1 number of elements and we want to divide this set X into n pairwise non empty disjoint subsets. We want to find out how many ways we can do the division. So, my claim is whatever way you divide this set X into n number of pairwise non empty disjoint subsets, the division can be of one of the following two categories. Category 1 division where the first m elements in the set X are divided into n - 1 number of pairwise non empty disjoint subsets and the last element $x_{m+1}$ is occupying a solitary position in a single subset. So, like that you have now total n number of subsets n - 1 number of subsets. Their union will give you $x_1$ to $x_m$ and you have an additional subset which has only element $x_{m+1}$.

That is one category of partition of the set X. How many partitions in this category we can have? The number of partitions in this category is nothing but, $S(m, n - 1)$ because basically the number of ways in which you can partition the first m elements into n - 1 number of pairwise disjoint subsets. In each such partition you just add one additional subset consisting of the solitary element $x_{m+1}$.

That will give you a valid partition for the bigger set X and the number of such partitions we can have here is nothing but $S(m, n - 1)$. That is one category of partition. The second category of partition that we can have for the set X will be as follows, I divide the first m elements into now n pairwise non-empty disjoints subsets. I can have $S(m, n)$ number of such subsets now what about the element $x_{m+1}$.

Well we can either include it in the first subset or in the second subset or in the third subset or in the last subset and that will give you an overall valid partition for the bigger set X. And clearly the partition in this category is disjoint from the partitions in the first category. Because in the, first category of partition the element $x_{m+1}$ is present alone in a single subset. Whereas in the, second category of partition the element $x_{m+1}$ is not the only element within its subset, it is present along with some other elements as well. And you cannot have any other third category of partition for the subset x.

You can have either partition of type 1 or partition of type 2. Now how many partitions of type 2, I can have? I can have $n \cdot S(m, n)$ number of partitions. This is because the element $x_{m+1}$ can occupy any of the n subsets and plus because the two categories of partitions are disjoint. And if I sum them I will get all possible ways of partitioning the set X.

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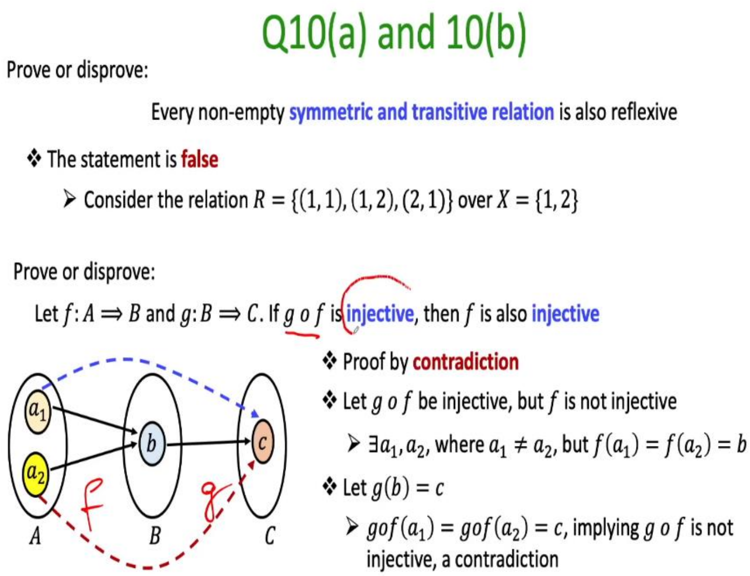


In question 10a, you are asked to either prove what is proof that every non-empty symmetric and transitive relation is also reflexive. Well we can give a very simple counter example to prove that the statement is false. Imagine you are given a relation R over this set X, the relation is clearly symmetric. It is also transitive. If you are wondering why it is transitive you have (1, 1) and (1, 1) present and also (1, 1) present and you have (1, 2), (2, 1) present.

So, you should have (1, 1) in the relation which is present in the relation but the relation is not reflexive because (2, 2) not present. In part 2 you are given two functions $f: A -> B$ and $g: B \rightarrow C$ respectively. And you are also given that $gof$ injective. Then the question is, is it necessary that f is also injective. The statement is true and we can prove it by contradiction.

So, imagine that g o f is injective but f is not injective. Since f is not injective that means I have a pair of distinct elements from the A set, say $a_1$ and $a_2$ getting mapped to the same image, say b and say the image of b as for the g function is c. Then I get a contradiction that $gof(a_1)$ and $gof(a_2)$ are same namely c, but $a_1 \neq a_2$ showing that $gof$ is not injective which is a contradiction to my premise here.

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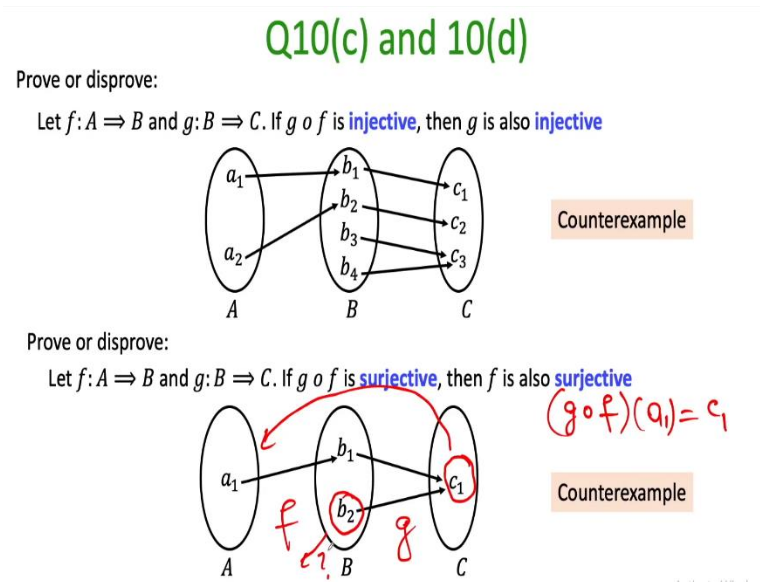


In question 10c you are given that $gof$ is injective, then the question is, is it necessary that g is also injective? Well we can give us counterexample to disprove this statement. So, take this f function and g function clearly $gof(a_1) = c_1$ and $gof(a_2) = c_2$. So, clearly my premise is satisfied here $gof$ function is injective, but what about the g function it is not injective you have $g(b_3)$ and $g(b_4)$ both mapping to $c_3$, g function is not injective.

So, this statement is not necessarily true. Part d you are given the f and g functions and your $gof$ function is surjective none is it necessary that the function f is also surjective again this is not necessary here is a very simple counterexample this is your f function this is your g function. Your $gof$ is surjective because indeed there is only one element in your set C, namely $c_1$ and the pre-image for that $c_1$ is $a_1$ because you have $gof(a_1) = c_1$.

So, the function $gof$ is indeed surjective, but the function f is not surjective because if you take the element $b_2$ it has no pre-image. So, that shows this statement is not necessarily true. So, that brings me to the end of part two of tutorial 4. Thank you!